

Identifying tunnel number one knots

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(Received July 29, 1994)

(Revised Oct. 19, 1994)

Let K be a knot in S^3 . The tunnel number $t(K)$ of K is the minimal number of mutually disjoint arcs $\{\tau_i\}$ “properly embedded” in the pair (S^3, K) such that the complement of an open regular neighbourhood of $K \cup (\cup \tau_i)$ is a handlebody. In the above, if the arc system consists of only one arc, it is called an *unknotting tunnel* for K . K is said to have a (g, b) -decomposition if there is a genus g Heegaard splitting $\{W_1, W_2\}$ of S^3 such that K intersects W_i ($i=1, 2$) in a b -string trivial arc system (cf. [D, MS]). If a knot K has a (g, b) -decomposition, then $t(K) \leq g+b-1$. In particular; if K admits a $(1, 1)$ -decomposition then it has tunnel number one; however, it is shown by [MR, MSY, Yo1] that the converse does not hold.

Kohno [Kh] gave an estimate of tunnel numbers of knots in terms of the quantum invariants (cf. [Wk, G]), and the third author [Yo1] gave a condition for a knot to admit a (g, b) -decomposition in terms of the quantum $SU(2)$ invariants. Kouzi Kodama [Kd] applied Kohno’s estimate to prime knots up to 10 crossings by using his computer program “Knot”, and determined the tunnel numbers of several such knots.

In this paper, we give another method to determine whether a given knot K has tunnel number one and whether it admits a $(1, 1)$ -decomposition, by using the idea due to Birman-Hilden [BH] and Viro [V] (cf. [BGM], [BM], [BoZe]). The method enables us to determine the tunnel numbers of prime knots up to 10 crossings (Theorem 2.5), and is potentially useful to the problem of detecting tunnel number one knots which do not admit $(1, 1)$ -decompositions. The idea is to look at the canonical 2-fold symmetry arising from an unknotting tunnel and to reduce the problem to that concerning symmetries of knots and that concerning spatial θ -curves (Theorem 1.2). Study of symmetries of knots has long history, and we now have enough information concerning symmetries of various kinds of knots, including the Montesinos knots and the prime knots up to 10 crossings (see [AHW, BoZm, HW, KS]). On the other hand, there is a naive but convenient method for the study of the problem concerning spatial θ -curves (Corollary 1.3). By using this method, we obtain a certain condition for a

Montesinos knot to have tunnel number one (Theorem 2.2), and determine the tunnel numbers of prime knots up to 10 crossings. However, for the θ -curves arising from certain knots, the above naive method does not work. In Section 3, we give another method for the problem concerning spatial θ -curves by means of special values of Yamada's invariants of spatial graphs (Theorem 3.2). The method depends on an idea similar to that due to Walker [Wk], Garoufalidis [G], Kohno [Ko] and the third author [Yo1]. By using the method, we show that the condition given by Theorem 2.2 is not complete (Example 3.8). In the final section, we discuss relation of our results with the reflection groups.

1. θ -curves associated with unknotting tunnels.

A knot K in S^3 is said to be *strongly invertible*, if there is an involution h of the pair (S^3, K) such that $\text{Fix}(h)$ is a circle intersecting K in two points. We call h a *strong inversion* of K . Let p be the projection $S^3 \rightarrow S^3/h$, and put $O = p(\text{Fix}(h))$, $\gamma = p(K)$, and $G(K, h) = O \cup \gamma$. Then, by [Wd], S^3/h is again a 3-sphere, O is a trivial knot, and γ is an arc such that $\gamma \cap O = \partial\gamma$. We call $G(K, h)$ the θ -curve associated with h .

DEFINITION 1.1. (1) $G(K, h)$ is said to have a *3-bridge decomposition*, if (S^3, O, γ) is a union of (B_1^3, t_1, γ) and (B_2^3, t_2, ϕ) along their boundaries, where (B_i^3, t_i) is a 3-strand trivial tangle for $i=1, 2$, and γ is a "trivial" arc in (B_1^3, t_1) as illustrated in Figure 1.1(a).

(2) $G(K, h)$ is said to have a *2-bridge decomposition*, if (S^3, O, γ) is a union of (B_1^3, t_1, γ_1) and (B_2^3, t_2, γ_2) along their boundaries, where (B_i^3, t_i) is a 2-strand trivial tangle and γ_i is a "trivial" arc in (B_i^3, t_i) as illustrated in Figure 1.1(b) for $i=1, 2$.

