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Another proof of the end curve theorem for normal surface singularities

Dedicated to Professor Kimio Watanabe on the occasion of his sixtieth birthday

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Abstract. Neumann and Wahl introduced the notion of splice-quotient singularities, which is a broad generalization of quasihomogeneous singularities with rational homology sphere links, and proved the End Curve Theorem that characterizes splice-quotient singularities. The purpose of this paper is to give another proof of the End Curve Theorem. We use combinatorics of "monomial cycles" and some basic ring theory, whereas they applied their theory of numerical semigroups.

1. Introduction.

Let (X,o) be a normal complex surface singularity with $\boldsymbol{Q}$-homology sphere link Σ and $\pi\colon \widetilde{X} \to X$ a good resolution with the exceptional set E. Then there uniquely exists a *universal abelian covering* $q\colon (Y,o) \to (X,o)$; by definition q is a finite morphism of normal surface singularities which induces an unramified Galois covering $Y \setminus \{o\} \to X \setminus \{o\}$ with covering transformation group $\boldsymbol{H} := H_1(\Sigma, \boldsymbol{Z})$. In [**5**] Neumann proved that if (X,o) is quasihomogeneous, then (Y,o) is a Brieskorn-Pham complete intersection, and that its system of equations and the $\boldsymbol{H}$-action are explicitly obtained from the weighted dual graph Γ of E. This situation is significantly generalized by Neumann and Wahl; see [**9**], [**8**].

They introduced the notion of *splice type* singularities whose equations are explicitly obtained from Γ satisfying the "semigroup condition." Suppose that we obtained a splice type singularity (Z,o) from Γ. They proved that (Z,o) is an isolated complete intersection surface singularity, and that if the "congruence condition" on Γ is satisfied and equations are chosen so that $\boldsymbol{H}$ naturally acts on (Z,o), then the quotient $Z/\boldsymbol{H}$ is a normal surface singularity with resolution graph Γ, and the quotient morphism $Z \to Z/\boldsymbol{H}$ is the universal abelian covering.

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We call $(Z/\boldsymbol{H}, o)$ a *splice-quotient singularity*; this gives a generalization of quasihomogeneous singularities with $\boldsymbol{Q}$-homology sphere links, and furthermore the class of these singularities includes rational singularities and minimally elliptic singularities with $\boldsymbol{Q}$-homology sphere link (see [**11**]).

A characterization of splice-quotient singularities (cf. Theorem 2.10) is given by

END CURVE THEOREM (Neumann-Wahl [**7**]). *Let $E_1, \dots, E_n$ denote the irreducible components at the end of E. If there exist $\boldsymbol{Q}$-Cartier prime divisors $H_1, \dots, H_n$ on X satisfying $\widetilde{H}_i \cdot E = \widetilde{H}_i \cdot E_i = 1$, where $\widetilde{H}_i$ denotes the strict transform of H_i, then (X, o) is a splice-quotient singularity. In fact, functions z_i with $\mathrm{div}_Y(z_i) = q^*H_i$ give "coordinates" of the splice type singularity Y.*

This theorem, first announced in [**6**], also plays a key role in the proofs of the formula for the geometric genus of splice-quotient singularities ([**12**]), the Casson invariant conjecture for splice type singularities ([**4**]) and the extension due to Némethi and Nicolaescu ([**3**]).

First, Neumann and Wahl proved the End Curve Theorem in the case where $\boldsymbol{H}$ is trivial, applying a theory of numerical semigroups which concerns irreducible curve singularities, that they developed ([**9**, Sections 3–4]). For the general case, they further extended the theory to study reducible curves with $\boldsymbol{H}$-action (for example, q^*H_i above).

The purpose of this paper is to give another proof of the End Curve Theorem. The author has studied splice-quotient singularities using a terminology "monomial cycle" ([**11**], [**12**]). We will take this approach. Although the notion of monomial cycles is elementary, it is useful for connecting the combinatorics of the resolution graph to analytic subjects such as the existence of the curves $\widetilde{H}_i$. For the proof, we will use these methods, some basic ring theory, and a formula for the delta invariant of curve singularities; the idea of computing the delta invariant was used in [**9**, 4.1].

This paper is organized as follows. In Section 2, we review the definition of splice-quotient singularities in terms of monomial cycles. In Section 3, we prove that the assumption of the End Curve Theorem implies the "monomial condition," which is equivalent to the semigroup and congruence conditions. In Section 4, we study the associated graded rings with respect to certain filtrations of the local rings of universal abelian covers, and compute the delta invariants of hyperplane sections. These results will conclude the proof.

