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An analysis of the nonlinear equation of motion of a vibrating membrane in the space of BV functions

Dedicated to Professor Kiyoshi Mochizuki on his sixtieth birthday

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Abstract. In this article the nonlinear equation of motion of vibrating membrane $u_{tt} - \operatorname{div}\{\sqrt{1+|\nabla u|^2}^{-1} \nabla u\} = 0$ is discussed in the space of functions having bounded variation. Approximate solutions are constructed in Rothe's method. It is proved that a subsequence of them converges to a function u and that, if u satisfies the energy conservation law, then it is a weak solution in the space of functions having bounded variation. The main tool is varifold convergence.

1. Introduction.

Let Ω be a bounded domain in $\boldsymbol{R}^n$ with Lipschitz continuous boundary $\partial\Omega$. Given a function u in Ω, we regard its graph as a membrane in $\Omega \times \boldsymbol{R}$. Longitudinal vibrations of this membrane is described by the following equation:

$$\frac{\partial^2 u}{\partial t^2}(t,x) - \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left\{(1+|\nabla u(t,x)|^2)^{-1/2}\frac{\partial u}{\partial x_j}\right\} = 0, \quad x \in \Omega, \tag{1.1}$$

$$u(0,x) = u_0(x), \quad \frac{\partial u}{\partial t}(0,x) = v_0(x), \quad x \in \Omega, \tag{1.2}$$

$$u(t,x) = 0, \quad x \in \partial\Omega. \tag{1.3}$$

In [4] D. Fujiwara and S. Takakuwa have investigated this equation in the class of functions having bounded variation. A function $u \in L^1(\Omega)$ is said to have bounded variation in Ω if

$$\sup\left\{\int_\Omega u \operatorname{div} g\,dx; g = (g_1, \dots, g_n) \in C_0^1(\Omega; \boldsymbol{R}^n), |g| \le 1\right\} < \infty$$

(see [2], [5], [12]). The set of all functions in $L^1(\Omega)$ having bounded variation is denoted by $BV(\Omega)$, and a member of $BV(\Omega)$ is often called a BV function. For each $u \in BV(\Omega)$ there are a Radon measure μ and a μ-measurable function $\nu = (\nu^1, \dots, \nu^n)$ with $|\nu| = 1, \mu$-a.e., such that

$$\int_\Omega u \operatorname{div} g\,dx = -\int_\Omega g \cdot \nu\,d\mu. \tag{1.4}$$

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We usually use such notations as $\mu = |Du|$ and $\mu \llcorner \nu = Du$. $BV(\Omega)$ is a Banach space equipped with the norm $\|u\|_{BV} = \|u\|_{L^1(\Omega)} + |Du|(\Omega)$. Equation (1.1) does not always have a classical solution globally in time (in [**6**] it is proved that in the two dimensional case (1.1) does not always have a classical solution globally in time even though the initial data is smooth and small). Thus a time global solution should be found in a weak sense. When a C^2 class function u satisfies (1.1), multiplying u_t to (1.1) and integrating with respect to x, we obtain the energy conservation law

$$\int_\Omega |u_t(t,x)|^2\,dx + J(u(t,\cdot)) = \text{const.},$$

where J is the area functional $J(u) = \int_\Omega \sqrt{1+|\nabla u|^2}\,dx$. J is finite for $u \in W^{1,1}(\Omega)$, and thus this space is expected to be the appropriate function space for weak solutions to (1.1). But it is not reflexive and thus does not guarantee the weak compactness of bounded sets. While, for a bounded set B in $BV(\Omega)$, there exist a subsequence $\{u_j\} \subset B$ and a function $u \in BV(\Omega)$ such that $u_j \to u$ strongly in $L^1(\Omega)$ and $Du_j \rightharpoonup Du$ in the sense of distributions. Thus $BV(\Omega)$ satisfies a kind of compactness for bounded sets. These facts suggest that this equation should be treated in the class of BV functions.

The graph of a BV function is possibly broken. Thus the area functional should measure not only the graph but the broken part. It is extended to $BV(\Omega)$ in the following manner:

$$J(u) = \sup\left\{\int_\Omega (g_0 + u \operatorname{div} g)\,dx; (g_0, g) \in C^1(\Omega, \boldsymbol{R}^{n+1}), |g_0|^2 + |g|^2 \le 1\right\}.$$

It is still convex in $BV(\Omega)$ and satisfies that, if $\{u_j\}$ converges strongly in $L^1_{loc}(\Omega)$, then $\liminf_{j\to\infty} J(u_j) \ge J(u)$ ([**5**] Theorem 14.2). We set $U = \Omega \times \boldsymbol{R}$. For $u \in BV(\Omega)$, we define $E_u \subset U$ by

$$E_u = \{(x,y); x \in \Omega, y > u(x)\}.$$

It is a set of finite perimeter in U, that is, $\chi_{E_u} \in BV(U)$. Hence, for χ_{E_u}, there exist μ and ν as in (1.4), which are in this article denoted by μ_{E_u} and ν_{E_u}, respectively. It holds that

$$J(u) = \mu_{E_u}(U) \tag{1.5}$$

([**5**] Theorem 14.6). The measure μ_{E_u} is characterized by the reduced boundary $\partial^* E_u$, which is defined by

$$\partial^* E_u = \Bigg\{ z \in U; \mu_{E_u}(B_\rho(z)) > 0 \text{ for all } \rho > 0,$$

$$\nu_{E_u}(z) = \lim_{\rho \searrow 0} \frac{\int_{B_\rho(z)} \nu_{E_u}\,d\mu_{E_u}}{\mu_{E_u}(B_\rho(z))}, \text{ and } |\nu_{E_u}(z)| = 1 \Bigg\},$$

where $B_\rho(z)$ denotes the closed ball with center z, radius ρ, ([**2**], [**5**], [**12**]). It is countably n-rectifiable and satisfies

$$\mu_{E_u} = \mathscr{H}^n \llcorner \partial^* E_u, \tag{1.6}$$

where $\mathscr{H}^n$ denotes the n-dimensional Hausdorff measure. In fact, for μ_{E_u}-a.e. $z \in \partial^* E_u$, the approximate tangent space $T_z(\partial^* E_u)$ exists and is given by

$$T_z(\partial^* E_u) = \{\zeta \in \boldsymbol{R}^{n+1}; \zeta \cdot \nu_{E_u} = 0\}. \tag{1.7}$$

Moreover ν_{E_u} is the inward pointing unit normal for E_u in a generalized sense ([**12**] Theorem 14.3, [**2**] Theorem 2 of Section 5.7.3).

Each BV function u has its trace γu (see [**2**] or [**5**]). γ is a bounded operator from $BV(\Omega)$ to $L^1(\partial\Omega)$ such that, for each $g \in C^1(\bar{\Omega}; \boldsymbol{R}^n)$,

$$\int_\Omega u \operatorname{div} g \, dx = -\int_\Omega g \cdot Du + \int_{\partial\Omega} \gamma u g \cdot \vec{n} \, d\mathscr{H}^{n-1}, \tag{1.8}$$

where $\vec{n}$ is the outer unit normal to $\partial\Omega$. Boundary condition (1.3) is regarded as $\gamma u = 0$.

In this article we call a weak solution in the class of functions having bounded variation a *BV solution*. In [**4**] it is asserted that *a sequence of approximate solutions to* (1.1)–(1.3) *constructed by Ritz-Galerkin method converges to a function* $u \in L^\infty_{loc}((-\infty, \infty); L^2(\Omega) \cap BV(\Omega))$, and that, *if u satisfies the energy conservation law and one more condition holds, it is a "BV solution" to* (1.1). However there are several problems in their theory. First they require high regularity for u_0. Second there is a technical condition which is closely related to their tool "varifold". The last and the most serious problem is that their formulation of a BV solution is not appropriate. Their BV solution is not suitable for calling a solution. The purpose of this article is to dissolve these problems and reestablish their result (Theorem 4.1). The first problem is caused by the way of approximation. Ritz-Galerkin method does not seem to be appropriate in treating BV functions. In this article we employ the method of semidiscretization in time variable. This approximating method is often called Rothe's method and at first introduced to construct weak solutions to parabolic equations ([**11**]). However many hyperbolic equations are also solved by this method (see [**10**] and references cited there). Thanks to this method we can treat our problem for $u_0(x) \in L^2(\Omega) \cap BV(\Omega)$ with $\gamma u_0 = 0$ and $v_0(x) \in L^2(\Omega)$. The technical condition mentioned in the second problem is needed for controlling BV functions in terms of varifolds. Varifolds are regarded as a kind of generalized surfaces (the theory of general varifold is precisely discussed in [**12**] Chapter 8). In [**4**] each BV function u is identified with its graph (or $\partial^* E_u$, more precisely) and regarded as a varifold in $U = \Omega \times \boldsymbol{R}$. It is proved that a subsequence of varifolds corresponding to approximate solutions converges under the topology of the space of varifolds. The above condition is imposed on this limit varifold. In this article we follow their strategy, but we can remove this condition by introducing "orientations" for varifolds. Let G_0 be the collection of all oriented n-dimensional vector subspaces of $\boldsymbol{R}^{n+1}$. Each element of G_0 is characterized by an n-vector ξ which is represented as $\xi = \tau_1 \wedge \cdots \wedge \tau_n$, where $\{\tau_1, \dots, \tau_n\}$ is an orthonormal basis of this element. Thus G_0 is often identified with the set of all simple n-vectors having unit norm ([**3**] 1.6.2). We say V an *oriented n-varifold* in U if V is a Radon measure on $U \times G_0$. We associate each BV function u with an oriented varifold V in the following way. Let $\xi(z)$ be the orientation of $\partial^* E_u$ which agrees with the inward pointing unit normal $\nu_{E_u}(z)$. More precisely $\xi(z)$ is an n-vector valued $\mathscr{H}^n$-measurable function on $\partial^* E_u$ such that, for μ_{E_u}-

a.e. $z \in \partial^* E_u$, $\xi(z) = \tau_1 \wedge \cdots \wedge \tau_n$ and $\xi(z) \wedge \nu_{E_u}(z) = e_1 \wedge \cdots \wedge e_{n+1}$, where $\{\tau_1, \ldots, \tau_n\}$ is an orthonormal basis of the approximate tangent space $T_z(\partial^* E_u)$ and $\{e_1, \ldots, e_{n+1}\}$ is the standard basis of $\boldsymbol{R}^{n+1}$. Now we define a continuous linear functional on $C_0^0(U \times G_0)$ by

$$L(\beta) = \int_{\partial^* E_u} \beta(z, \xi(z))\, d\mathscr{H}^n \quad (\beta \in C_0^0(U \times G_0)).$$

It follows from the Riesz representation theorem (see, for example, [**12**] Theorem 4.1) that there exists a Radon measure V on $U \times G_0$ (thus an oriented varifold V in U) such that

$$L(\beta) = \int_{U \times G_0} \beta(z, \xi)\, dV(z, \xi).$$

In this article we write

$$V = \boldsymbol{v}_+(u).$$

For each oriented n-varifold V in U a Radon measure μ_V on U is defined by

$$\mu_V(A) = V(A \times G_0) \quad \text{for a Borel set } A \subset U.$$

When $V = \boldsymbol{v}_+(u)$, it immediately follows from the definition of $\boldsymbol{v}_+(u)$ that

$$\mu_V = \mu_{E_u}. \tag{1.9}$$

Now we can associate each BV function u with an oriented varifold $\boldsymbol{v}_+(u)$. It is the same as in [**4**] that a subsequence of varifolds corresponding to approximate solutions converges in the space of varifolds. We achieve our purpose by investigating the structure of the limit varifold. The key point is whether it satisfies a relation such as (1.9). This varifold convergence method is very useful. There is another application of this method to a theory of parabolic equations ([**7**]).

In Section 2 we present a suitable formulation of a BV solution. Readers possibly feel our definition fairly weak. Since the area functional J is convex, we can regard (1.1) as an evolution equation $u_{tt} + \partial J(u) \ni 0$. In the appendix we show that, if $\partial\Omega$ is of C^2 class, our definition of a BV solution is equivalent to the definition of a weak solution to $u_{tt} + \partial J(u) \ni 0$. Abstract theories of hyperbolic evolution equations are discussed by several authors (for example [**8**]). However it seems that there are few works on abstract hyperbolic evolution equations in the space of functions having bounded variations.