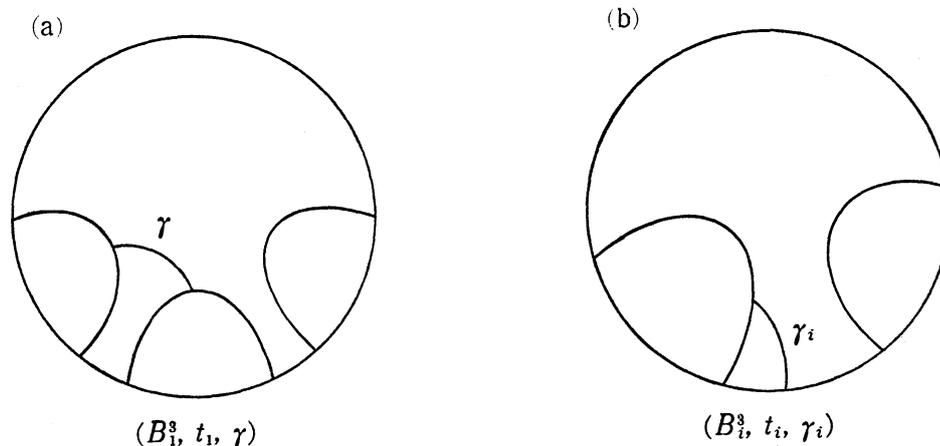


Figure 1.1.

By using the argument of [BH, V], we obtain the following result.

THEOREM 1.2. (1) *A knot K in S^3 has tunnel number one, if and only if K admits a strong inversion h such that $G(K, h)$ has a 3-bridge decomposition.*

(2) *A knot K in S^3 admits a (1, 1)-decomposition, if and only if K admits a strong inversion h such that $G(K, h)$ has a 2-bridge decomposition.*

PROOF. We prove (1). Suppose K admits an unknotting tunnel, say τ . Put $V_1=N(K\cup\tau)$ and $V_2=cl(S^3-V_1)$, where $N(K\cup\tau)$ is a regular neighbourhood of $K\cup\tau$ in S^3 . Then $\{V_1, V_2\}$ determines a genus 2 Heegaard splitting of S^3 . Let h_1 be the involution of the triple (V_1, K, τ) as illustrated in Figure 1.2(a). Then as described in [BGM, BH, BM], the restriction of h_1 to $\partial V_1=\partial V_2$ extends to an involution, say h_2 , of V_2 as illustrated in Figure 1.2(b). Let h be the involution of S^3 determined by h_1 and h_2 . Then h is a strong inversion of K , and its quotient $(S^3, O, \gamma)=(S^3, \text{Fix}(h), K)/h$ is a union of the quotients $(V_1, \text{Fix}(h_1), K)/h_1$ and $(V_2, \text{Fix}(h_2), \phi)/h_2$, which are illustrated in Figures 1.2(c) and (d). This proves the only if part of (1). The if part of (1) can be proved by tracing backward the above argument, and (2) can be proved by a similar argument.

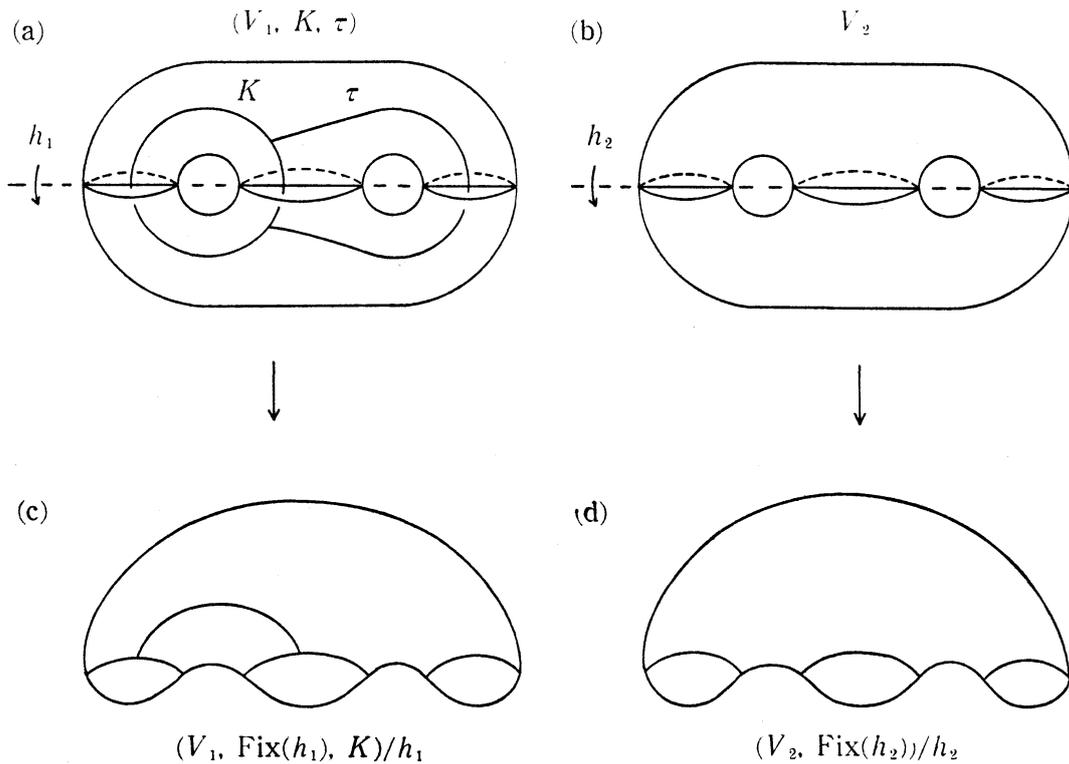


Figure 1.2.