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2. Splice-quotient singularities.

We recall the definition of splice-quotient singularities in terms of "monomial cycles." We refer [**12**, Section 2] for more details.

Let (X, o) be a normal complex surface singularity whose link is a $\boldsymbol{Q}$-homology sphere, and $\pi\colon \widetilde{X} \to X$ a good resolution with exceptional divisor $E = \pi^{-1}(o)$. Then E is a tree of rational curves. Let $\{E_v\}_{v\in\mathscr{V}}$ denote the set of irreducible components of E and $\boldsymbol{L}$ the group of divisors supported in E; i.e.,

$$\boldsymbol{L} = \sum_{v\in\mathscr{V}} \boldsymbol{Z}E_v.$$

We call an element of $\boldsymbol{L}$ (resp. $\boldsymbol{L}\otimes\boldsymbol{Q}$) a cycle (resp. $\boldsymbol{Q}$-cycle). Since the intersection matrix $I(E) := (E_v \cdot E_w)$ is negative definite, for each $v \in \mathscr{V}$ there exists an effective $\boldsymbol{Q}$-cycle E_v^* such that $E_v^* \cdot E_w = -\delta_{vw}$ for every $w \in \mathscr{V}$, where δ_{vw} denotes the Kronecker delta. Let

$$\boldsymbol{L}^* = \sum_{v\in\mathscr{V}} \boldsymbol{Z}E_v^*.$$

Let δ_v denote the number of irreducible components of E intersecting E_v, i.e., $\delta_v = (E - E_v)\cdot E_v$. A curve E_v is called an *end* (resp. a *node*) if $\delta_v = 1$ (resp. $\delta_v \geq 3$). Let $\mathscr{E}$ (resp. $\mathscr{N}$) denote the set of indices of ends (resp. nodes). A connected component of $E - E_v$ is called a *branch* of E_v. Let $\boldsymbol{C}\{z\} := \boldsymbol{C}\{z_w; w\in\mathscr{E}\}$ be the convergent power series ring in $\#\mathscr{E}$ variables.

Definition 2.1. An element of a semigroup $\mathscr{M} := \sum_{w\in\mathscr{E}} \boldsymbol{Z}_{\geq 0}E_w^*$, where $\boldsymbol{Z}_{\geq 0}$ is the set of nonnegative integers, is called a *monomial cycle*. For a monomial cycle $D = \sum_{w\in\mathscr{E}} \alpha_w E_w^*$, we associate a monomial $z(D) := \prod_{w\in\mathscr{E}} z_w^{\alpha_w} \in \boldsymbol{C}\{z\}$.

Definition 2.2. We say that E (or its weighted dual graph) satisfies the *monomial condition* if for any node E_v and any branch C of E_v, there exists a monomial cycle D such that $D - E_v^*$ is an effective cycle supported on C. The monomial $z(D)$ is called an *admissible monomial* belonging to the branch C.

The monomial condition is equivalent to the *semigroup* and *congruence condition* (see [**8**, Section 13]). The original definition of admissible monomials requires only the semigroup condition ([**8**, Section 2]); the congruence condition is needed to obtain "appropriate" splice diagram functions (cf. Theorem 2.7).

Definition 2.3. Assume that the monomial condition is satisfied. Let E_v be a node with branches $C_1, \ldots, C_{\delta_v}$, and let m_i denote an admissible monomial belonging to C_i. Let (c_{ij}) be an arbitrary $(\delta_v - 2) \times \delta_v$ matrix with $c_{ij} \in \boldsymbol{C}$ such that every maximal minor of it has rank $\delta_v - 2$. We define polynomials $f_1, \dots, f_{\delta_v - 2}$ by $f_i = \sum_{j=1}^{\delta_v} c_{ij} m_j$. We call $\mathscr{F}_v := \{f_1, \dots, f_{\delta_v - 2}\}$ a *Neumann-Wahl system* at E_v, and $\bigcup_{v \in \mathscr{N}} \mathscr{F}_v$ a *Neumann-Wahl system* associated with E.