2. Definition of a BV solution.

In the original physical meaning Equation (1.1) is derived as the Euler-Lagrange equation of the action integral

$$\int_0^T \left(\int_\Omega |u_t(t,x)|^2\, dx - J(u) \right) dt \quad (J \text{ is the area functional, } T > 0).$$

However the area functional J is not always Gâteaux differentiable on $BV(\Omega)$. Thus we cannot calculate $(d/d\varepsilon)J(u+\varepsilon\varphi)|_{\varepsilon=0}$ directly. This is a variation of the area of the graph of u (or $\partial^* E_u$, more precisely). Usually variations of areas of surfaces in a domain U are calculated by the use of a one parameter family of diffeomorphisms of U (see, for example, [**12**] Section 9). In our case the domain U is $\Omega \times \boldsymbol{R}$, and, since the equation describes the longitudinal vibration, we may suppose that each diffeomorphism of the one parameter family is written as $U \ni (x, y) \mapsto (x, y+\varepsilon\varphi(x, y)) \in U$, where ε is the parameter and φ is a given function on U.

Taking account of these facts, we present a definition of a BV solution to (1.1)–(1.3) in the following way. Let T be any positive number.

Definition 1. A function u is said to be a BV solution to (1.1)–(1.3) in $(0,T)\times\Omega$ if and only if

i) $u \in L^\infty((0,T); BV(\Omega))$, $u_t \in L^2((0,T)\times\Omega)$

ii) $u(0,x) = u_0(x)$

iii) $\gamma u = 0$ for $\mathscr{L}^1$-a.e. $t \in (0,T)$

iv) for any $\varphi \in C_0^1([0,T)\times U)$,

$$\int_0^T \left\{ -\int_\Omega u_t(\varphi_t(t,x,u) + \varphi_y(t,x,u)u_t)\,dx + \frac{d}{d\varepsilon}J(u+\varepsilon\varphi(t,x,u))|_{\varepsilon=0} \right\} dt$$
$$= \int_\Omega v_0(x)\varphi(0,x,u_0(x))\,dx.$$

Remark. The second relation of i) implies $u \in C^{0,1/2}((0,T); L^1(\Omega))$. Hence $\text{s-lim}_{t\searrow 0} u(t,\cdot)$ exists in $L^1(\Omega)$ and then we define $u(0,\cdot)$ by this limit. Further, if $u(0,\cdot) \in L^2(\Omega)$, then $u \in C^{0,1/2}((0,T); L^2(\Omega))$.

First of all we should justify Definition 1. We should show that $d/d\varepsilon \cdot J(u+\varepsilon\varphi(x,u))|_{\varepsilon=0}$ describes the variation of the area of $\partial^* E_u$ and that it exists when $\varphi \in C_0^1(U)$.

Let u be a function in $BV(\Omega)$ and let φ be a function in $C^1(U)$. Suppose that all first derivatives of φ are bounded. Here we do not assume the boundedness of φ itself. Let $\Phi^{\varepsilon,\varphi}: U \to U$ denote the one parameter family of diffeomorphisms defined by $\Phi^{\varepsilon,\varphi}(x,y) = (x, y+\varepsilon\varphi(x,y))$. Then the variation of the area is given by

$$\frac{d}{d\varepsilon}\mathscr{H}^n(\Phi^{\varepsilon,\varphi}(\partial^* E_u))|_{\varepsilon=0}.$$

Note that the area formula (see [**2**], [**12**]) implies, for each n-rectifiable set M with $\mathscr{H}^n(M) < \infty$,

$$\mathscr{H}^n(\Phi^{\varepsilon,\varphi}(M)) = \int_M J_S\Phi^{\varepsilon,\varphi}\,d\mathscr{H}^n, \tag{2.1}$$

where $S = T_z M$ and

$$J_S\Phi^{\varepsilon,\varphi}(x) = \sqrt{\det((d\Phi_z^{\varepsilon,\varphi}|_S)^* \circ (d\Phi_z^{\varepsilon,\varphi}|_S))}.$$

It immediately follows from (2.1) that, if $\mathscr{H}^n(M) = 0$, then $\mathscr{H}^n(\Phi^{\varepsilon,\varphi}(M)) = 0$. It also follows from (2.1) that, when $M = M_1 \cup M_2$ with $M_1 \cap M_2 = \emptyset$, then

$$\mathscr{H}^n(\Phi^{\varepsilon,\varphi}(M_1 \cup M_2)) = \mathscr{H}^n(\Phi^{\varepsilon,\varphi}(M_1)) + \mathscr{H}^n(\Phi^{\varepsilon,\varphi}(M_2)). \tag{2.2}$$

First we claim

Theorem 2.1. *If ε is sufficiently small, then* $J(u + \varepsilon\varphi(x,u)) = \mathscr{H}^n(\Phi^{\varepsilon,\varphi}(\partial^* E_u))$.

Proof. By (1.5) and (1.6) we have

$$J(u + \varepsilon\varphi(x,u)) = \mathscr{H}^n(\partial^* E_{u+\varepsilon\varphi(x,u)}).$$

When ε is sufficiently small, $y + \varepsilon\varphi(x,y)$ is monotone increasing with respect to y. Thus $y + \varepsilon\varphi(x,y) > u(x) + \varepsilon\varphi(x,u(x))$ if and only if $y > u(x)$. This implies $E_{u+\varepsilon\varphi(x,u)} = \Phi^{\varepsilon,\varphi}(E_u)$. In particular, we obtain

$$\mathscr{H}^n(\partial^* E_{u+\varepsilon\varphi(x,u)}) = \mathscr{H}^n(\partial^* \Phi^{\varepsilon,\varphi}(E_u)).$$

Let ∂_* denote the measure theoretic boundary (for the definition, refer to [**2**] Section 5.8). In [**2**] Lemma 1 of Section 5.8 it is proved that, for each set E of locally finite perimeter, $\partial^* E \subset \partial_* E$ and

$$\mathscr{H}^n(\partial_* E \backslash \partial^* E) = 0. \tag{2.3}$$

Thus we have

$$\mathscr{H}^n(\partial^* \Phi^{\varepsilon,\varphi}(E_u)) = \mathscr{H}^n(\partial_* \Phi^{\varepsilon,\varphi}(E_u)).$$

Generally, for a set E of locally finite perimeter in U and a diffeomorphism $\Phi : U \to U$ which satisfies

$$\begin{cases} C_0 \le \left|\det \dfrac{\partial \Phi}{\partial z}\right| \le C_1 \\ C_2|z - z'| \le |\Phi(z) - \Phi(z')| \le C_3|z - z'| \end{cases} \tag{2.4}$$

with positive constants C_j $(j = 0, 1, 2, 3)$, it holds that $\partial_*\Phi(E) = \Phi(\partial_* E)$. When ε is sufficiently small, $\Phi^{\varepsilon,\varphi}$ satisfies (2.4) since first derivatives of φ are bounded. Hence we have

$$\mathscr{H}^n(\partial_* \Phi^{\varepsilon,\varphi}(E_u)) = \mathscr{H}^n(\Phi^{\varepsilon,\varphi}(\partial_* E_u)).$$

By (2.2) and (2.3) we have

$$\mathscr{H}^n(\Phi^{\varepsilon,\varphi}(\partial_* E_u)) = \mathscr{H}^n(\Phi^{\varepsilon,\varphi}(\partial^* E_u)) + \mathscr{H}^n(\Phi^{\varepsilon,\varphi}(\partial_* E_u \backslash \partial^* E_u)) = \mathscr{H}^n(\Phi^{\varepsilon,\varphi}(\partial^* E_u)).$$

Thus the assertion is verified. □

Let z be the point at which the approximate tangent space for $\partial^* E_u$ exists. Let $\{\tau_1, \dots, \tau_n\}$ be an orthonormal basis of $S = T_z(\partial^* E_u)$. Put $C' = (\tau_1, \dots, \tau_n)$ and $C = (\tau_1, \dots, \tau_n, v_{E_u}(z))$. By (1.7) we have $\det C = 1$. Then $d\Phi_z^{\varepsilon,\varphi}|_S = (d\Phi_z^{\varepsilon,\varphi})C'$ and thus

$$J_S \Phi^{\varepsilon,\varphi}(z) = \sqrt{\det({}^t C'(d\Phi_z^{\varepsilon,\varphi})^* \circ (d\Phi_z^{\varepsilon,\varphi})C')}.$$

Since $d\Phi_z^{\varepsilon,\varphi} = \begin{pmatrix} I & 0 \\ \varepsilon {}^t\nabla_x \varphi & 1+\varepsilon\varphi_y \end{pmatrix}$, the (i,j)-element of ${}^tC'(d\Phi_z^{\varepsilon,\varphi})^* \circ (d\Phi_z^{\varepsilon,\varphi})C'$ is

$$\begin{aligned} {}^t\tau_i {}^t(d\Phi_z^{\varepsilon,\varphi})(d\Phi_z^{\varepsilon,\varphi})\tau_j = \tau_i \cdot \tau_j + \varepsilon(\tau_i^{n+1}\nabla_x\varphi \cdot (\tau_j)' \\ + \tau_j^{n+1}\nabla_x\varphi \cdot (\tau_i)' + \tau_i^{n+1}\tau_j^{n+1} + 2\tau_i^{n+1}\tau_j^{n+1}\varphi_y) + O(\varepsilon^2). \end{aligned}$$

Noting $\tau_i \cdot \tau_j = \delta_{ij}$, we have by the use of relations $\det(I+\varepsilon A) = 1 + \varepsilon \mathrm{tr} A + O(\varepsilon^2)$ and $\sqrt{1+x} = 1 + x/2 + O(x^2)$

$$J_S\Phi^{\varepsilon,\varphi} = 1 + \varepsilon\{|v'_{E_u}|^2\varphi_y - (\nabla_x\varphi \cdot v'_{E_u})v_{E_u}^{n+1}\} + O(\varepsilon^2).$$

Thereby we obtain the following theorem by Theorem 2.1 and (2.1).

Theorem 2.2. *Let $u \in BV(\Omega)$ and $\varphi \in C^1(U)$. Suppose that all first derivatives of φ are bounded. Then $(d/d\varepsilon)J(u+\varepsilon\varphi(x,u))|_{\varepsilon=0}$ exists and it holds that*

$$\frac{d}{d\varepsilon}J(u+\varepsilon\varphi(x,u))|_{\varepsilon=0} = \int_{\partial^* E_u} [-(\nabla_x\varphi \cdot v'_{E_u})v_{E_u}^{n+1} + |v'_{E_u}|^2\varphi_y]\, d\mathscr{H}^n.$$

In [**4**] test functions φ depend only on x variables. Hence in their definition of a BV solution the second term of the right hand side of the above disappears. This is not appropriate because this term describes the variation of the area of the broken part of the graph.

3. Approximate solutions and their limit.

In this section we construct approximate solutions with Rothe's method, prove that this approximating sequence converges to a function u, and investigate its properties in terms of varifolds. In Rothe's method we should solve elliptic equations with respect to space variables. Here we solve them by a direct variational method.

Suppose that $u_0 \in L^2(\Omega) \cap BV(\Omega)$ with $\gamma u_0 = 0$ and $v_0 \in L^2(\Omega)$. For a positive number h we construct a sequence $\{u_\ell\}_{\ell=-1}^{\infty}$ in the following way. For $\ell = 0$ we let u_0 be as above and for $\ell = -1$ we set $u_{-1} = u_0 - hv_0$. For $\ell \geq 1$ it is usual to define u_ℓ as the minimizer of the functional

$$\mathscr{F}_\ell(v) = \frac{1}{2}\int_\Omega \frac{|v - 2u_{\ell-1} + u_{\ell-2}|^2}{h^2}\,dx + J(v) \quad (J \text{ is the area functional})$$

in the class $\{v \in L^2(\Omega) \cap BV(\Omega); \gamma v = 0\}$. However the existence of the minimizer of $\mathscr{F}_\ell$ in this class is not assured. Then we introduce another sequence of functionals in $L^2(\Omega) \cap BV(\Omega)$:

$$\mathscr{G}_\ell(v) = \frac{1}{2}\int_\Omega \frac{|v - 2u_{\ell-1} + u_{\ell-2}|^2}{h^2}\,dx + J(v) + \|\gamma v\|_{L^1(\partial\Omega)} \quad (v \in L^2(\Omega) \cap BV(\Omega)).$$

First we remark the following fact.