COROLLARY 1.3. *Let K be a knot with tunnel number one. Then K admits a strong inversion h such that the set of the constituent knots of $G(K, h)$ consists of two trivial knots and a knot with a 2-bridge decomposition.*

2. Montesinos knots and prime knots up to 10 crossings.

A Montesinos link $K=M(b; (\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r))$ with r branches is a link in S^3 as illustrated in Figure 2.1(a).

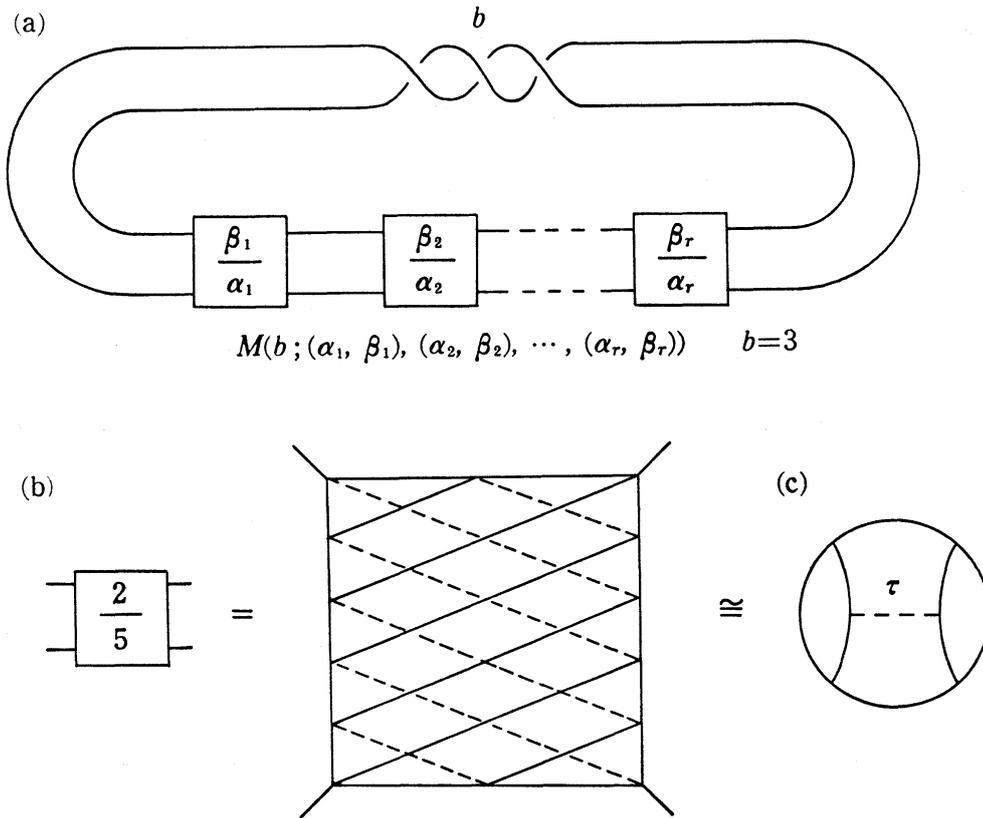


Figure 2.1.

Here r, b, α_i and β_i are integers such that $r \geq 0, \alpha_i \geq 2$, and $g.c.d.(\alpha_i, \beta_i) = 1$. A box $\frac{\beta/\alpha}$ stands for a rational tangle of slope β/α (see Fig. 2.1(b)). If we forget the chart on the boundary, a rational tangle is merely a 2-strand trivial tangle as illustrated in Figure 2.1(c); we call the image of the arc τ in Figure 2.1(c) in a rational tangle the *core* of the rational tangle. The following proposition is well-known (see [Z], [BuZ, Chapter 12]):

PROPOSITION 2.1. (1) Suppose $r=2$. Then K is a 2-bridge link $S(p, q)$ of type (p, q) , where $p = |b\alpha_1\alpha_2 - \alpha_1\beta_2 - \alpha_2\beta_1|$ and q is an integer relatively prime to p . In particular, K is a trivial knot, if and only if $b\alpha_1\alpha_2 - \alpha_1\beta_2 - \alpha_2\beta_1 = \pm 1$.