Let a_{vw} denote the (v, w)-element of the matrix $-I(E)^{-1}$ and $d(E)$ the absolute value of $\det I(E)$. We define positive integers e_v, ℓ_{vw}, and m_{vw} as follows:

$$\ell_{vw} = d(E) a_{vw}, \quad e_v = d(E)/\gcd\{\ell_{vw} \mid w \in \mathscr{V}\}, \quad m_{vw} = e_v a_{vw}.$$

DEFINITION 2.4. For any $v \in \mathscr{V}$, we define the *v-weight* of z_w to be m_{vw}. Therefore, the *v-degree* of a monomial $\prod_{w \in \mathscr{E}} z_w^{\alpha_w}$ is $\sum_{w \in \mathscr{E}} \alpha_w m_{vw}$. The leading form of $f \in \boldsymbol{C}\{z\}$ with respect to the v-weight is called the *v-leading form* of f and denoted by $\mathrm{LF}_v(f)$. The v-degree of $\mathrm{LF}_v(f)$ is called the *v-order* of f.

Note that for $D = \sum_{w \in \mathscr{V}} \alpha_w E_w \in \mathscr{M}$ we have

$$v\text{-}\deg z(D) = e_v \alpha_v = -e_v D \cdot E_v^*.$$

DEFINITION 2.5. We call a set

$$\mathscr{F} := \{f_{vj_v} \mid v \in \mathscr{N},\ j_v = 1, \dots, \delta_v - 2\} \subset \boldsymbol{C}\{z\}$$

a *system of splice diagram functions* associated with E if $\{\mathrm{LF}_v(f_{vj_v}) \mid f_{vj_v} \in \mathscr{F}\}$ is a Neumann-Wahl system associated with E. A germ of a singularity in $(\boldsymbol{C}^{\#\mathscr{E}}, o)$ defined by a system of splice diagram functions is called a *splice type singularity.*

THEOREM 2.6 (Neumann-Wahl [**8**, 2.6]). *A splice type singularity is an isolated complete intersection surface singularity.*

Let us recall that the first homology group of the link of (X, o) is isomorphic to a finite group $\boldsymbol{H} := \boldsymbol{L}^*/\boldsymbol{L}$ of order $d(E)$. The intersection form $\boldsymbol{L}^* \times \boldsymbol{L}^* \to \boldsymbol{Q}$ induces a pairing

$$\theta \colon \boldsymbol{H} \times \boldsymbol{L}^* \to \boldsymbol{Q}/\boldsymbol{Z} \xrightarrow{e} \boldsymbol{C}^*,$$

where $e(x) = \exp(2\pi\sqrt{-1}x)$. We denote by $\theta(D)$ the character determined by $\theta(\,\cdot\,, D)$. The group $\boldsymbol{H}$ acts on the power series ring $\boldsymbol{C}\{z\}$ as follows. For any $(h, D) \in \boldsymbol{H} \times \mathscr{M}$, we define $h \cdot z(D) \in \boldsymbol{C}\{z\}$ by

$$h \cdot z(D) = \theta(h, D) z(D).$$

If $f \in \boldsymbol{C}\{z\}$ and $\chi \in \hat{\boldsymbol{H}} := \mathrm{Hom}(\boldsymbol{H}, \boldsymbol{C}^*)$ satisfies $h \cdot f = \chi(h) f$ for all $h \in \boldsymbol{H}$, we call f a *χ-eigenfunction*; for example, a Neumann-Wahl system at a node E_v consists of $\theta(E_v^*)$-eigenfunctions.

THEOREM 2.7 (Neumann-Wahl [**8**, 7.2]). *Suppose that (Z, o) is a singularity defined by a system of splice diagram functions $\{f_{vj_v}\}$ such that every f_{vj_v} is a $\theta(E_v^*)$-eigenfunction. Then the quotient $Z/\boldsymbol{H}$ is a normal surface singularity whose weighted dual graph is the same as that of X, and the quotient map $Z \to Z/\boldsymbol{H}$ is the universal abelian covering.*

DEFINITION 2.8. A singularity whose universal abelian cover is a splice type singularity is called a *splice-quotient singularity.*

DEFINITION 2.9. We say that $\widetilde{X}$ satisfies the *end curve condition* if for each $w \in \mathscr{E}$ there exists an irreducible curve $H_w \subset \widetilde{X}$, not contained in E, such that $H_w \cdot E = H_w \cdot E_w = 1$ and $e_w(E_w^* + H_w) \sim 0$. We call H_w an *end curve* at E_w.