Proposition 3.1. *Suppose that $\partial\Omega$ is of C^1 class. Then it holds that*

$$\inf\{\mathscr{F}_\ell(v); v \in L^2(\Omega) \cap BV(\Omega), \gamma v = 0\} = \inf\{\mathscr{G}_\ell(v); v \in L^2(\Omega) \cap BV(\Omega)\}.$$

Proof. It is sufficient to show that the left hand side is not greater than the right hand side.

Let v be a function in $L^2(\Omega) \cap BV(\Omega)$. First we suppose that $\gamma v \in L^2(\partial\Omega)$. In the same way as in the proof of [**5**] Theorem 2.16 (refer also to Remark 2.17) we can obtain the following: for any $\varepsilon > 0$, there exists a function $w_\varepsilon \in L^2(\Omega) \cap W^{1,1}(\Omega)$ such that

$$\gamma w_\varepsilon = \gamma v, \tag{3.1}$$

$$\|w_\varepsilon\|_{L^2(\Omega)} \leq \varepsilon \|\gamma v\|_{L^2(\partial\Omega)}, \tag{3.2}$$

and

$$\int_\Omega |\nabla w_\varepsilon|\, dx \leq (1+\varepsilon)\|\gamma v\|_{L^1(\partial\Omega)}. \tag{3.3}$$

Then we put $v_\varepsilon = v - w_\varepsilon$. By (3.1) $\gamma v_\varepsilon = \gamma v - \gamma w_\varepsilon = 0$. By (3.3) we have

$$J(v_\varepsilon) \leq J(v) + \int_\Omega |\nabla w_\varepsilon|\, dx \leq J(v) + (1+\varepsilon)\|\gamma v\|_{L^1(\partial\Omega)}.$$

Thus, using (3.2), we have

$$\begin{aligned} \mathscr{F}_\ell(v_\varepsilon) &\leq \frac{1}{2}\int_\Omega \frac{|v - 2u_{\ell-1} + u_{\ell-2}|^2}{h^2}\, dx + \frac{\varepsilon}{2}\int_\Omega \frac{|v - 2u_{\ell-1} + u_{\ell-2}|^2}{h^2}\, dx \\ &\quad + \frac{1}{2h^2}\left(1 + \frac{1}{\varepsilon}\right)\int_\Omega |w_\varepsilon|^2\, dx + J(v) + (1+\varepsilon)\|\gamma v\|_{L^1(\partial\Omega)} \\ &\leq \mathscr{G}_\ell(v) + \frac{\varepsilon}{2}\int_\Omega \frac{|v - 2u_{\ell-1} + u_{\ell-2}|^2}{h^2}\, dx \\ &\quad + \frac{\varepsilon(1+\varepsilon)}{2h^2}\|\gamma v\|^2_{L^2(\partial\Omega)} + \varepsilon\|\gamma v\|_{L^1(\partial\Omega)}. \end{aligned} \tag{3.4}$$

When v is an arbitrary function in $L^2(\Omega) \cap BV(\Omega)$, we set $v_R(x) = R$ for $v(x) \geq R$, $= v(x)$ for $|v(x)| < R$, $= -R$ for $v(x) \leq -R$, where R is a positive number. Then v_R is a function of $L^2(\Omega) \cap BV(\Omega)$ with $\gamma v_R \in L^2(\partial\Omega)$ and satisfies $\mathscr{G}_\ell(v_R) \to \mathscr{G}_\ell(v)$ as $R \to \infty$. Applying (3.4) for v_R, we obtain

$$\begin{aligned} \mathscr{F}_\ell((v_R)_\varepsilon) &\leq \mathscr{G}_\ell(v) + (\mathscr{G}_\ell(v_R) - \mathscr{G}_\ell(v)) + \frac{\varepsilon}{2}\int_\Omega \frac{|v_R - 2u_{\ell-1} + u_{\ell-2}|^2}{h^2}\, dx \\ &\quad + \frac{\varepsilon(1+\varepsilon)}{2h^2}\|\gamma v_R\|^2_{L^2(\partial\Omega)} + \varepsilon\|\gamma v_R\|_{L^1(\partial\Omega)}. \end{aligned}$$

Since ε and R are arbitrary, we have the conclusion. □

Remark. The assumption that $\partial\Omega$ is of C^1 class is needed for the proof of (3.3).

The existence of the minimizer of $\mathscr{G}_\ell$ can be obtained in the same way as in the proof of [**5**] Theorem 14.5. Note that in the proof of this fact $\partial\Omega$ is assumed to be only Lipschitz continuous. Taking account of Proposition 3.1, we define u_ℓ as the minimizer of $\mathscr{G}_\ell$.

The next lemma follows from the convexity of the functional $J(v) + \|\gamma v\|_{L^1(\partial\Omega)}$ and the minimality of each u_ℓ ([**9**] Lemma 4.1).

Lemma 3.2 (Energy inequality).

$$\frac{1}{2}\int_\Omega \frac{|u_\ell - u_{\ell-1}|^2}{h^2}\,dx + J(u_\ell) + \|\gamma u_\ell\|_{L^1(\partial\Omega)} \le \frac{1}{2}\int_\Omega |v_0|^2\,dx + J(u_0).$$

Next we define approximate solutions $u^h(t,x)$ and $\bar{u}^h(t,x)$ for $(t,x) \in (-h,\infty)\times\Omega$ as follows: for $(\ell-1)h < t \le \ell h$

$$u^h(t,x) = \frac{t-(\ell-1)h}{h}u_\ell(x) + \frac{\ell h - t}{h}u_{\ell-1}(x)$$

and

$$\bar{u}^h(t,x) = u_\ell(x).$$

Then Lemma 3.2 shows

$$\frac{1}{2}\int_\Omega |u_t^h(t,x)|^2\,dx + J(\bar{u}^h(t,\cdot)) + \|\gamma\bar{u}^h(t,\cdot)\|_{L^1(\partial\Omega)} \le \frac{1}{2}\int_\Omega |v_0|^2\,dx + J(u_0) \tag{3.5}$$

for each $t \in \bigcup_{\ell=0}^\infty((\ell-1)h,\ell h)$.

Theorem 3.3. *It holds that*

1) $\{\|u_t^h\|_{L^\infty((0,\infty);L^2(\Omega))}\}$ *is uniformly bounded with respect to* h

2) *for any* $T>0$, $\{\|u^h\|_{L^\infty((0,T);L^2(\Omega)\cap BV(\Omega))}\}$ *is uniformly bounded with respect to* h

3) *for any* $T>0$, $\{\|\bar{u}^h\|_{L^\infty((0,T);L^2(\Omega)\cap BV(\Omega))}\}$ *is uniformly bounded with respect to h.*

Then there exist a sequence $\{h_j\}$ *with* $h_j \to 0$ *as* $j\to\infty$ *and a function u such that*

4) *for any* $T>0$, $\bar{u}^{h_j}$ *converges to u as* $j\to\infty$ *weakly star in* $L^\infty((0,T);L^2(\Omega))$

5) $u_t^{h_j}$ *converges to* u_t *as* $j\to\infty$ *weakly star in* $L^\infty((0,\infty);L^2(\Omega))$

6) *for any* $T>0$, u^{h_j} *converges to u as* $j\to\infty$ *strongly in* $L^p((0,T)\times\Omega)$ *for each* $1\le p<1^*$

7) *for any* $T>0$, $\bar{u}^{h_j}$ *converges to u as* $j\to\infty$ *strongly in* $L^p((0,T)\times\Omega)$ *for each* $1\le p<1^*$

8) $u\in L^\infty((0,\infty);BV(\Omega))$

9) *for* $\mathscr{L}^1$*-a.e.* $t\in(0,\infty)$, $D\bar{u}^{h_j}(t,\cdot)$ *converges to* $Du(t,\cdot)$ *as* $j\to\infty$ *in the sense of distributions*

10) $\text{s-lim}_{t\searrow 0}\, u(t) = u_0$ *in* $L^2(\Omega)$.

Proof. Assertion 1) immediately follows from (3.5). Moreover it also follows that $\{\|J(\bar{u}^h(t,\cdot))\|_{L^\infty(-h,\infty)}\}$ is uniformly bounded with respect to h. Since J is convex, we have

$$J(u^h(t,\cdot)) \le \frac{t-(\ell-1)h}{h}J(\bar{u}^h(t,\cdot)) + \frac{\ell h - t}{h}J(\bar{u}^h(t-h,\cdot)).$$

Thus $\{\|J(u^h(t,\cdot))\|_{L^\infty(0,\infty)}\}$ is also uniformly bounded with respect to h. Then Assertion 2) follows from Assertion 1) because

$$u^h(t,x) = u_0(x) + \int_0^t u_t^h(s,x)\,ds.$$

In the same way as in the proof of [**9**] Lemma 4.2 we can obtain

$$\|u^h - \bar{u}^h\|^2_{L^\infty((0,T);L^2(\Omega))} \to 0 \quad \text{as} \quad h \to 0. \tag{3.6}$$

Using this fact, we obtain Assertion 3) by Assertion 2). Assertions 4) and 5) are direct consequences of Assertions 3) and 1), respectively. By Sobolev's theorem $BV(\Omega) \subset L^p(\Omega)$ compactly for each $1 \le p < 1^*$. Then in the same way as in the proof of [**4**] Proposition 5.1 we obtain Assertions 6) and 7). The limits in 6) and 7) are the same because of (3.6). Assertion 8) immediately follows from 3), 4), and 7). Assertion 9) follows from (1.8) and 7). Assertion 10) is obtained in the same way as in the proof of [**9**] Theorem 4.1. □

Remark. In the sequel $\{u^{h_j}\}$ and $\{\bar{u}^{h_j}\}$ are often denoted by $\{u^h\}$ and $\{\bar{u}^h\}$ for simplicity.

Theorem 3.3, 5) and 8) imply i) of Definition 1 and 10) implies ii). Thus, if we show iii) and iv) of Definition 1, then u is a BV solution. In the next section we prove them with the assumption that u satisfies the energy conservation law. In this section we investigate the properties of u which hold without assuming the energy conservation law.

Since u_ℓ is the minimizer of $\mathscr{G}_\ell(v)$, we have, for any $\varphi \in C_0^1(U)$,

$$\begin{aligned} 0 &= \frac{d}{d\varepsilon}\mathscr{G}_\ell(u_\ell + \varepsilon\varphi(x,u_\ell))|_{\varepsilon=0} \\ &= \int_\Omega \frac{u_\ell(x) - 2u_{\ell-1}(x) + u_{\ell-2}(x)}{h^2}\varphi(x,u_\ell)\,dx + \frac{d}{d\varepsilon}J(u_\ell + \varepsilon\varphi(x,u_\ell))|_{\varepsilon=0}. \end{aligned}$$

By Theorem 2.2 we have

$$\frac{d}{d\varepsilon}J(u_\ell + \varepsilon\varphi(x,u_\ell))|_{\varepsilon=0} = \int_{\partial^* E_\ell} [-(\nabla_x\varphi \cdot v'_{E_\ell})v^{n+1}_{E_\ell} + |v'_{E_\ell}|^2\varphi_y]\,d\mathscr{H}^n,$$

where $E_\ell = E_{u_\ell}$. Then, noting that, for $(\ell-1)h < t < \ell h$, $u_t^h(t,x) = (u_\ell(x) - u_{\ell-1}(x))/h$, we have for any $T > 0$ and for any $\varphi \in C_0^1([0,T) \times U)$

$$\begin{aligned} \int_0^T \bigg\{ &\int_\Omega \frac{u_t^h(t,x) - u_t^h(t-h,x)}{h}\varphi(t,x,\bar{u}^h(t,x))\,dx \\ &+ \int_{\partial^* E_t^h} [-(\nabla_x\varphi \cdot v'_{E_t^h})v^{n+1}_{E_t^h} + |v'_{E_t^h}|^2\varphi_y]\,d\mathscr{H}^n \bigg\}\,dt = 0, \end{aligned} \tag{3.7}$$

where $E_t^h = E_{\bar{u}^h(t,\cdot)}$. In the sequel notation $E_t = E_{u(t,\cdot)}$ is also used. By the use of this notation and by Theorem 2.2 the equality of iv) of Definition 1 is rewritten as

$$\begin{aligned} \int_0^T \bigg\{ &-\int_\Omega u_t(\varphi_t(t,x,u) + \varphi_y(t,x,u)u_t)\,dx + \int_{\partial^* E_t} [-(\nabla_x\varphi \cdot v'_{E_t})v^{n+1}_{E_t} \\ &+ |v'_{E_t}|^2\varphi_y]\,d\mathscr{H}^n \bigg\}\,dt = \int_\Omega v_0(x)\varphi(0,x,u_0(x))\,dx. \end{aligned} \tag{3.8}$$