(2) Suppose $r \geq 3$. Then K is not a 2-bridge link, and it is classified by the Euler number

$$e(K) = b - \sum_{i=1}^r \beta_i/\alpha_i,$$

and the vector

$$v(K) = (\beta_1/\alpha_1, \dots, \beta_r/\alpha_r) \in (\mathbf{Q}/\mathbf{Z})^r$$

up to cyclic permutation and reversal of the order.

In this section, we prove the following theorem:

THEOREM 2.2. *Let $K=M(b; (\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r))$ be a Montesinos knot (not a link) with r branches. Suppose K has tunnel number one, then one of the following conditions holds up to cyclic permutation of the indices:*

- (1) $r = 2$.
- (2) $r = 3$, $\alpha_1 = 2$, and $\alpha_2 \equiv \alpha_3 \equiv 1 \pmod{2}$.
- (3) $r = 3$, $\beta_2/\alpha_2 \equiv \beta_3/\alpha_3 \in \mathbf{Q}/\mathbf{Z}$, and $e(K) = \pm 1/(\alpha_1\alpha_2)$.

Conversely, if the condition (1), (2), or the following (3') holds, then K has tunnel number one:

- (3') $r = 3$, $\beta_2/\alpha_2 \equiv \beta_3/\alpha_3 \equiv \pm 1/3 \in \mathbf{Q}/\mathbf{Z}$, and $e(K) = \pm 1/(3\alpha_1)$.

First, we prove the second part of Theorem 2.2. If the condition (1) is satisfied, then K is a 2-bridge knot, and therefore it has tunnel number one. Suppose the condition (2) is satisfied. For $i=2, 3$, let τ_i be the core of the rational tangle $[\beta_i/\alpha_i]$. Then we see each τ_i is an unknotting tunnel for K as illustrated in Figure 2.2. (A core of the genus 2 handlebody $cl(S^3 - N(K \cup \tau_i))$ is as illustrated in Figure 2.2(a). From this figure, we see that τ_1 and τ_2 are (1, 1)-tunnels and are dual to each other (see [MS, 1.1-1.3]).) Suppose the condition (3') is satisfied. Then, up to reflection, K is equivalent to $M(0; (3n+2, -2n-1), (3, 1), (3, 1))$ for some integer n . Let τ and τ' be arcs as illustrated in Figure 2.3(a). Then we see each of them is an unknotting tunnel as illustrated in Figure 2.3. (A core of $cl(S^3 - N(K \cup \tau))$ is as illustrated in Figure 2.3(a). From this figure, we see that τ and τ' are (1, 1)-tunnels and are dual to each other.)

In the following we prove the first half of Theorem 2.2. If K is an elliptic Montesinos knot, i.e., $2 - \sum_{i=1}^r (1 - 1/\alpha_i) \geq 0$, then it satisfies either (1) or (2), and it admits an unknotting tunnel. So, in the following, we may assume K is non-elliptic. Let $\text{Sym}(S^3, K)$ be the symmetry group of K , i.e., the group of all pairwise isotopy classes of diffeomorphisms of the pair (S^3, K) . Let $\text{Sym}_+(S^3, K)$ be the subgroup of $\text{Sym}(S^3, K)$ generated by diffeomorphisms which preserve the orientation of S^3 . The following result is proved by Boileau-Zimmermann [BoZm].

PROPOSITION 2.3. *There is an exact sequence*

$$1 \longrightarrow \mathbf{Z}_2 \longrightarrow \text{Sym}_+(S^3, K) \xrightarrow{\psi} D_+(v(K)) \longrightarrow 1,$$