Let $\mathrm{Div}(\widetilde{X})$ denote the group of divisors on $\widetilde{X}$ and $c_1 \colon \mathrm{Div}(\widetilde{X}) \to \boldsymbol{L}^*$ a map defined by $c_1(D) = \sum_{v \in \mathscr{V}} (-D \cdot E_v) E_v^*$. We fix a section

$$\sigma \colon \boldsymbol{L}^* \to \mathrm{Div}(\widetilde{X})$$

of the homomorphism c_1, which satisfies that $d(E)\sigma(D) \sim d(E)D$ for every $D \in \boldsymbol{L}^*$. Using σ, the $\theta(D)$-eigenspace of $H^0(\mathscr{O}_Y)$ is expressed as $H^0(\mathscr{O}_{\widetilde{X}}(-\sigma(D - [D])))$, where $[D]$ denotes the integral part (see [**12**, Section 3.2]).

If the end curve condition is satisfied, then for each $w \in \mathscr{E}$ there exists a section $s_w \in H^0(\mathscr{O}_{\widetilde{X}}(-\sigma(E_w^*)))$ which defines a divisor $\sigma(E_w^*) + H_w$, and hence we obtain an $\boldsymbol{H}$-equivariant $\boldsymbol{C}$-algebra homomorphism

$$\psi \colon \boldsymbol{C}\{z\} \to \mathscr{O}_{Y,o}, \quad \psi(z_w) = s_w. \tag{2.1}$$

Our goal of this article is to prove the following:

THEOREM 2.10 (End Curve Theorem). *If $\widetilde{X}$ satisfies the end curve condition, then X is a splice-quotient singularity; in fact, the homomorphism ψ is surjective, and its kernel is generated by a system of splice diagram functions with $\boldsymbol{H}$-action.*

REMARK 2.11. If (X, o) is splice-quotient, then every $\theta(D)$-eigenfunction in $H^0(\mathscr{O}_Y)$ is a series of monomials $z(D')$ with $D' \in \boldsymbol{L} + D$.

3. The monomial condition.

In this section we will prove the following.

PROPOSITION 3.1. *The end curve condition implies the monomial condition.*

From now on, we always assume that $\widetilde{X}$ satisfies the end curve condition, and fix a homomorphism $\psi\colon \boldsymbol{C}\{z\} \to \mathscr{O}_{Y,o}$ in (2.1). For any section $f \in H^0(\mathscr{O}_{\widetilde{X}}(-\sigma(D)))$ with $D \in \boldsymbol{L}^*$, there exists $D' \in \boldsymbol{L}$ such that $\operatorname{div}(f) - \sigma(D) - D'$ has no component of E. Then we define the $\boldsymbol{Q}$-cycle $(f)_E$ to be $D + D'$. For example, $(\psi(z_w))_E = E_w^*$. For any $\boldsymbol{Q}$-cycle $D = \sum_{i \in \mathscr{V}} c_i E_i$, we write $\operatorname{cff}_{E_i} D = c_i$.

LEMMA 3.2 (cf. [**12**, 2.14, 2.16]). *The end curve condition is satisfied on*
(1) *a sufficiently small neighborhood of any connected reduced cycle E', and*
(2) *a surface obtained by blowing up $\widetilde{X}$ at a singular point of E or at the intersection of E and an end curve.*

PROOF OF PROPOSITION 3.1. By [**11**, 5.10], Theorem 2.10 is valid in case $\#\mathscr{N} = 1$. Therefore we may assume that it is valid in case the number of nodes is less than $\#\mathscr{N}$. Let E_v be an arbitrary node and E' an arbitrary branch of E_v. Let $\boldsymbol{L}'$ denote the group of cycles supported in E'. We have to show that there exists a cycle $D \in \boldsymbol{L}'$ such that $\operatorname{Supp}(D) = E'$ and that $E_v^* + D$ is a monomial cycle. We may assume that E' is not a chain (cf. [**11**, 3.5]). By Lemma 3.2, we may also assume that E_v has only three branches E_1, E_2, and E'. Let $a_i = -E_i^2$ for each i. Then $z_1^{a_1}$ and $z_2^{a_2}$ are admissible monomials at E_v; indeed $(\psi(z_i^{a_i}))_E = E_v^* + E_i$. Since the images of $\psi(z_1^{a_1})$ and $\psi(z_2^{a_2})$ in $H^0(\mathscr{O}_{E_v}(-\sigma(E_v^*)))$ span this vector space, we may assume that the image of $f := \psi(z_1^{a_1} + z_2^{a_2})$ in the space has a zero at $E_v \cap E'$. Let E_0 be the irreducible component of E' intersecting E_v, and $F = (f)_E - E_v^* \in \boldsymbol{L}$. Since $\operatorname{cff}_{E_i}(f)_E = \operatorname{cff}_{E_i} E_v^*$ for $i = 1, 2, v$, and since $-(f)_E$ is nef, we have $\operatorname{Supp}(F) = E'$, $(f)_E \cdot E_v = 0$, $\operatorname{cff}_{E_0}(F) = 1$, and that $-F$ is nef on E'. Let $\widetilde{X}'$ be a sufficiently small neighborhood of E'. By a natural isomorphism $\mathscr{O}_{\widetilde{X}'}(-\sigma(E_v^*)) \to \mathscr{O}_{\widetilde{X}'}$, the section $f|_{\widetilde{X}'}$ corresponds to a regular function f' on $\widetilde{X}'$ with $(f')_{E'} = F$. By assumption, $\widetilde{X}'$ is a good resolution of a splice-quotient singularity. Therefore f' can be represented by a series of monomials associated with monomial cycles in $\boldsymbol{L}'$ (cf. Remark 2.11). Hence there exists a monomial cycle $D \in \boldsymbol{L}'$ which is reduced at E_0.