Now, for $\mathscr{L}^1$-a.e. t, we associate $\bar{u}^h(t,\cdot)$ with oriented varifolds and write $V_t^h = \boldsymbol{v}_+(\bar{u}^h(t,\cdot))$. For each $\xi \in G_0$ there exists a unique vector v such that $\xi \wedge v = e_1 \wedge \cdots \wedge e_{n+1}$. This map $\xi \mapsto v = v(\xi)$ is a homeomorphism from G_0 to the n-dimensional unit sphere S^n. For each $v \in BV(\Omega)$ we have by the definition of $\boldsymbol{v}_+(v)$ that, for any $\beta \in C_0^0(U \times G_0)$,

$$\int_{U\times G_0} \beta(z,\xi)\, d\boldsymbol{v}_+(v) = \int_{\partial^* E_v} \beta(z, v^{-1}(v_{E_v}(z)))\, d\mathscr{H}^n. \tag{3.9}$$

Note that $v(\xi)$ is the unit normal to the vector subspace associated with ξ and then $\operatorname{spt} \boldsymbol{v}_+(v) \subset U \times \{\xi \in G_0; v^{n+1}(\xi) \geq 0\}$. Let $\varphi(z) = \varphi(x,y)$ be an arbitrary function in $C_0^1(U)$. Applying (3.9) to $\bar{u}^h$ and $-(\nabla_x \varphi \cdot v'(\xi))v^{n+1}(\xi) + |v'(\xi)|^2 \varphi_y$ for v and $\beta(z,\xi)$, respectively, we obtain

$$\begin{aligned}\int_{\partial^* E_t^h} &[-(\nabla_x \varphi \cdot v'_{E_t^h})v_{E_t^h}^{n+1} + |v'_{E_t^h}|^2 \varphi_y]\, d\mathscr{H}^n \\ &= \int_{U\times G_0} [-(\nabla_x \varphi \cdot v'(\xi))v^{n+1}(\xi) + |v'(\xi)|^2 \varphi_y]\, dV_t^h(z,\xi).\end{aligned} \tag{3.10}$$

By Theorem 3.3 3) there exists a constant M which is independent of h such that

$$\operatorname*{ess.\,sup}_{t>0} \int_\Omega \sqrt{1 + |D\bar{u}^h(t,x)|^2} \leq M. \tag{3.11}$$

It follows from (1.5) and (1.9) that $V_t^h(U \times G_0)(= \mu_{V_t^h}(U)) = J(\bar{u}^h)$. Then (3.11) implies

$$\operatorname*{ess.\,sup}_{t>0} \left| \int_{U\times G_0} \beta(z,\xi)\, dV_t^h(z,\xi) \right| \leq M \sup|\beta| \tag{3.12}$$

for any $\beta \in C_0^0(U \times G_0)$. By the use of (3.12) we obtain the following theorem in the same way as in the proof of [**4**] Proposition 4.3.

Theorem 3.4. *There exists a subsequence of $\{V_t^h\}$ (still denoted by $\{V_t^h\}$) and a one parameter family of oriented varifolds V_t in $U = \Omega \times \boldsymbol{R}$, $t \in (0,\infty)$, such that, for each $\psi(t) \in L^1(0,\infty)$ and $\beta \in C_0^0(U \times G_0)$,*

$$\lim_{h\to 0} \int_0^\infty \psi(t) \int_{U\times G_0} \beta(z,\xi)\, dV_t^h(z,\xi)\, dt = \int_0^\infty \psi(t) \int_{U\times G_0} \beta(z,\xi)\, dV_t(z,\xi)\, dt.$$

The following lemma corresponds to [**4**] Proposition 6.2. Proof is the same as that of this proposition.

Lemma 3.5. *For $\mathscr{L}^1$-a.e. $t \in (0,\infty)$,*
1) $\limsup_{h\to 0} \mu_{V_t^h}(\omega) \geq \mu_{V_t}(\omega)$ *for each open set ω in U*
2) $\liminf_{h\to 0} \mu_{V_t^h}(K) \leq \mu_{V_t}(K)$ *for each compact set K in U.*

From [**1**] Theorem 10 of page 14 there exists a probability Radon measure $\eta_{V_t}^{(z)}$ on G_0 for μ_{V_t}-a.e. $z \in U$ such that

$$(3.13)\qquad \int_{U\times G_0}\beta(z,\xi)\,dV_t=\int_U\left(\int_{G_0}\beta(z,\xi)\,d\eta_{V_t}^{(z)}\right)d\mu_{V_t}\quad (\beta\in C_0^0(U\times G_0)).$$

The following lemma corresponds to [**4**] Propositions 6.4 and 6.5. However, owing to introducing orientations in varifolds, we can simplify the proof and refine the result.

Lemma 3.6. *$\int_U g(z)\nu_{E_t}(z)\,d\mu_{E_t}=\int_U g(z)\left(\int_{G_0}\nu(\xi)\,d\eta_{V_t}^{(z)}\right)d\mu_{V_t}$ for any $g\in C_0^0(U;\boldsymbol{R}^{n+1})$, for $\mathscr{L}^1$-a.e. $t\in(0,\infty)$.*

Proof. For any $\varphi\in C_0^0(U)$

$$\left|\int_U(\chi_{E_t^h}(z)-\chi_{E_t}(z))\varphi(z)\,dz\right|\le\left|\int_\Omega\left(\int_{u(t,x)}^{\bar u^h(t,x)}\varphi(x,y)\,dy\right)dx\right|$$
$$\le\sup|\varphi|\int_\Omega|\bar u^h(t,x)-u(t,x)|\,dx.$$

Since by Theorem 3.3 7) $\bar u^h$ converges to u strongly in $L^1(\Omega)$ for $\mathscr{L}^1$-a.e. t, we have $\chi_{E_t^h}$ converges to $\chi_{E_t}(z)$ in the sense of distributions in U for $\mathscr{L}^1$-a.e. t.

For any $g\in C_0^1(U;\boldsymbol{R}^{n+1})$

$$\int_U g(z)\nu_{E_t}(z)\,d\mu_{E_t}=\int_U\chi_{E_t}(z)\operatorname{div}g(z)\,dz=\lim_{h\to0}\int_U\chi_{E_t^h}(z)\operatorname{div}g(z)\,dz$$
$$=\lim_{h\to0}\int_U g(z)\nu_{E_t^h}(z)\,d\mu_{E_t^h}=\lim_{h\to0}\int_{U\times G_0}g(z)\nu(\xi)\,dV_t^h(z,\xi)$$
$$=\int_{U\times G_0}g(z)\nu(\xi)\,dV_t(z,\xi).$$

Since $C_0^1(U;\boldsymbol{R}^{n+1})$ is dense in $C_0^0(U;\boldsymbol{R}^{n+1})$, the conclusion follows. □

Now we sum up the properties of the limit varifold V_t. This theorem is closely related to [**4**] Proposition 6.3 and Theorem 2.

Theorem 3.7. *For $\mathscr{L}^1$-a.e. $t\in(0,\infty)$,*

1) *$\mu_{V_t}(A)\ge\mu_{E_t}(A)$ for each Borel set $A\subset U$*

2) *$\mu_{V_t}(A)=\int_A D_{\mu_{E_t}}\mu_{V_t}(z)\,d\mu_{E_t}+(\mu_{V_t}\llcorner Z)(A)$ for $A\subset U$, where $D_{\mu_{E_t}}\mu_{V_t}$ is the derivative of μ_{V_t} with respect to μ_{E_t} and Z is the μ_{E_t}-null set defined by $Z=\{z;D_{\mu_{E_t}}\mu_{V_t}(z)=\infty\}$*

3) *$\int_{G_0}\nu(\xi)\,d\eta_{V_t}^{(z)}=0$ for $\mu_{V_t}\llcorner Z$-a.e. z*

4) *$\operatorname{spt}\eta_{V_t}^{(z)}\subset\operatorname{irr}(G_0)$ for $\mu_{V_t}\llcorner Z$-a.e. z, where $\operatorname{irr}(G_0)=\{S;\nu^{n+1}(\xi)=0\}$.*

Proof. 1) First we consider the case that A is an open set. By Lemma 3.6 we have, for any $g\in C_0^0(A;\boldsymbol{R}^{n+1})$,

$$\left|\int_A g(z)\nu_{E_t}(z)\,d\mu_{E_t}\right|\le\int_A|g(z)|\,d\mu_{V_t}\le\sup|g|\mu_{V_t}(A).$$

Taking supremum with respect to $g\in C_0^0(A;\boldsymbol{R}^{n+1})$ with $|g|\le1$, we obtain $\mu_{E_t}(A)\le\mu_{V_t}(A)$.

Let A be any Borel set. For each open set ω with $A \subset \omega$, $\mu_{E_t}(A) \le \mu_{E_t}(\omega) \le \mu_{V_t}(\omega)$. Thus, since $\inf_{A \subset \omega} \mu_{V_t}(\omega) = \mu_{V_t}(A)$, we have $\mu_{E_t}(A) \le \mu_{V_t}(A)$.

2) It is the direct consequence of the differentiation theory for Radon measures (see, for example, [**12**] Theorem 4.7).

3) By Lemma 3.6 and Assertion 2) we have, for any $g(z) \in C_0^0(U; \boldsymbol{R}^{n+1})$,

$$0 = \int_Z g(z) \nu_{E_t}(z)\, d\mu_{E_t} = \int_Z g(z) \left(\int_{G_0} \nu(\xi)\, d\eta_{V_t}^{(z)} \right) d\mu_{V_t}.$$

This shows Assertion 3).

4) By 3), in particular, we have $\int_{G_0} \nu^{n+1}(\xi)\, d\eta_{V_t}^{(z)} = 0$ for $\mu_{V_t} \llcorner Z$-a.e. z. For each h, $\operatorname{spt} V_t^h$ is contained in $U \times \{\xi \in G_0; \nu^{n+1}(\xi) \ge 0\}$. Then $\operatorname{spt} V_t$ is also contained in this set. Thus Assertion 4) immediately follows. $\square$

4. Main Theorem.

Theorem 4.1. *Let T be a positive number. Suppose that $u_0 \in L^2(\Omega) \cap BV(\Omega)$ with $\gamma u_0 = 0$ and $v_0 \in L^2(\Omega)$. If u as in* Theorem 3.3 *satisfies the energy conservation law*

$$\frac{1}{2} \int_\Omega |u_t(t,x)|^2\, dx + J(u(t,\cdot)) = \frac{1}{2} \int_\Omega |v_0(x)|^2\, dx + J(u_0) \tag{4.1}$$

for $\mathscr{L}^1$-a.e. $t \in (0,T)$, then u is a BV solution to (1.1)–(1.3) *in* $(0,T) \times \Omega$.

Proof. *1st step.* By Theorem 3.3 5) and 7) we have, for $\mathscr{L}^1$-a.e. $t \in (0,T)$,

$$\liminf_{h \searrow 0} \int_\Omega |u_t^h(t,x)|^2\, dx \ge \int_\Omega |u_t(t,x)|^2\, dx$$

and

$$\liminf_{h \searrow 0} J(\bar{u}^h(t,\cdot)) \ge J(u(t,\cdot)).$$

Thus energy inequality (3.5) and energy conservation law (4.1) imply, for $\mathscr{L}^1$-a.e. $t \in (0,T)$,

$$\lim_{h \searrow 0} \int_\Omega |u_t^h(t,x)|^2\, dx = \int_\Omega |u_t(t,x)|^2\, dx, \tag{4.2}$$

$$\lim_{h \searrow 0} J(\bar{u}^h(t,\cdot)) = J(u(t,\cdot)), \tag{4.3}$$

and

$$\lim_{h \searrow 0} \|\gamma \bar{u}^h(t,\cdot)\|_{L^1(\partial\Omega)} = 0. \tag{4.4}$$

Writing (1.8) for $u = \bar{u}^h$ and letting $h \to 0$, we obtain iii) of Definition 1 by Theorem 3.3 9) and (4.4).