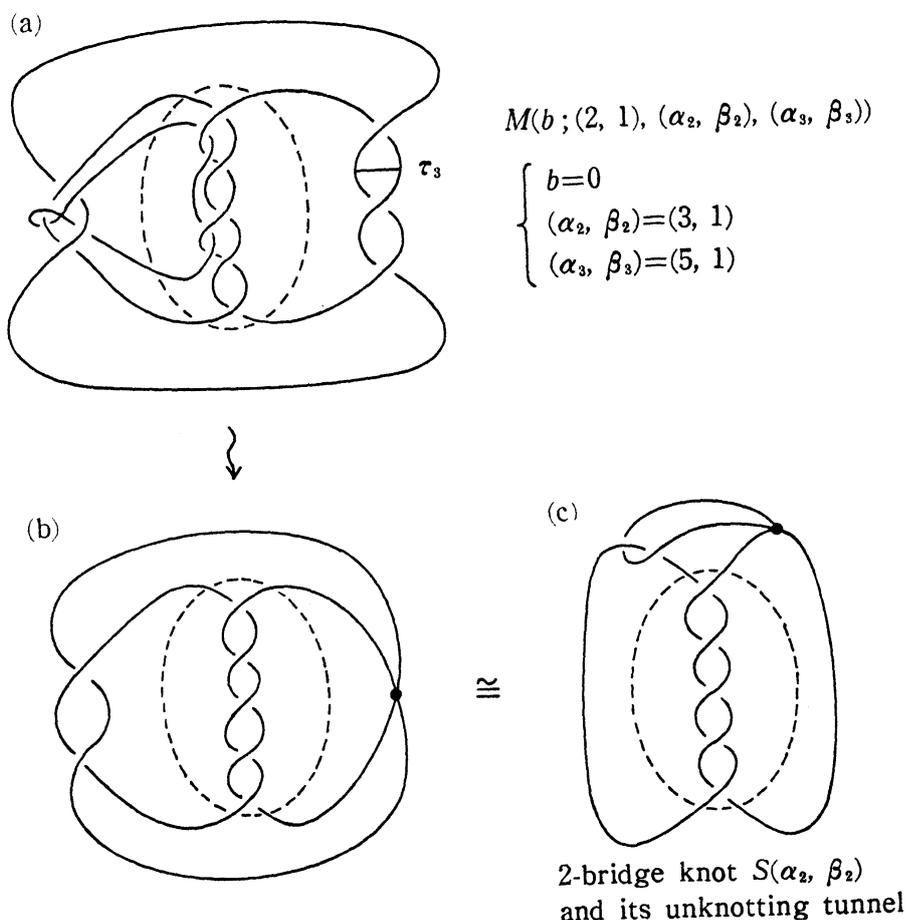


Figure 2.2.

where $D_+(\mathbf{v}(K))$ is the group of those dihedral permutations of the vector $\mathbf{v}(K) = (\beta_1/\alpha_1, \dots, \beta_r/\alpha_r) \in (\mathbf{Q}/\mathbf{Z})^r$ which preserve $\mathbf{v}(K)$.

Let (B, t) be a rational tangle of slope β/α , and let f [resp. g] be the π -rotation of (B, t) about the horizontal [resp. vertical] axis (see Figures 2.4 and 2.5).

LEMMA 2.4. (1) Suppose α [resp. β] is odd. Then f [resp. g] interchanges the two components of t , and the quotient pair $(B, t \cup \text{Fix}(f))/f$ [resp. $(B, t \cup \text{Fix}(g))/g$] is naturally identified with a rational tangle of slope $2\beta/\alpha$ [resp. $\beta/2\alpha$].

(2) Suppose α [resp. β] is even. Then f [resp. g] preserves the two components of t , and $t \cap \text{Fix}(f)$ [resp. $t \cap \text{Fix}(g)$] consists of two points, say v_1 and v_2 . Put $s = cl(\text{Fix}(f) - \overline{v_1 v_2})$ [resp. $s = cl(\text{Fix}(g) - \overline{v_1 v_2})$], where $\overline{v_1 v_2}$ is the subarc of $\text{Fix}(f)$ [resp. $\text{Fix}(g)$] bounded by v_1 and v_2 . Then $(B, t \cup s)/f$ [resp. $(B, t \cup s)/g$] is naturally identified with a rational tangle of slope $\beta/(\alpha/2)$ [resp. $(\beta/2)/\alpha$]. The image of $\text{Fix}(f)$ [resp. $\text{Fix}(g)$] in $(B, t \cup s)/f$ [resp. $(B, t \cup s)/g$] is parallel to an arc of slope 0.