In case $a_0 = 1$, it is clear that $D \cdot E_0 = 0$, and hence $E_v^* + D$ is a monomial cycle. For general case, consider the blowing up at $E_v \cap E'$. By Lemma 3.2, we can apply the argument above, and thus obtain the cycle D. Then the direct image of D on $\widetilde{X}$ is a desired cycle. □

4. Filtrations.

In the following we will apply some results of [**12**, Section 3], which are obtained without using the end curve theorem.

Recall that the universal abelian cover $q\colon Y \to X$ fits into the following commutative diagram, where p is finite and unramified over $\widetilde{X} \setminus E$, ρ is a partial resolution, and $\widetilde{Y}$ has only cyclic quotient singularity at the fiber of the singular points of E (cf. [**10**, Section 3.2]):

$$\begin{tikzcd} \widetilde{Y} \arrow[r, "p"] \arrow[d, "\rho"'] & \widetilde{X} \arrow[d, "\pi"] \\ Y \arrow[r, "q"'] & X \end{tikzcd}$$

Then $F := p^{-1}(E)$ is the ρ-exceptional set on $\widetilde{Y}$. Let $v \in \mathscr{V}$ and $F_v = p^{-1}(E_v)$. For each $n \in \boldsymbol{Z}_{\geq 0}$, we define the ideal I_n of $\mathscr{O}_{Y,o}$ by

$$I_n = \bigl(\rho_* \mathscr{O}_{\widetilde{Y}}(-nF_v)\bigr)_o.$$

We call $\{I_n\}_{n\geq 0}$ the *v-filtration*. Let $G(Y,v)$ denote the graded ring $\bigoplus_{n\geq 0} I_n/I_{n+1}$. It follows from [**12**, 3.3] that $G(Y,v)$ is a finitely generated $\boldsymbol{C}$-algebra. Clearly $G(Y,v)$ is reduced and two-dimensional, and has a natural $\boldsymbol{H}$-action. Since $p^*E_v = e_vF_v$ by [**10**, 3.4], it follows from the definition of the v-weight that I_n contains all functions $\psi(f)$ with v-ord$(f) \geq n$.

4.1. By virtue of Proposition 3.1, we can discuss Neumann-Wahl systems associated with E. Let E_v be an arbitrary node, and let $\{m_i\}$ be the admissible monomials at E_v as in Definition 2.3. We consider the vector space $V := \sum \boldsymbol{C}m_i \subset \boldsymbol{C}\{z\}$. Then $\psi(V)$ is a subspace of $H^0(\mathscr{O}_{\widetilde{X}}(-\sigma(E_v^*)))$. Consider the composite

$$\alpha\colon V \xrightarrow{\psi} H^0\bigl(\mathscr{O}_{\widetilde{X}}(-\sigma(E_v^*))\bigr) \to H^0\bigl(\mathscr{O}_{E_v}(-\sigma(E_v^*))\bigr).$$

Since α is surjective and $h^0(\mathscr{O}_{E_v}(-\sigma(E_v^*))) = 2$, the kernel of α determines a Neumann-Wahl system at E_v (cf. [**11**, 5.6]). Applying this argument to every node, we obtain a Neumann-Wahl system $\mathscr{F}$ associated with E. We denote by Y' the splice type singularity defined by $\mathscr{F}$. Then $Y'/\boldsymbol{H}$ and X have the same weighted dual graphs by Theorem 2.7; in fact, we can conclude that Y is an equisingular deformation of Y' when the main theorem is verified (cf. [**11**, 5.10]).