By (3.7) and (3.8) we obtain iv) of Definition 1 if we show, as $h \to 0$, passing to a subsequence if necessary,

$$(4.5)\qquad \int_0^T\int_\Omega \frac{u_t^h(t,x)-u_t^h(t-h,x)}{h}\varphi(t,x,\bar{u}^h(t,x))\,dxdt$$

$$\to \int_0^T\left\{-\int_\Omega u_t(\varphi_t(t,x,u)+\varphi_y(t,x,u)u_t)\,dx\right\}dt-\int_\Omega v_0(x)\varphi(0,x,u_0(x))\,dx$$

and

$$(4.6)\qquad \int_0^T\int_{\partial^* E_t^h}[-(\nabla_x\varphi\cdot \nu'_{E_t^h})\nu_{E_t^h}^{n+1}+|\nu'_{E_t^h}|^2\varphi_y]\,d\mathscr{H}^n\,dt$$

$$\to \int_0^T\int_{\partial^* E_t}[-(\nabla_x\varphi\cdot \nu'_{E_t})\nu_{E_t}^{n+1}+|\nu'_{E_t}|^2\varphi_y]\,d\mathscr{H}^n\,dt.$$

2nd step (Proof of (4.5)). First we rewrite

$$(4.7)\qquad \int_0^T\int_\Omega \frac{u_t^h(t,x)-u_t^h(t-h,x)}{h}\varphi(t,x,\bar{u}^h(t,x))\,dxdt$$

$$=\int_0^\infty\int_\Omega \frac{u_t^h(t,x)-u_t^h(t-h,x)}{h}\varphi(t,x,\bar{u}^h(t,x))\,dxdt$$

$$=\int_0^\infty\int_\Omega \frac{u_t^h(t,x)}{h}\varphi(t,x,\bar{u}^h(t,x))\,dxdt$$

$$-\int_{-h}^\infty\int_\Omega \frac{u_t^h(s,x)}{h}\varphi(s+h,x,\bar{u}^h(s+h,x))\,dxds$$

$$=-\left\{\int_0^\infty\int_\Omega u_t^h(t,x)\frac{\varphi(t+h,x,\bar{u}^h(t+h,x))-\varphi(t,x,\bar{u}^h(t,x))}{h}\,dxdt\right.$$

$$\left.+\frac{1}{h}\int_{-h}^0\int_\Omega u_t^h(s,x)\varphi(s+h,x,\bar{u}^h(s+h,x))\,dxds\right\}$$

$$=: -(I+II).$$

Noting that, for $-h<s\le 0$, $u_t^h(s,x)=v_0(x)$ and $\bar{u}^h(s+h,x)=u_1(x)$, we have

$$II=\frac{1}{h}\int_{-h}^0\int_\Omega v_0(x)\varphi(s+h,x,u_1(x))\,dxds=\frac{1}{h}\int_0^h\int_\Omega v_0(x)\varphi(t,x,u_1(x))\,dxdt.$$

Since, for $0<t<h$,

$$\varphi(t,x,u_1(x))=\varphi(t,x,u_0(x))+\int_0^1\varphi_y(t,x,\theta(u_1(x)-u_0(x)))(u_1(x)-u_0(x))\,d\theta$$

$$=\varphi(t,x,u_0(x))+\int_0^1\varphi_y(t,x,u_0(x)+h\theta u_t^h(t,x))hu_t^h(t,x)\,d\theta,$$

it holds that

$$II = \frac{1}{h}\int_0^h \int_\Omega v_0(x)\varphi(t,x,u_0(x))\,dxdt$$
$$+ \int_0^h \int_\Omega v_0(x) \int_0^1 \varphi_y(t,x,u_0(x) + h\theta u_t^h(t,x))u_t^h(t,x)\,d\theta dxdt.$$

Thus, noting Theorem 3.3 1), we have

$$\lim_{h\searrow 0} II = \int_\Omega v_0(x)\varphi(0,x,u_0(x))\,dx. \tag{4.8}$$

On the other hand, since

$$\frac{\varphi(t+h,x,\bar{u}^h(t+h,x)) - \varphi(t,x,\bar{u}^h(t,x))}{h}$$
$$= \int_0^1 \Big[\varphi_t(t+\theta h,x,\bar{u}^h(t+h,x))$$
$$+ \varphi_y(t,x,\bar{u}^h(t,x) + \theta(\bar{u}^h(t+h,x) - \bar{u}^h(t,x)))\frac{\bar{u}^h(t+h,x) - \bar{u}^h(t,x)}{h}\Big]\,d\theta$$
$$= \int_0^1 [\varphi_t(t+\theta h,x,\bar{u}^h(t+h,x)) + \varphi_y(t,x,\bar{u}^h(t,x) + \theta h u_t^h(t+h,x))u_t^h(t+h,x)]\,d\theta,$$

we have

$$I = \int_0^T \int_\Omega u_t^h(t,x) \int_0^1 [\varphi_t(t+\theta h,x,\bar{u}^h(t+h,x))$$
$$+ \varphi_y(t,x,\bar{u}^h(t,x) + \theta h u_t^h(t+h,x))u_t^h(t+h,x)]\,d\theta\,dxdt.$$

By (4.2) $\{u_t^h(t,\cdot)\}$ converges to $u_t(t,\cdot)$ strongly in $L^2(\Omega)$ for $\mathscr{L}^1$-a.e. $t \in (0,T)$. By (3.5) and (4.1) $\|u_t^h(t,\cdot)\|_{L^2(\Omega)}$ and $\|u_t(t,\cdot)\|_{L^2(\Omega)}$ are uniformly bounded with respect to t and h. Thus the dominated convergence theorem implies $\{u_t^h\}$ converges to u_t strongly in $L^2((0,T)\times\Omega)$. Let T' be any number with $0 < T' < T$. If $0 < h < T - T'$, we have

$$\|u_t^h(\cdot+h) - u_t(\cdot+h)\|_{L^2((0,T')\times\Omega)} = \|u_t^h - u_t\|_{L^2((h,T'+h)\times\Omega)} \le \|u_t^h - u_t\|_{L^2((0,T)\times\Omega)},$$

the right hand side of which converges to 0 as $h \to 0$. It follows from Lusin's theorem that, as $h \to 0$,

$$\|u_t(\cdot+h) - u_t\|_{L^2((0,T')\times\Omega)} \to 0.$$

Thus, writing

$$\|u_t^h(\cdot+h) - u_t\|_{L^2((0,T')\times\Omega)}$$
$$\le \|u_t^h(\cdot+h) - u_t(\cdot+h)\|_{L^2((0,T')\times\Omega)} + \|u_t(\cdot+h) - u_t\|_{L^2((0,T')\times\Omega)},$$

we obtain that $u_t^h(\cdot+h) \to u_t$ strongly in $L^2((0,T')\times\Omega)$. By Theorem 3.3 7) we also obtain that $\bar{u}^h(\cdot+h) \to u$ strongly in $L^1((0,T')\times\Omega)$. Noting that the support of φ

with respect to the t variable is a compact subset of $[0,T)$, we further see that

$$\varphi_t(t+\theta h, x, \bar{u}^h(t+h,x)) \to \varphi_t(t,x,u(t,x))$$

and

$$\varphi_y(t,x,\bar{u}^h(t,x)+\theta h u_t^h(t+h,x)) \to \varphi_y(t,x,u(t,x))$$

strongly in $L^1((0,T)\times\Omega\times(0,1))$. Hence there exist subsequences of $\{\varphi_t(t+\theta h, x, \bar{u}^h(t+h,x))\}$ and $\{\varphi_y(t,x,\bar{u}^h(t,x)+\theta h u_t^h(t+h,x))\}$ which converge at $\mathscr{L}^{n+2}$-a.e. $(t,x,\theta)\in(0,T)\times\Omega\times(0,1)$. Generally we can prove that, when $\{\psi_j\}$ converges to ψ a.e., $\{\|\psi_j\|_{L^\infty}\}$ is uniformly bounded, and $\{v_j\}$ converges to v strongly in L^1, then $\{\psi_j v_j\}$ converges to $\{\psi v\}$ strongly in L^1. Thus, passing to the subsequence, we have

$$\lim_{h\searrow 0} I = \int_0^\infty\int_\Omega u_t(\varphi_t(t,x,u)+\varphi_y(t,x,u)u_t)\,dxdt. \tag{4.9}$$

Now (4.5) follows from (4.7), (4.8), and (4.9).

3rd step (Proof of (4.6)). If we obtain, for $\mathscr{L}^1$-a.e. t,

$$V_t = \boldsymbol{v}_+(u(t,\cdot)), \tag{4.10}$$

then (4.6) follows from (3.10) and Theorem 3.4. Thus we have only to show (4.10). Proof of (4.10) is essentially the same as that of [**4**] (6.57). However by introducing oriented varifolds we can make it clearer.

It follows from Lemma 3.5 1) and Theorem 3.7 1) that, for $\mathscr{L}^1$-a.e. t,

$$\limsup_{h\to 0}\mu_{V_t^h}(U) \ge \mu_{V_t}(U) \ge \mu_{E_t}(U). \tag{4.11}$$

On the other hand (4.3) means

$$\lim_{h\to 0}\mu_{V_t^h}(U) = \mu_{E_t}(U). \tag{4.12}$$

Thus, for each t at which both (4.11) and (4.12) hold, we have $\mu_{V_t}(U) = \mu_{E_t}(U)$. Further Theorem 3.7 1) implies

$$\mu_{V_t} = \mu_{E_t}. \tag{4.13}$$

It follows from Lemma 3.6 and (4.13) that

$$\nu_{E_t}(z) = \int_{G_0}\nu(\xi)\,d\eta_{V_t}^{(z)} \tag{4.14}$$

for μ_{V_t}-a.e. $z\in U$. By [**4**] Lemma 6.8 we have, for μ_{V_t}-a.e. $z\in U$, for $\eta_{V_t}^{(z)}$-a.e. $\xi\in G_0$,

$$\nu(\xi) = \nu_{E_t}(z).$$

Thus, for μ_{V_t}-a.e. $z\in U$,

$$\operatorname{spt}\eta_{V_t}^{(z)} = \text{one point} = \{\nu^{-1}(\nu_{E_t}(z))\}.$$

By the definition of ν we see that $\xi(z)\equiv\nu^{-1}(\nu_{E_t}(z))$ is the orientation of $\partial^* E_t$ which agrees with $\nu_{E_t}(z)$. By (3.13) and (4.13) we obtain

$$\int_{U\times G_0} \beta(z,\xi)\,dV_t = \int_U \beta(z,\xi(z))\,d\mu_{E_t} \quad (\beta \in C_0^0(U\times G_0)).$$

Now (4.10) is verified. $\square$

Appendix.

In this appendix we suppose that $\partial\Omega$ is of C^2 class. This fact is used in the proof of Lemma A.4.

The purpose of this appendix is to show the following theorem.

THEOREM A.1. *A function u is a BV solution to* (1.1)–(1.3) *in* $(0,T)\times\Omega$ *if and only if u satisfies* i)–iii) *of Definition* 1 *and*

iv)' *for any* $\phi \in C_0^1([0,T);L^2(\Omega))\cap L^\infty((0,T);BV(\Omega))$ *with* $\gamma\phi = 0$ *for* $\mathscr{L}^1$*-a.e.* $t\in(0,T)$,

$$\int_0^T \{J(u+\phi)-J(u)\}\,dt \geq \int_0^T\int_\Omega u_t\phi_t(t,x)\,dxdt + \int_\Omega v_0(x)\phi(0,x)\,dx.$$

PROOF OF 'IF' PART. Suppose that u satisfies i)–iii) of Definition 1 and iv)'. For each $\varphi \in C_0^1([0,T)\times U)$ we have $\varphi(t,x,u)\in C_0^1([0,T);L^2(\Omega))\cap L^\infty((0,T);BV(\Omega))$. Thus, since J is convex and by Theorem 2.2 differentiable to the direction $\varphi(t,x,u)$ for each t, iv)' yields

$$\int_0^T \frac{d}{d\varepsilon}J(u+\varepsilon\varphi(x,u))|_{\varepsilon=0}\,dt = \int_0^T\int_\Omega u_t[\varphi(t,x,u)]_t\,dxdt + \int_\Omega v_0(x)\varphi(0,x,u_0(x))\,dx,$$

which shows iv) of Definition 1. $\square$

Before the proof of 'only if' part we prepare several lemmas.