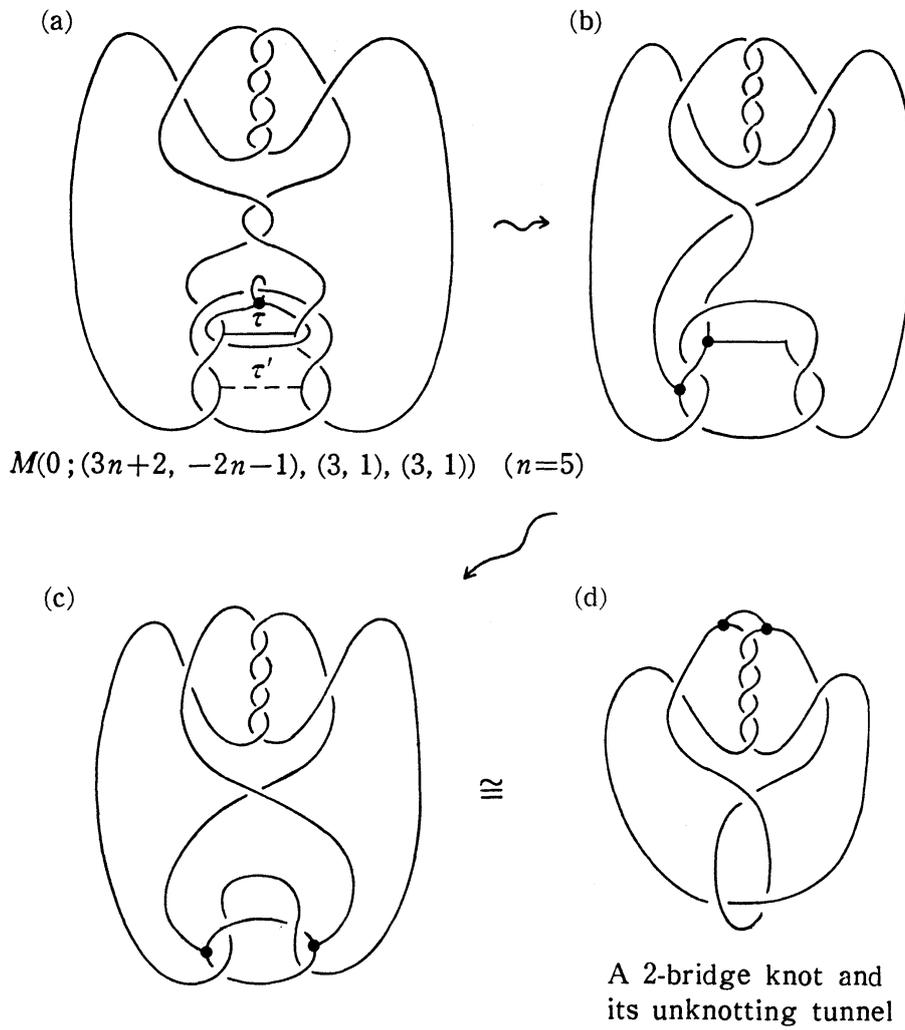


Figure 2.3.

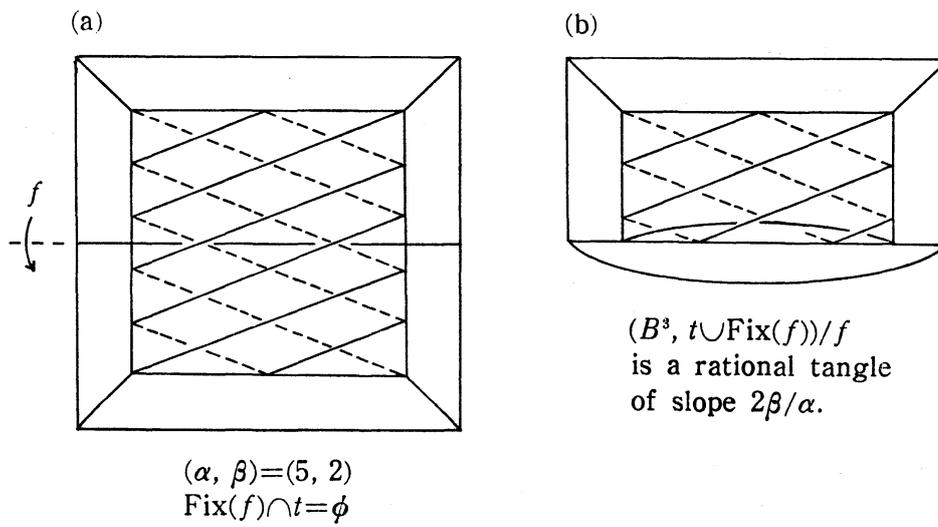


Figure 2.4.

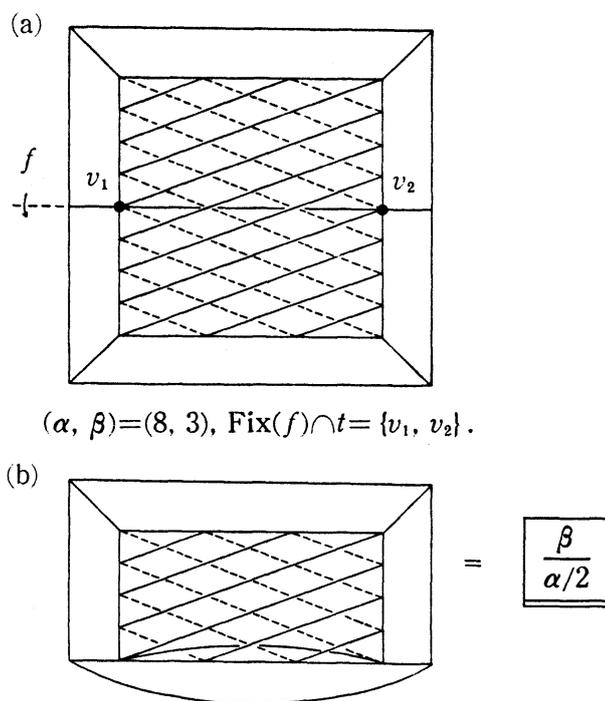


Figure 2.5.

PROOF. Clear from Figures 2.4 and 2.5.

