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# A note on a conjecture of Xiao

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**Abstract.** We prove that the image of the relative dualizing sheaf of a fibration from a smooth projective surface onto a smooth projective curve is ample under some extra conditions.

When $f : S \to B$ is a surjective morphism of a complex, smooth surface $S$ onto a complex, smooth, genus $b$ curve $B$, such that the fibre $F$ of $f$ has genus $g$, it is well known that $f_*\omega_{S/B} = \mathscr{E}$ is a locally free sheaf of rank $g$ and degree $d = \mathscr{X}\mathcal{O}_S - (b-1)(g-1)$ and that $f$ is not an holomorphic fibre bundle if and only if $d > 0$. In this case the *slope*, $\lambda(f) = \{K_S^2 - 8(b-1)(g-1)\}/d$, is a natural invariant associated by Xiao to $f$ (cf. [**7**]). In [**7**, Conjecture 2] he conjectured that $\mathscr{E}$ has no locally free quotient of degree zero (i.e., $\mathscr{E}$ is ample) if $\lambda(f) < 4$. We give a partial affirmative answer to this conjecture:

Theorem 1. *Let $f : S \to B$ be a relatively minimal fibration with general fibre $F$. Let $b = g(B)$ and assume that $g = g(F) \geq 2$ and that $f$ is not locally trivial.*

*If $\lambda(f) < 4$ then $\mathscr{E} = f_*\omega_{S/B}$ is ample provided one of the following conditions hold.*

(i) *$F$ is non hyperelliptic.*

(ii) $b \leq 1$.

(iii) $g(F) \leq 3$.

Proof. (i) If $q(S) > b$ the result follows from [**7**, Corollary 2.1]. Now assume $q(S) = b$. By Fujita's decomposition theorem (see [**3**], [**4**] and also [**5**] for a proof)

$$\mathscr{E} = \mathscr{A} \oplus \mathscr{F}_1 \oplus \cdots \oplus \mathscr{F}_r$$

where $h^0(B, (\mathscr{A} \oplus \mathscr{F}_1 \oplus \cdots \oplus \mathscr{F}_r)^*) = 0$, $\mathscr{A}$ is an ample sheaf and $\mathscr{F}_i$ are non trivial stable degree zero sheaves. Then we only must prove that $\mathscr{F}_i = 0$. If $F$ is not hyperelliptic and rank $(\mathscr{F}_i) \geq 2$ the claim is the content of [**7**, Proposition 3.1]. If

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$\mathrm{rank}(\mathscr{F}_i) = 1$ we can use [**2**, §4.2] or [**1**, Theorem 3.4] to conclude that $\mathscr{F}_i$ is torsion in $\mathrm{Pic}^0(B)$. Hence it induces an étale base change:

$$\begin{tikzcd} \tilde{S} \arrow[r] \arrow[d, "\tilde{f}"] & S \arrow[d, "f"] \\ \tilde{B} \arrow[r, "\sigma"] & B \end{tikzcd}$$

By flatness $\tilde{f}_*\omega_{\tilde{S}/\tilde{B}} = \sigma^*(f_*\omega_{S/B})$. Since $\sigma$ is étale $\lambda(f) = \lambda(\tilde{f})$ and $\sigma^*(\mathscr{F}_i) = \mathscr{O}_{\tilde{B}}$ is a direct summand of $\tilde{f}_*\omega_{\tilde{S}/\tilde{B}}$. In particular by [**3**] $q(\tilde{S}) > \tilde{b} = g(\tilde{B})$ hence $\lambda(\tilde{f}) \geq 4$ by [**7**, Theorem 3.3]: a contradiction.

(ii) If $b = 0$ the claim is trivial. If $b = 1$, any stable degree zero sheaf has rank one, then as in (i) we conclude.

(iii) If $g = 2$ and $\mathscr{E} \neq \mathscr{A}$, then $\mathscr{E} = \mathscr{A} \oplus \mathscr{L}$ where $\mathscr{L}$ torsion and we are done. The only non trivial case if $g = 3$ is $\mathscr{E} = \mathscr{A} \oplus \mathscr{F}$ where $\mathscr{A}$ an ample line bundle and $\mathscr{F}$ a stable, degree zero, rank two vector bundle. Then $K^2_{S/B} \geq (2g-2)\deg\mathscr{A} = 4d$ and we are done by [**7**, Theorem 2]. □

Theorem 3.3 of [**7**] Xiao says that if $q(S) > b$ and $\lambda(f) = 4$ then $\mathscr{E} = \mathscr{F} \oplus \mathscr{O}_B$, where $\mathscr{F}$ is a semistable sheaf. We have the following improvement:

Theorem 2. *Let $f: S \to B$ be a relatively minimal non locally trivial fibration. If $\lambda(f) = 4$ then $\mathscr{E} = f_*\omega_{S/B}$ has at most one degree zero, rank one quotient $\mathscr{L}$.*

*Moreover, in this case $\mathscr{E} = \mathscr{A} \oplus \mathscr{L}$ with $\mathscr{A}$ semistable and $\mathscr{L}$ torsion.*

Proof. As in the previous theorem the torsion subsheaf $\mathscr{L}$ becomes the trivial one after an étale base change; thus

$$\tilde{f}_*\omega_{\tilde{S}/\tilde{B}} = \tilde{\mathscr{A}} \oplus \mathscr{O}_{\tilde{B}}, \quad \tilde{\mathscr{A}} = \sigma^*\mathscr{A}.$$

By [**7**, Theorem 3.3], $\tilde{\mathscr{A}}$ is semistable. Then $\mathscr{A}$ is also semistable by [**6**, Proposition 3.2]. □

## References


[1] M.A. Barja, On the slope and geography of fibred surfaces and threefolds, Ph.D. Thesis. 1998.

[2] P. Deligne, Théorie de Hodge II, Publ. Math. I.H.E.S., **40** (1971), 5–58.

[3] T. Fujita, On Kahler fibre spaces over curves, J. Math. Soc. Japan, **30** (1978), 779–794.

[4] T. Fujita, The sheaf of relative canonical forms of a Kahler fiber space over a curve, Proc. Japan. Acad., **54**, Ser. A (1978), 183–184.

[5] J. Kollár, Subadditivity of the Kodaira dimension: fibers of general type, Adv. studies in Pure Math., **10** (1987), Algebraic Geometry, Sendai 1985, 361–398.

[6] Y. Miyaoka, The Chern classes and Kodaira dimension of a minimal variety. Algebraic Geometry, Sendai (1985); Adv. Study in Pure Math., **10** (1987), 449–476.
[7] G. Xiao, Fibered algebraic surfaces with low slope, Math. Ann., **276** (1987), 449–466.

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