Let $G = G(Y, v)$ and $G' = G(Y', v)$. Let I^v be the ideal of the polynomial ring $\boldsymbol{C}[z] := \boldsymbol{C}[z_w; w \in \mathscr{E}]$ generated by the v-leading forms $\mathrm{LF}_v \mathscr{F} := \{\mathrm{LF}_v(f) \mid f \in \mathscr{F}\}$.

Proposition 4.1. *G' is a reduced complete intersection ring, naturally isomorphic to $\boldsymbol{C}[z]/I^v$, and regarded as an $\boldsymbol{H}$-equivariant graded subring of G.*

Proof. By [**12**, 3.3], the v-filtration of $\mathscr{O}_{Y',o}$ coincides with the filtration induced by v-order. Hence [**8**, 2.6] implies the first claim. Via ψ we have a natural homomorphism $\boldsymbol{C}[z] \to G$. By the definitions of $\mathscr{F}$ and I^v, we obtain a natural $\boldsymbol{H}$-equivariant homomorphism $\beta : G' \to G$ of reduced graded rings. Since $\boldsymbol{H}$ acts transitively on the set of the irreducible components of $\operatorname{Spec} G'$, $\operatorname{Im}\beta$ is pure two-dimensional. Hence $\operatorname{Ker}\beta = (0)$. $\square$

For any $\chi \in \hat{\boldsymbol{H}}$, let $G^\chi \subset G$ denote the χ-eigenspace, and let $G^\chi_n = G^\chi \cap G_n$. In case $\chi = 1$, we also write $G^\chi = G^{\boldsymbol{H}}$; this is the invariant subring.

The map α induces the surjective map $V^{\otimes n} \to H^0(\mathscr{O}_{E_v}(-\sigma(nE_v^*)))$. Regarding $H^0(\mathscr{O}_{\widetilde{X}}(-\sigma(e_v E_v^*)))$ as an $\boldsymbol{H}$-invariant subspace of $H^0(\mathscr{O}_{\widetilde{Y}})$, from the argument of [**12**, Section 3.2] we obtain the following.

Lemma 4.2. *For every $n \in \boldsymbol{N}$, we have*

$$G'^{\boldsymbol{H}}_{ne_v m_{vv}} \cong G^{\boldsymbol{H}}_{ne_v m_{vv}} \cong H^0(\mathscr{O}_{E_v}(-\sigma(ne_v E_v^*))).$$

From [**12**, 3.7], $G'^{\boldsymbol{H}}_n = 0$ if $n \not\equiv 0 \pmod{e_v}$. The following lemma shows that $G'^{\boldsymbol{H}}_{ne_v} \neq 0$ for every sufficiently large n.

Lemma 4.3. *For every sufficiently large n, there exists a monomial cycle $D \in \boldsymbol{L}$ such that $\mathrm{cff}_{E_v} D = n$.*

Proof. Let $C_1, \dots, C_{\delta_v}$ be the branches of E_v, and let $E_i \le C_i$ be the component intersecting E_v. It follows from [**8**, 5.1] that for each C_i there exists a monomial cycle $\widetilde{C}_i \in \boldsymbol{L}$ supported on C_i such that $\widetilde{C}_i \cdot E_i = -1$. Let $c_i = \mathrm{cff}_{E_i} \widetilde{C}_i$. For each $k \in \boldsymbol{Z}_{\ge 0}$ and each $0 \le l < m_{vv}$, put $D_{k,l} = ke_v E_v^* + lE_v + l\sum_{i=1}^{\delta_v} \widetilde{C}_i$ and $c_{k,l} = -D_{k,l} \cdot E_v$. We assume that k is sufficiently large so that $c_{k,l} \ge 0$. Let D_1 be a cycle supported on C_1 such that $E_v^* + D_1$ is a monomial cycle. Then $D_{k,l} + c_{k,l} D_1$ is a monomial cycle with $\mathrm{cff}_{E_v} = km_{vv} + l$. $\square$

4.2. We fix an end E_w and the node E_v nearest to w. Let $\mathscr{M}' = \sum_{i \in \mathscr{E} \setminus \{w\}} \boldsymbol{Z}_{\ge 0} E_i^*$. Let $\psi' : \boldsymbol{C}\{z\} \to \mathscr{O}_{Y',o}$ be the natural surjective homomorphism. To simplify notation, for $f \in \boldsymbol{C}\{z\}$ let us also denote by f the functions $\psi(f)$ and

$\psi'(f)$. Let $\mathscr{G} = G/z_wG$ and $\mathscr{G}' = G'/z_wG'$. From [**11**, 4.4] we obtain the following.