In case the condition (2) in the above lemma holds, we denote the quotient pair $(B, t \cup s)/f$ by the symbol $\boxed{\frac{\beta}{\alpha/2}}$. By [BS, Proposition 9.19] (cf. [BoZm, Proposition 2.1]), the exterior of the non-elliptic Montesinos knot (not a link) K is not Seifert fibred. Thus, by using [T], we see the equivalence class of a strong inversion h is uniquely determined by the order 2 element $[h]$ of $\text{Sym}_+(S^3, K)$ represented by h . Since $\text{Fix}(h)$ is a circle intersecting K in two points, we see that $\Psi([h])$ is either trivial or a reflection. Hence, by using Proposition 2.3 and Lemma 2.4, we obtain the following classification of strong inversions of non-elliptic Montesinos knots.

Case 1. $\Psi([h]) = 1$. Then we may assume $\alpha_i \equiv 0 \pmod{2}$ and $\alpha_i \equiv 1 \pmod{2}$ ($2 \leq i \leq r$), and h is as illustrated in Figure 2.6.

Case 2. $\Psi([h])$ is a reflection. Then h is equivalent to an involution as illustrated in Figure 2.7. Here $\alpha'_i \geq 2$ ($2 \leq i \leq s-1$), $\beta'_1 \equiv 1 \pmod{2}$, and $\beta'_s \equiv 0 \pmod{2}$.

REMARK 2.5. For each reflection $\gamma \in D_+(\mathbf{v}(K))$, there are two involutions, say h_1 and h_2 , of (S^3, K) such that $[h_1]$ and $[h_2]$ are mutually distinct elements of $\text{Sym}_+(S^3, K)$ whose image under Ψ are equal to γ . If we fix a projection of K in which $\text{Fix}(h_1)$ is a planar axis, then $\text{Fix}(h_2)$ is not planar. However, in

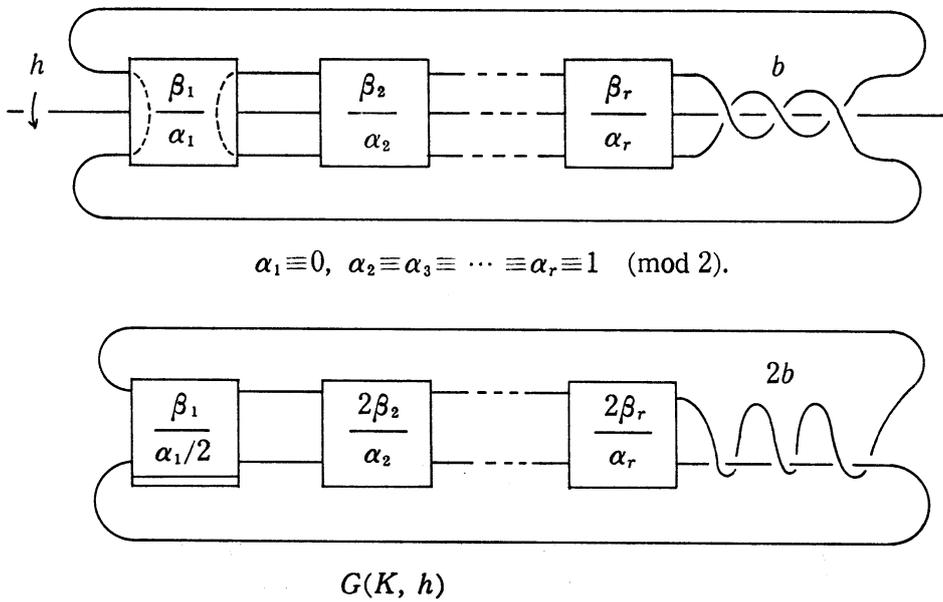


Figure 2.6.