LEMMA A.2. *Suppose that u is a BV solution in* $(0,T)\times\Omega$ *and* $u_t(T,\cdot)\equiv \operatorname{ap\,lim}_{t\nearrow T} u_t(t,\cdot)$ *exists in* $L^2(\Omega)$. *Then, for any* $\varphi\in C_0^1([0,T]\times U)$, *it holds that*

$$\text{(A.1)}\qquad \int_0^T\left\{-\int_\Omega u_t(\varphi_t(t,x,u)+\varphi_y(t,x,u)u_t)\,dx + \frac{d}{d\varepsilon}J(u+\varepsilon\varphi(t,x,u))|_{\varepsilon=0}\right\}dt$$
$$= \int_\Omega v_0(x)\varphi(0,x,u_0(x))\,dx - \int_\Omega u_t(T,x)\varphi(T,x,u(T,x))\,dx,$$

where $u(T,\cdot) = \text{s-}\lim_{t\nearrow T} u(t,\cdot)$ *in* $L^2(\Omega)$.

PROOF. Let η be a one dimensional mollifier, that is, $\eta\in C_0^\infty(\boldsymbol{R})$, $\operatorname{spt}\eta\subset[-1,1]$, $0\le\eta\le 1$, and $\int_{-\infty}^\infty \eta(t)\,dt = 1$. For $\sigma>0$ we put $\rho_\sigma(t) = 1 - Y_\sigma(t-T+2\sigma)$, where

$$Y_\sigma(t) = \int_{-\infty}^t \sigma^{-1}\eta(\sigma^{-1}s)\,ds.$$

Then $\rho_\sigma(t)\varphi(t,x,y)$ belongs to $C_0^1([0,T)\times U)$ and $\rho_\sigma(0)=1$ when $\sigma\le T/3$. Thus iv) of Definition 1 yields

$$\text{(A.2)}\qquad \int_0^T \left\{ -\int_\Omega u_t(\rho_\sigma(t)\varphi_t(t,x,u) + \rho_\sigma'(t)\varphi(t,x,u) + \rho_\sigma(t)\varphi_y(t,x,u)u_t)\,dx \right.$$
$$\left. + \frac{d}{d\varepsilon}J(u+\varepsilon\rho_\sigma(t)\varphi(t,x,u))|_{\varepsilon=0} \right\} dt = \int_\Omega v_0(x)\varphi(0,x,u_0(x))\,dx.$$

Since $\rho_\sigma(t) \to 1$ as $\sigma \to 0$ on $[0,T)$, we obtain by Theorem 2.2 that, as $\sigma \to 0$,

$$\int_0^T \frac{d}{d\varepsilon}J(u+\varepsilon\rho_\sigma(t)\varphi(t,x,u))|_{\varepsilon=0}\,dt \to \int_0^T \frac{d}{d\varepsilon}J(u+\varepsilon\varphi(t,x,u))|_{\varepsilon=0}\,dt.$$

The first and third terms of the left hand side of (A.2) also converge to corresponding terms of iv) of Definition 1.

Now we mention about the second term. Note that

$$\rho_\sigma'(t) = -(Y_\sigma(t-T+2\sigma))' = -\sigma^{-1}\eta(\sigma^{-1}(t-T+2\sigma)).$$

We put

$$f(t) = \begin{cases} \int_\Omega u_t(t,x)\varphi(t,x,u(t,x))\,dx & \text{for } t \in (0,T] \\ 0 & \text{otherwise.} \end{cases}$$

Then the second term of the left hand side of (A.2) coincides with $\int_0^T \sigma^{-1}\eta(\sigma^{-1}(t-T+2\sigma))f(t)\,dt$. Since $f \in L^\infty(0,T)$ by i) of Definition 1, we have, for any $\delta > 0$,

$$\begin{aligned} &\left| \int_0^T \sigma^{-1}\eta(\sigma^{-1}(t-T+2\sigma))f(t)\,dt - f(T) \right| \\ &\quad = \left| \int_{-\infty}^{\infty} \sigma^{-1}\eta(\sigma^{-1}s)(f(s+T-2\sigma) - f(T))\,ds \right| \\ &\quad \le \delta \int_{-\infty}^{\infty} \sigma^{-1}\eta(\sigma^{-1}s)\,ds + 2\|f\|_{L^\infty(0,T)} \int_{A_{\delta,\sigma}} \sigma^{-1}\eta(\sigma^{-1}s)\,ds, \end{aligned}$$

where $A_{\delta,\sigma} = \{t; |f(t+T-2\sigma) - f(T)| \ge \delta\} \cap \{t; -\sigma \le t \le \sigma\}$. Since $0 \le \eta \le 1$, we have

$$\text{(A.3)}\qquad \left| \int_0^T \sigma^{-1}\eta(\sigma^{-1}(t-T+2\sigma))f(t)\,dt - f(T) \right| \le \delta + \sigma^{-1}\mathscr{L}^1(A_{\delta,\sigma}).$$

Note that

$$\text{(A.4)}\qquad \mathscr{L}^1(A_{\delta,\sigma}) = \mathscr{L}^1(\{t; |f(t) - f(T)| \ge \delta\} \cap \{t; T-3\sigma \le t \le T-\sigma\}).$$

By the definition of f there exists a constant $C = C(\Omega, \varphi, \|u_t\|_{L^\infty((0,T),L^2(\Omega))})$ such that

$$|f(t) - f(T)| \le C(\|u_t(t,\cdot) - u_t(T,\cdot)\|_{L^2(\Omega)} + \|u(t,\cdot) - u(T,\cdot)\|_{L^2(\Omega)}).$$

Hence we obtain

$$\text{(A.5)} \qquad \{t; |f(t) - f(T)| \geq \delta\} \subset \{t; \|u_t(t,\cdot) - u_t(T,\cdot)\|_{L^2(\Omega)} \geq (2C)^{-1}\delta\}$$
$$\cup \{t; \|u(t,\cdot) - u(T,\cdot)\|_{L^2(\Omega)} \geq (2C)^{-1}\delta\}.$$

Since $u(T,\cdot) = \text{s-}\lim_{t\nearrow T} u(t,\cdot)$ in $L^2(\Omega)$, we have, if σ is sufficiently small,

$$\mathscr{L}^1(\{t; \|u(t,\cdot) - u(T,\cdot)\|_{L^2(\Omega)} \geq (2C)^{-1}\delta\} \cap \{t; T - 3\sigma \leq t \leq T - \sigma\}) = 0.$$

Further, since $u_t(T,\cdot) = \text{ap}\lim_{t\nearrow T} u_t(t,\cdot)$, we have

$$\lim_{\sigma\to 0} \sigma^{-1}\mathscr{L}^1(\{t; \|u_t(t,\cdot) - u_t(T,\cdot)\|_{L^2(\Omega)} \geq (2C)^{-1}\delta\} \cap \{t; T - 3\sigma \leq t \leq T - \sigma\}) = 0.$$

Now by (A.4) and (A.5) we obtain $\lim_{\sigma\to 0} \sigma^{-1}\mathscr{L}^1(A_{\delta,\sigma}) = 0$. Thus, since δ is arbitrary, (A.3) yields

$$\lim_{\sigma\to 0} \int_0^T \sigma^{-1}\eta(\sigma^{-1}(t - T + 2\sigma))f(t)\,dt = f(T).$$

This shows the conclusion. □

Lemma A.3. *If u satisfies the same conditions as in* Lemma A.2, *then* (A.1) *holds for each function φ having the form $\varphi(t,x,y) = \tilde{\varphi}(t,x) + ay$, where $\tilde{\varphi} \in C_0^1([0,T]\times\Omega)$ and a is a real constant.*

Proof. Note that the function φ as in the statement of lemma satisfies the assumption of Theorem 2.2. Then by Theorem 2.2 it is sufficient to show that

$$\text{(A.6)} \qquad \int_0^T \left\{ -\int_\Omega u_t(\varphi_t(t,x,u) + \varphi_y(t,x,u)u_t)\,dx \right.$$
$$\left. + \int_{\partial^* E_t} [-(\nabla_x\varphi \cdot \nu'_{E_t})\nu_{E_t}^{n+1} + |\nu'_{E_t}|^2\varphi_y]\,d\mathscr{H}^n \right\} dt$$
$$= \int_\Omega v_0(x)\varphi(0,x,u_0(x))\,dx - \int_\Omega u_t(T,x)\varphi(T,x,u(T,x))\,dx.$$

Let $\mathscr{C}$ denote the set $\{\varphi \in C^1([0,T]\times U);\ \varphi,\ \varphi_t,\ \nabla_x\varphi,\ \varphi_y$ are all bounded, $\operatorname{spt}\varphi \subset [0,T]\times K \times \boldsymbol{R}$, where K is a compact subsets of $\Omega\}$. First we show that (A.6) holds for each $\varphi \in \mathscr{C}$. Let ζ be a C^∞ function on $\boldsymbol{R}$ such that $\zeta(r) = 1$ for $r \leq 0$, $= 0$ for $r \geq 1$, and $0 \leq \zeta(r) \leq 1$ for $r \in \boldsymbol{R}$. Now we put $\zeta_R(r) = \zeta(r - R)$ $(R > 0)$. Suppose that $\varphi \in \mathscr{C}$. Then, since $\zeta_R(|y|)\varphi(t,x,y) \in C_0^1([0,T]\times U)$ and u is a BV solution, we have by Lemma A.2

$$\text{(A.7)} \qquad \int_0^T \left\{ -\int_\Omega u_t(\zeta_R(|u|)\varphi_t(t,x,u) + \zeta'_R(|u|)\varphi(t,x,u)u_t + \zeta_R(|u|)\varphi_y(t,x,u)u_t)\,dx \right.$$
$$\left. + \int_{\partial^* E_t} [-(\zeta_R(|y|)\nabla_x\varphi \cdot \nu'_{E_t})\nu_{E_t}^{n+1} + |\nu'_{E_t}|^2(\zeta'_R(|y|)\varphi + \zeta_R(|y|)\varphi_y)]\,d\mathscr{H}^n \right\} dt$$
$$= \int_\Omega v_0(x)\zeta_R(|u_0(x)|)\varphi(0,x,u_0(x))\,dx$$
$$- \int_\Omega u_t(T,x)\zeta_R(|u(T,x)|)\varphi(T,x,u(T,x))\,dx.$$

Since $u_t \in L^2((0,T)\times\Omega)$ and φ, φ_t, $\nabla_x\varphi$, and φ_y are bounded and $\mathscr{H}^n(\partial^* E_t) < \infty$, the both side of (A.7) converges to the both side of (A.6), respectively, as $R \to \infty$.

Next we prove (A.6) for a function $\varphi(t,x,y) = \tilde{\varphi}(t,x) + ay$, where $\tilde{\varphi} \in C_0^1([0,T]\times\Omega)$ and a is a real constant. We define a nondecreasing function $\eta_L \in C^1(\boldsymbol{R})$ $(L > 0)$ as $\eta_L(y) = y$ for $0 \le y \le L$, $= \sin(y-L) + L$ for $L < y < L + \pi/2$, $= L+1$ for $y \ge L + \pi/2$, and $\eta_L(y) = -\eta_L(-y)$ for $y < 0$. Putting $\varphi_L(t,x,y) = \tilde{\varphi}(t,x) + a\eta_L(y)$, we have $\varphi_L \in \mathscr{C}$ and thus

$$\begin{aligned}\text{(A.8)}\qquad &\int_0^T \left\{ -\int_\Omega u_t((\varphi_L)_t(t,x,u) + (\varphi_L)_y(t,x,u)u_t)\,dx \right. \\ &\qquad \left. + \int_{\partial^* E_t} [-(\nabla_x\varphi_L \cdot v'_{E_t})v_{E_t}^{n+1} + |v'_{E_t}|^2(\varphi_L)_y]\,d\mathscr{H}^n \right\} dt \\ &= \int_\Omega v_0(x)\varphi_L(0,x,u_0(x))\,dx - \int_\Omega u_t(T,x)\varphi_L(T,x,u(T,x))\,dx.\end{aligned}$$

By the definition of φ_L we have $\nabla_x\varphi_L = \nabla_x\varphi$, $(\varphi_L)_t = \varphi_t$, and $(\varphi_L)_y = (\eta_L)_y = 1$ for $|y| \le L$, $= 0$ for $|y| > L + \pi/2$. Thus, since $(\varphi_L)_y \nearrow 1$ as $L \to \infty$, (A.6) holds for this φ by the dominated convergence theorem. □

For $\delta > 0$ we set

$$\text{(A.9)}\qquad \Omega_\delta = \{x \in \Omega; \operatorname{dist}(x,\partial\Omega) > \delta\}.$$

Let γ_δ^+ and γ_δ^- denote the trace operators in $BV(\Omega_\delta)$ and $BV(\boldsymbol{R}^n\backslash\bar{\Omega}_\delta)$, respectively.