Lemma 4.4. *$\mathscr{G}'$ is a reduced complete intersection ring, naturally isomorphic to $\boldsymbol{C}[z]/I^v + (z_w)$. Moreover $z(D) \in \mathscr{G}'$ is regular and the support of $\mathscr{G}'/(z(D))$ is the origin for every $D \in \mathscr{M}'$.*

Lemma 4.5. *There exists $D \in \mathscr{M}'$ such that $z(D)G \subset G'$.*

Proof. For each $\chi \in \hat{\boldsymbol{H}}$, there exists a monomial cycle D_χ such that $\chi = \theta(-D_\chi)$. Then $z(D_\chi)G^\chi \subset G^{\boldsymbol{H}}$. It follows from Lemma 4.3 that for each $0 \le l < m_{vv}$ there exist $k_l \in \boldsymbol{N}$ and a monomial $m_l \in G'^{\boldsymbol{H}}_{(k_l m_{vv} - l)e_v}$. Let $m = z(\sum_{\chi \in \hat{\boldsymbol{H}}} D_\chi) \prod_{l=1}^{m_{vv}-1} m_l$. Using Lemma 4.2, we have $mG_k^\chi \subset G'$ for every $(k,\chi) \in \boldsymbol{N} \times \hat{\boldsymbol{H}}$. Hence $mG \subset G'$. Suppose $z_w \mid m$. For each $x \in G$, we have $(mx)^2 = m(mx^2) \in z_wG'$. By Lemma 4.4, $mx \in z_wG'$ and z_w is regular. Therefore we have $(mz_w^{-1})G \subset G'$. This implies the assertion. □

Corollary 4.6. *$\mathscr{G}'$ and $\mathscr{G}$ have the same normalizations.*

Proof. Let m be a monomial described in Lemma 4.5. Then the inclusion $G' \subset G$ implies the isomorphism $G'[m^{-1}] \cong G[m^{-1}]$ of localizations. Hence $\operatorname{Spec} \mathscr{G}'$ and $\operatorname{Spec} \mathscr{G}$ are isomorphic outside the origin, and hence $\mathscr{G}$ is reduced as $\mathscr{G}'$ is. Therefore the natural homomorphism $\mathscr{G}' \to \mathscr{G}$ is injective (cf. Proposition 4.1). By Lemma 4.2, for every $(k,\chi) \in \boldsymbol{Z}_{\ge 0} \times \hat{\boldsymbol{H}}$ and every $x \in \mathscr{G}_k^\chi$, we have that $x^n \in \mathscr{G}'$ for some $n \in \boldsymbol{N}$. Hence $\mathscr{G}$ is integral over $\mathscr{G}'$. □

Let H (resp. H') denote the hypersurface of Y (resp. Y') defined by $z_w = 0$, and $\{J_n\}_{n \ge 0}$ (resp. $\{J'_n\}_{n\ge 0}$) the filtration of $\mathscr{O}_{H,o}$ (resp. $\mathscr{O}_{H',o}$) induced by the v-filtration. Then $\mathscr{G} = \bigoplus_{n \ge 0} J_n/J_{n+1}$ and $\mathscr{G}' = \bigoplus_{n\ge 0} J'_n/J'_{n+1}$.

Remark 4.7. We can see that $\mathscr{G}' = \boldsymbol{C}[z]/I^w + (z_w)$ and that $\mathscr{G}'$ is also obtained from the w-filtration. This filtration measures the vanishing order at $o \in H$.

Next we give a formula for the delta invariant of the curve singularity (H, o).

Proposition 4.8. *We have the following formula.*

$$\delta(H,o) = \frac{1}{2} p^* E_w^* \cdot (K_{\widetilde{Y}} - p^*E_w^*) = \frac{|\boldsymbol{H}|}{2e_w} \left(\sum_{v \in \mathscr{V} \setminus \{w\}} (\delta_v - 2) m_{wv} + 1 \right).$$