Lemma A.4. *There exists a constant $\delta_0 = \delta_0(\Omega)$ such that, if $\delta < \delta_0$, then $\partial\Omega_\delta$ is of C^1 class. In addition there exists a constant $C = C(\Omega)$ such that*

$$\text{(A.10)}\qquad \|\gamma_\delta^- v\|_{L^1(\partial\Omega_\delta)} \le C(\|v\|_{L^1(\Omega\backslash\bar{\Omega}_\delta)} + |Dv|(\Omega\backslash\bar{\Omega}_\delta) + \|\gamma v\|_{L^1(\partial\Omega)})$$

for any $v \in BV(\Omega)$ and any δ with $0 < \delta < \delta_0$.

Proof. *1st step.* Given $x = (x_1,\dots,x_n)$, let us write $x = (x', x_n)$ for $x' = (x_1,\dots,x_{n-1}) \in \boldsymbol{R}^{n-1}, x_n \in \boldsymbol{R}$. Similar notations are used for other points. Given $x \in \boldsymbol{R}^n$ and r, $h > 0$, we define the open cylinder

$$C(x,r,h) = \{y \in \boldsymbol{R}^n; |y' - x'| < r, |y_n - x_n| < h\}$$

and the $(n-1)$-dimensional open ball

$$B(x',r) = \{y' \in \boldsymbol{R}^{n-1}; |y' - x'| < r\}.$$

Since $\partial\Omega$ is of C^2 class, for each point $x \in \partial\Omega$ there exist r, $h > 0$ and a C^2 function f such that, upon rotating and relabeling the coordinate axis if necessary,

$$\text{(A.11)}\qquad \Omega \cap C(x,2r,h) = \{y \in \boldsymbol{R}^n; |y' - x'| < 2r, x_n - h < y_n < f(y')\}$$

and

$$\text{(A.12)}\qquad \max_{|x'-y'|\le r} |f(y') - x_n| \le \frac{h}{2}.$$

Suppose that

$$\delta < \min\left\{\frac{h}{2}, r\right\}. \tag{A.13}$$

For $y \in \partial\Omega$ we define a point z by

$$z = y - \delta\vec{n}_y,$$

where $\vec{n}_y$ is the outer unit normal to $\partial\Omega$ at y. If $y \in \partial\Omega \cap C(x, 2r, h)$, then

$$\vec{n}_y = {}^t\!\left(-\frac{\nabla f(y')}{\sqrt{1 + |\nabla f(y')|^2}}, \frac{1}{\sqrt{1 + |\nabla f(y')|^2}}\right).$$

Now we set $z' = \tau(y')$. It is a C^1 map from $B(x', 2r)$ into $\boldsymbol{R}^{n-1}$ and given by

$$\tau(y') = y' + \frac{\delta\nabla f(y')}{\sqrt{1 + |\nabla f(y')|^2}}.$$

Since f is of C^2 class, $\det(\partial\tau/\partial y') \neq 0$ if δ is sufficiently small. Thus, noting that $\tau(B(x', 2r)) \supset B(x', 2r - \delta)$, we find by (A.13) that the inverse τ^{-1} can be defined at least for $z' \in B(x', r)$. It is also a C^1 map, and we define another C^1 function $f_\delta : B(x', r) \to \boldsymbol{R}$ by

$$f_\delta(z') = f(\tau^{-1}(z')) - \frac{\delta}{\sqrt{1 + |\nabla f(\tau^{-1}(z'))|^2}}. \tag{A.14}$$

It is clear by (A.12) and (A.13) that $x_n - h < f_\delta(z')$. For $z' \in B(x', r)$, putting $y' = \tau^{-1}(z')$ and $y = (y', f(y'))$, we have $z \equiv (z', f_\delta(z')) = y - \delta\vec{n}_y$. Moreover we can show that there exists a positive number $\delta^* < \min\{h/2, r\}$ which depends on f such that, if $\delta < \delta^*$,

$$\Omega_\delta \cap C(x, r, h) = \{y \in \boldsymbol{R}^n; |y' - x'| < r, x_n - h < y_n < f_\delta(y')\}. \tag{A.15}$$

Note that f is determined by x and $\partial\Omega$. Hence δ^* is determined by x and $\partial\Omega$.

2nd step. For each $x \in \partial\Omega$ there exist r_x, h_x such that (A.11) holds and δ_x^* as above. Since $\partial\Omega$ is compact, there are finitely many points $\{x_j\}_{j=1}^N \subset \partial\Omega$ such that

$$\partial\Omega \subset \bigcup_{j=1}^N C(x_j, r_j - \delta_j^*, h_j),$$

where $r_j = r_{x_j}$, $h_j = h_{x_j}$, and $\delta_j^* = \delta_{x_j}^*$.

Now we put

$$\delta_0 = \min\{\delta_j^*; j = 1, \dots, N\}.$$

Let δ be a number with $0 < \delta < \delta_0$. For each $x \in \partial\Omega_\delta$ there exists a point $y \in \partial\Omega$ such that $\operatorname{dist}(x, y) = \delta$. Then y belongs to one of cylinders $\{C(x_j, r_j - \delta_j^*, h_j)\}$ and hence x, which coincides with $y - \delta\vec{n}_y$, belongs to $C(x_j, r_j, h_j)$. Since (A.15) holds for each x_j, we see that $\partial\Omega_\delta$ is of C^1 class.

3rd step (Proof of (A.10)). Let v be a function in $BV(\Omega)$. By the use of a partition of unity subordinate to the $\{C(x_j, r_j, h_j)\}$ we may suppose that $\operatorname{spt} v$ is contained in one of these cylinders. Omitting the index j, we write $C(x, r, h)$ for this cylinder. By (A.11) and (A.15), for each δ with $0 < \delta < \delta_0$,

$$(\Omega\backslash\bar{\Omega}_\delta) \cap C(x, r, h) = \{y \in \boldsymbol{R}^n; |y' - x'| < r, f_\delta(y') < y_n < f(y')\}.$$

For the time we assume that $v \in BV(\Omega) \cap C^\infty(\Omega\backslash\bar{\Omega}_\delta)$. When ε is a sufficiently small positive number, then

$$v(y', f(y') - \varepsilon) - v(y', f_\delta(y') + \varepsilon) = \int_{f_\delta(y')+\varepsilon}^{f(y')-\varepsilon} \frac{\partial v}{\partial x_n}(y', s)\, ds.$$

It is easy to prove that $v(y', f(y') - \varepsilon) \to \gamma v(y', f(y'))$ and $v(y', f_\delta(y') + \varepsilon) \to \gamma_\delta^- v(y', f_\delta(y'))$ in $L^1(B(x', r))$ (refer to the 2nd step of the proof of [**2**] Theorem 1 of Section 5.3). Integrating over $B(x', r)$ and letting $\varepsilon \to 0$, we have

$$\int_{B(x',r)} |\gamma_\delta^- v(y', f_\delta(y'))|\, dy' \le \int_{B(x',r)} \int_{f_\delta(y')}^{f(y')} \left|\frac{\partial v}{\partial x_n}(y', s)\right| dy' ds + \int_{B(x',r)} |\gamma v(y', f(y'))|\, dy'.$$

Now note that

$$\int_{\partial\Omega_\delta \cap C(x,r,h)} |\gamma_\delta^- v(y)|\, d\mathscr{H}^{n-1} = \int_{B(x',r)} |\gamma_\delta^- v(y', f_\delta(y'))| \sqrt{1 + |\nabla f_\delta(y')|^2}\, dy'.$$

Since f is of C^2 class, we find by (A.14) that there exists a constant C which depends only on x and Ω such that $\sqrt{1 + |\nabla f_\delta(y')|^2} \le C$. Then

$$\begin{aligned}
\int_{\partial\Omega_\delta \cap C(x,r,h)} |\gamma_\delta^- v(y)|\, d\mathscr{H}^{n-1} &\le C \int_{B(x',r)} |\gamma_\delta^- v(y', f_\delta(y'))|\, dy' \\
&\le C\left(\int_{B(x',r)} \int_{f_\delta(y')}^{f(y')} \left|\frac{\partial v}{\partial x_n}(y', s)\right| ds dy' + \int_{B(x',r)} |\gamma v(y', f(y'))|\, dy'\right) \\
&\le C\left(\int_{(\Omega\backslash\bar{\Omega}_\delta)\cap C(x,r,h)} |\nabla v(y)|\, dy + \int_{B(x',r)} |\gamma v(y', f(y'))| \sqrt{1 + |\nabla f(y')|^2}\, dy'\right) \\
&\le C\left(|Dv|((\Omega\backslash\bar{\Omega}_\delta) \cap C(x, r, h)) + \int_{\partial\Omega\cap C(x,r,h)} |\gamma v(y)|\, d\mathscr{H}^{n-1}\right).
\end{aligned}$$

Thus (A.10) holds for $v \in BV(\Omega) \cap C^\infty(\Omega\backslash\bar{\Omega}_\delta)$.

Now we assume only $v \in BV(\Omega)$. For each fixed δ with $0 < \delta < \delta_0$ there exists a sequence $\{v_k\} \subset BV(\Omega) \cap C^\infty(\Omega\backslash\bar{\Omega}_\delta)$ such that $v_k \to v$ in $L^1(\Omega\backslash\bar{\Omega}_\delta)$ and $|Dv_k|(\Omega\backslash\bar{\Omega}_\delta) \to |Dv|(\Omega\backslash\bar{\Omega}_\delta)$ ([**5**] Theorem 1.17 or [**2**] Theorem 2 of Section 5.2). Furthermore we have $\gamma v_k \to \gamma v$ in $L^1(\partial\Omega)$ and $\gamma_\delta^- v_k \to \gamma_\delta^- v$ in $L^1(\partial\Omega_\delta)$ (see the proof of [**2**] Theorem 1 of Section 5.3). Since (A.10) holds for each v_k, our passing to the limit as $k \to \infty$ yields the conclusion. $\square$

LEMMA A.5. *Let T be a positive number and v be a function on $(0,T)\times\Omega$. Suppose that $v \in L^\infty((0,T); L^2(\Omega)\cap BV(\Omega))$, $v_t \in L^2((0,T)\times\Omega)$, and $\gamma v = 0$ for $\mathscr{L}^1$-a.e. $t \in (0,T)$. Then there exists a sequence $\{v_j\}_{j=1}^\infty \subset C_0^1([0,T]\times\Omega)$ such that, as $j\to\infty$, $v_j \to v$, $(v_j)_t \to v_t$ strongly in $L^2((0,T)\times\Omega)$, and $J(v_j)\to J(v)$ strongly in $L^1(0,T)$.*

PROOF. *1st step.* For the time we suppose that the support of v is contained in a compact subset of $[0,T]\times\Omega$.

Let $\eta^{(t)}$ and $\eta^{(x)}$ be positive symmetric mollifiers ([**5**] 1.14) with respect to t and x variables, respectively, and put $\eta(t,x) = \eta^{(t)}(t)\eta^{(x)}(x)$. Now we define

$$v_\sigma = \eta_\sigma * v.$$

Then $v_\sigma \in C_0^1([0,T]\times\Omega)$ if σ is sufficiently small. Further it holds that, as $\sigma\to 0$, $v_\sigma \to v$ and $(v_\sigma)_t \to v_t$ strongly in $L^2((0,T)\times\Omega)$.

Let $(g_0, g) \in C_0^1(\Omega; \boldsymbol{R}^{n+1})$ be a vector valued function with $g_0^2 + |g|^2 \le 1$. Since $v(t,\cdot)\in BV(\Omega)$ for $\mathscr{L}^1$-a.e. t, we have

$$\begin{aligned}\int_\Omega (g_0 + v_\sigma \operatorname{div} g)\,dx &= \int_\Omega [g_0 + \eta_\sigma * v \operatorname{div} g]\,dx = \int_{-\infty}^{\infty} \eta_\sigma^{(t)}(t-s)\int_\Omega [g_0 + \eta_\sigma^{(x)} * v \operatorname{div} g]\,dxds \\ &= \int_{-\infty}^{\infty} \eta_\sigma^{(t)}(t-s)\int_\Omega [g_0 + v\operatorname{div}(\eta_\sigma^{(x)} * g)]\,dxds \le \eta_\sigma^{(t)} * J(v).\end{aligned}$$

This means

$$J(v_\sigma) \le \eta_\sigma^{(t)} * J(v) \tag{A.16}$$

for $t\in(0,T)$. On the other hand, since $v_\sigma \to v$ strongly in $L^2(\Omega)$ for $\mathscr{L}^1$-a.e. $t\in(0,T)$, we have $\liminf_{\sigma\to 0} J(v_\sigma) \ge J(v)$. By Fatou's lemma

$$\liminf_{\sigma\to 0}\int_0^T J(v_\sigma)\,dt \ge \int_0^T \liminf_{\sigma\to 0} J(v_\sigma)\,dt \ge \int_0^T J(v)\,dt. \tag{A.17}$$

It follows from (A.16) and (A.17) that

$$\begin{aligned}\limsup_{\sigma\to 0}\int_0^T |J(v) - J(v_\sigma)|\,dt &\le \limsup_{\sigma\to 0}\int_0^T |\eta_\sigma^{(t)} * J(v) - J(v_\sigma)|\,dt \\ &= \limsup_{\sigma\to 0}\int_0^T (\eta_\sigma^{(t)} * J(v) - J(v_\sigma))\,dt \\ &= \int_0^T J(v)\,dt - \liminf_{\sigma\to 0}\int_0^T J(v_\sigma)\,dt \le 0.\end{aligned}$$

Thus $J(v_\sigma)\to J(v)$ strongly in $L^1(0,T)$ as $\sigma\to 0$.

2nd step. Now we do not assume the compactness of the support of v. Let Ω_δ be as in (A.9) and put $w_\delta = \chi_\delta v$, where χ_δ denotes the characteristic function of Ω_δ. It is clear that, as $\delta\to 0$, $w_\delta \to v$ and $(w_\delta)_t \to v_t$ strongly in $L^2((0,T)\times\Omega)$. Since

$$J(w_\delta) = \int_{\Omega_\delta}\sqrt{1+|Dv|^2} + \mathscr{L}^n(\Omega\backslash\bar{\Omega}_\delta) + \int_{\partial\Omega_\delta} |\gamma_\delta^+ v|\,d\mathscr{H}^{n-1}$$

and

$$\mathscr{L}^n(\Omega\backslash\bar{\Omega}_\delta) \le \int_{\Omega\backslash\Omega_\delta} \sqrt{1+|Dv|^2} \le \mathscr{L}^n(\Omega\backslash\bar{\Omega}_\delta) + |Dv|(\Omega\backslash\bar{\Omega}_\delta),$$

we have

$$\text{(A.18)} \qquad -|Dv|(\Omega\backslash\bar{\Omega}_\delta) + \int_{\partial\Omega_\delta} |\gamma_\delta^+ v| \, d\mathscr{H}^{n-1} - |Dv|(\partial\Omega_\delta)$$
$$\le J(w_\delta) - J(v) \le \int_{\partial\Omega_\delta} |\gamma_\delta^+ v| \, d\mathscr{H}^{n-1} - |Dv|(\partial\Omega_\delta).$$

It is easy to find that $|Dv|(\partial\Omega_\delta) = \int_{\partial\Omega_\delta} |\gamma_\delta^+ v(x) - \gamma_\delta^- v(x)| \, d\mathscr{H}^{n-1}$. Thus, since $\gamma v = 0$ on $\partial\Omega$, it follows from Lemma A.4 that, for any $\delta < \delta_0$ and for $\mathscr{L}^1$-a.e. $t \in (0,T)$,

$$\left| \int_{\partial\Omega_\delta} |\gamma_\delta^+ v| \, d\mathscr{H}^{n-1} - |Dv|(\partial\Omega_\delta) \right| \le \int_{\partial\Omega_\delta} |\gamma_\delta^- v| \, d\mathscr{H}^{n-1} \le C(\|v\|_{L^1(\Omega\backslash\bar{\Omega}_\delta)} + |Dv|(\Omega\backslash\bar{\Omega}_\delta)),$$

where δ_0 and C are as in the statement of Lemma A.4. This and (A.18) imply

$$\int_0^T |J(w_\delta) - J(v)| \, dt \le C\|v\|_{L^1((0,T)\times(\Omega\backslash\bar{\Omega}_\delta))} + (C+1)\int_0^T |Dv|(\Omega\backslash\bar{\Omega}_\delta) \, dt.$$

Hence we obtain $J(w_\delta) \to J(v)$ strongly in $L^1(0,T)$ as $\delta \to 0$.

3rd step. Since $\operatorname{spt} w_\delta$ is a compact subset of $[0,T] \times \Omega$, we have by the result of the 1st step that, as $\sigma \to 0$, $\eta_\sigma * w_\delta \to w_\delta$, $(\eta_\sigma * w_\delta)_t \to (w_\delta)_t$ strongly in $L^2((0,T) \times \Omega)$, and $J(\eta_\sigma * w_\delta) \to J(w_\delta)$ strongly in $L^1(0,T)$. Then, combining the result of the 2nd step, we can select a subsequence $\{\sigma_j, \delta_j\}$ by the use of the diagonal argument such that $v_j \equiv \eta_{\sigma_j} * w_{\delta_j}$ satisfies, as $j \to \infty$, $v_j \to v$, $(v_j)_t \to v_t$ strongly in $L^2((0,T) \times \Omega)$, and $J(v_j) \to J(v)$ strongly in $L^1(0,T)$. □

Proof of 'only if' part of Theorem A.1. Suppose that $u \in L^\infty((0,T); L^2(\Omega) \cap BV(\Omega))$ is a BV solution to (1.1)–(1.3) in $(0,T) \times \Omega$. Let $\phi \in C_0^1([0,T); L^2(\Omega)) \cap L^\infty((0,T); BV(\Omega))$ with $\gamma\phi = 0$ for $\mathscr{L}^1$-a.e. $t \in (0,T)$. Put $T_\phi = \sup\{t; \|\phi(t,\cdot)\|_{L^2(\Omega)} \neq 0\}$. In the inequality of iv)', T can be replaced with any of $[T_\phi, T]$. We choose $T' \in [T_\phi, T]$ such that the $L^2(\Omega)$ valued function $u_t : (0,T) \to L^2(\Omega)$ is approximately continuous at T'. Such points exist $\mathscr{L}^1$ almost everywhere in $[0,T]$. Then, u satisfies all assumptions of Lemmas A.2 and A.3 in $(0,T') \times \Omega$.

Since T' and $u + \phi$ satisfy the all assumptions for T and v in Lemma A.5, there is a sequence $\{v_j\}$ in $C_0^1([0,T'] \times \Omega)$ such that, as $j \to \infty$, $v_j \to u + \phi$, $(v_j)_t \to (u+\phi)_t$ strongly in $L^2((0,T') \times \Omega)$, and $J(v_j) \to J(u+\phi)$ strongly in $L^1(0,T')$. Now we put

$$\text{(A.19)} \qquad \varphi_j(t,x,y) = -y + v_j(t,x).$$

Then φ_j satisfies the assumption of Lemma A.3 in $[0,T'] \times \Omega \times \boldsymbol{R}$. Thus we have

$$\int_0^{T'} \left\{ -\int_\Omega u_t((\varphi_j)_t(t,x,u) + (\varphi_j)_y(t,x,u)u_t) \, dx + \frac{d}{d\varepsilon} J(u + \varepsilon\varphi_j(t,x,u))|_{\varepsilon=0} \right\} dt$$
$$= \int_\Omega v_0(x)\varphi_j(0,x,u_0(x)) \, dx - \int_\Omega u_t(T',x)\varphi_j(T',x,u(T',x)) \, dx.$$

On the other hand, since J is convex,

$$J(u+\varphi_j(t,x,u)) - J(u) \geq \frac{d}{d\varepsilon} J(u+\varepsilon\varphi_j(t,x,u))|_{\varepsilon=0}.$$

Hence

$$\text{(A.20)} \quad \int_0^{T'} \{J(u+\varphi_j(t,x,u)) - J(u)\}\, dt$$
$$\geq \int_0^{T'} \int_\Omega u_t((\varphi_j)_t(t,x,u) + (\varphi_j)_y(t,x,u)u_t)\, dxdt$$
$$+ \int_\Omega v_0(x)\varphi_j(0,x,u_0(x))\, dx - \int_\Omega u_t(T',x)\varphi_j(T',x,u(T',x))\, dx.$$

By (A.19) and Lemma A.5 we have, as $j \to \infty$,

$$\text{(A.21)} \quad \int_0^{T'} J(u+\varphi_j(t,x,u))\, dt = \int_0^{T'} J(v_j)\, dt \to \int_0^{T'} J(u+\phi)\, dt.$$

Lemma A.5 also implies

$$\text{(A.22)} \quad (\varphi_j)_t(t,x,u) + (\varphi_j)_y(t,x,u)u_t (= [\varphi_j(t,x,u)]_t)$$
$$= -u_t + (v_j)_t \to -u_t + (u+\phi)_t = \phi_t \quad \text{in } L^2((0,T')\times\Omega).$$

Further, integrating over $[0,t]$ for each $t \in [0,T']$, we have

$$\int_0^t [\varphi_j(s,x,u)]_t\, ds \to \int_0^t \phi_t(s,x)\, ds = \phi(t,x) - \phi(0,x) \quad \text{in } L^2(\Omega),$$

while

$$\int_0^t [\varphi_j(s,x,u)]_t\, ds = \varphi_j(t,x,u(t,x)) - \varphi_j(0,x,u_0(x)) = -u(t,x) + v_j(t,x) - \varphi_j(0,x,u_0(x)).$$

Since $v_j \to u+\phi$ in $L^2((0,T')\times\Omega)$, we have

$$\text{(A.23)} \quad \varphi_j(0,x,u_0(x)) \to \phi(0,x) \quad \text{in } L^2(\Omega).$$

In the same way we see that

$$\text{(A.24)} \quad \varphi_j(T',x,u(T',x)) \to \phi(T',x) = 0 \quad \text{in } L^2(\Omega).$$

By (A.20), (A.21), (A.22), (A.23), and (A.24) we obtain iv)'. □

References

[1] L. C. Evans, Weak convergence Methods for Nonlinear Partial Differential Equations, CBMS, Amer. Math. Soc. No. 74, 1990.

[2] L. C. Evans and R. F. Gariepy, Measure Theory and Fine Properties of Functions, CRC Press, 1992.

[3] H. Federer, Geometric Measure Theory, Springer-Verlag, New York, 1969.

[4] D. Fujiwara and S. Takakuwa, A varifold solution to the nonlinear equation of motion of a vibrating membrane, Kodai Math. J., **9** (1986), 84–116; correction, ibid., **14** (1991), 310–311.
[5] E. Giusti, Minimal Surfaces and Functions of Bounded Variation, Birkhäuser, Boston-Basel-Stuttgart, 1984.
[6] A. Hoshiga, The asymptotic behaviour of the radially symmetric solutions to quasilinear wave equations in two space dimensions, Hokkaido Math. J., **24** (1995), 575–615.
[7] K. Kikuchi, Constructing weak solutions in a direct variational method and an application of varifold theory, J. Differential Equations, **150** (1998), 1–23.
[8] K. Maruo, Existence of solutions to some nonlinear wave equations, Osaka J. Math., **22** (1985), 21–30.
[9] T. Nagasawa, Discrete Morse semiflows and evolution equations (in Japanese), Proceedings of the 16th Young Japanese Mathematicians' Seminar on Evolution Equations, 1994, 1–20.
[10] T. Nagasawa and A. Tachikawa, Existence and asymptotic behavior of weak solutions to semilinear hyperbolic systems with damping term, Tsukuba J. Math., **20** (1996), 51–64.
[11] E. Rothe, Zweidimensionale parabolische Randwertaufgaben als Grenzfall eindimensionaler Randwertaufgaben, Math. Ann., **102** (1930), 650–670.
[12] L. Simon, Lectures on Geometric Measure Theory, Proceedings of the Centre for Mathematical Analysis, Australian National University, Canberra Vol. 3, 1983.

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