Proof. Let $\widetilde{H} \subset \widetilde{Y}$ be the strict transform of H. Then $\mathrm{div}_{\widetilde{Y}}(z_w) = p^*E_w^* + \widetilde{H}$, and $\widetilde{H}$ intersects the exceptional set F transversally; indeed, $\widetilde{H} = p^*H_w$ for an end curve H_w at E_w. Therefore applying [**2**, 2.1.4], we have the first equality. By log-ramification formula and the relation $p^*E_v = e_vF_v$, we have $K_{\widetilde{Y}} \equiv p^*(K_{\widetilde{X}} + E - \sum_{v\in\mathscr{V}} \frac{1}{e_v} E_v)$. Using the equality $c_1(K_{\widetilde{X}} + E) = \sum_{v\in\mathscr{V}}(2-\delta_v)E_v^*$, we obtain

$$p^*E_w^* \cdot (K_{\widetilde{Y}} - p^*E_w^*) = |\boldsymbol{H}|E_w^* \cdot \left(\sum_{v\in\mathscr{V}\setminus\{w\}} (2-\delta_v)E_v^* - \sum_{v\in\mathscr{V}} \frac{1}{e_v} E_v\right).$$

Hence the formula follows from $e_wE_w^* \cdot E_v^* = -m_{wv}$. □

This proposition implies $\delta(H, o) = \delta(H', o)$. On the other hand, H (resp. H') is a δ-constant deformation of $\mathrm{Spec}\,\mathscr{G}$ (resp. $\mathrm{Spec}\,\mathscr{G}'$) (cf. [**13**, (5.15)], [**1**]). Therefore $\delta(\mathrm{Spec}\,\mathscr{G}, o) = \delta(\mathrm{Spec}\,\mathscr{G}', o)$. By Corollary 4.6, we obtain the following.

Corollary 4.9. *$\mathscr{G}' = \mathscr{G}$.*

Corollary 4.10. *$\psi\colon \boldsymbol{C}\{z\} \to \mathscr{O}_{Y,o}$ is surjective.*

Proof. Since H is a deformation of $\mathrm{Spec}\,\mathscr{G}$ and $\boldsymbol{C}[z] \to \mathscr{G}' = \mathscr{G}$ is surjective, Y is embedded by the functions $\{z_i\}_{i\in\mathscr{E}}$. □

Proof of Theorem 2.10. Let v be an arbitrary node. For each $n \in \boldsymbol{Z}_{\geq 0}$, let $\widetilde{I}_n$ denote the ideal of $\mathscr{O}_{Y,o}$ generated by the images of the elements of $\boldsymbol{C}\{z\}$ having v-order $\geq n$. Let $\widetilde{G}$ denote the graded ring $\bigoplus_{n\geq 0} \widetilde{I}_n/\widetilde{I}_{n+1}$. By Corollary 4.10 we have $\widetilde{G} = \boldsymbol{C}[z]/\widetilde{I}$ for a suitable homogeneous ideal $\widetilde{I} \subset \boldsymbol{C}[z]$ with respect to v-weight. Since $\widetilde{I}_n \subset I_n$ for $n \geq 0$, we obtain a natural homomorphism $\phi\colon \widetilde{G} \to G$. Since $\widetilde{I}$ contains the set $\{\mathrm{LF}_v(f) \mid f \in \mathscr{F}\}$, it follows from Proposition 4.1 that there is a surjective homomorphism $G' \to \widetilde{G}$, and the composite $G' \to \widetilde{G} \xrightarrow{\phi} G$ is injective. Therefore $G' = \widetilde{G}$, and hence $\widetilde{G}$ is reduced. Applying [**13**, (2.2)] and the proof of [**12**, 3.3] we obtain that $\widetilde{I}_n = I_n$ for every $n \geq 0$, and hence $G' = G$. Let $\mathscr{F}_v \subset \mathscr{F}$ be a Neumann-Wahl system at E_v. Recall that $\mathscr{F}_v \subset G^{\theta(E_v^*)}$. Then for each $f \in \mathscr{F}_v$ there exists a $\theta(E_v^*)$-eigenfunction $f^+ \in \boldsymbol{C}\{z\}$ with $v\text{-ord}(f^+) > v\text{-ord}(f)$ such that $f + f^+ \in \mathrm{Ker}\,\psi$ (cf. [**11**, Section 5]). Since v is an arbitrary node, $\mathrm{Ker}\,\psi$ contains a system of splice diagram functions $\widetilde{\mathscr{F}} := \{f + f^+ \mid f \in \mathscr{F}\}$. Since $\mathscr{O}_{Y,o}$ is a two-dimensional domain, it follows from Theorem 2.6 that $\mathrm{Ker}\,\psi$ is just generated by $\widetilde{\mathscr{F}}$. □

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