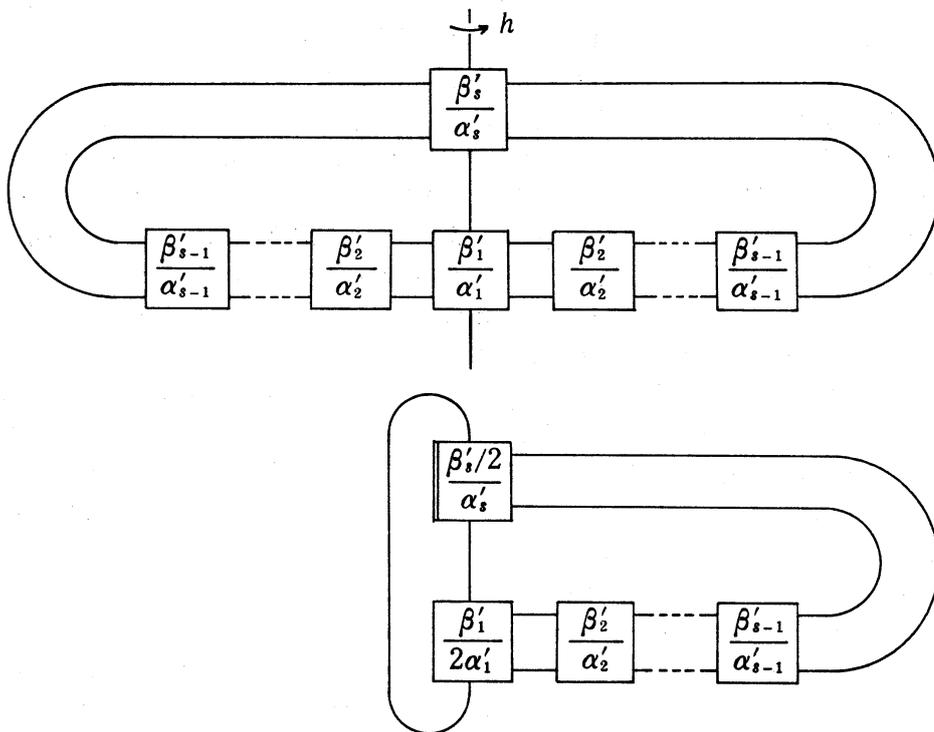


Figure 2.7.

another projection, $\text{Fix}(h_2)$ become planar as illustrated in Figure 2.8 (cf. [GS, Figure 2.6], [Sa]).

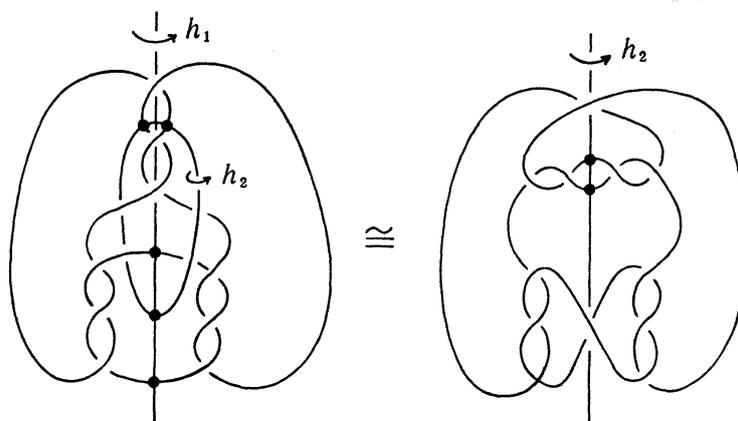


Figure 2.8.

Now, suppose that h is associated with an unknotting tunnel.

Case 1. Then the constituent knots of $G(K, h)$ are two trivial knots and

$$M(2b; (\alpha_1/2, \beta_1), (\alpha_2, \beta_2), \dots, (\alpha_r, \beta_r)).$$

The latter knot is a 2-bridge knot, if and only if $r=3$ and $\alpha_1=2$ by Proposition 2.1 (2). Hence, the condition (2) in Theorem 2.2 holds by Corollary 1.3.

Case 2. Then the constituent knots of $G(K, h)$ are a trivial knot, $\#_{i=2}^s S(\alpha'_i, \beta'_i)$, and $K' = M(0; (2\alpha'_1, \beta'_1), (\alpha'_2, \beta'_2), \dots, (\alpha'_{s-1}, \beta'_{s-1}), (\alpha'_s, \beta'_s/2))$. Hence, by Corollary 1.3, $s=3$ and K' is a trivial knot; so it follows that $\alpha'_3=1$ by Proposition 2.1 (1). This means that (1) $K = M(b; (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_2, \beta_2))$, where $(\alpha_i, \beta_i) = (\alpha'_i, \beta'_i)$ ($i=1, 2$) and $b = -\beta'_3$, and (2) $K' = M(b; (2\alpha_1, \beta_1), (\alpha_2, \beta_2))$ is a trivial knot. By Proposition 2.1 (1), the latter condition holds, if and only if $b\alpha_1\alpha_2 - 2\alpha_1\beta_2 - \alpha_2\beta_1 = \pm 1$. This is equivalent to the condition $e(K) = \pm 1/(\alpha_1\alpha_2)$. Hence the condition (3) of Theorem 2.2 holds.

At the end of this section, we apply our method to the prime knots up to 10 crossings. The bridge indices of these knots are equal to 2 or 3. So, their tunnel numbers are equal to 1 or 2. On the other hand, the symmetry groups of these knots are determined by [AHW, HW, KS]. So, by Theorem 1.2, the problem to determine the tunnel numbers of these knots is reduced to a problem concerning spatial θ -curves. However, for the θ -curves arising from these knots, the problem is settled by the method described in Corollary 1.3, except 10_{152} and 10_{154} . For these exceptional knots, we use the result of Scharlemann [Sc] that tunnel number one knots are doubly prime, and conclude that their tunnel numbers are not one.

THEOREM 2.6. *Let K be a prime knot with ≤ 10 crossings. Then $t(K)=2$ if and only if K is equivalent to one the following knots; otherwise, $t(K)=1$:*

- 8_n with $n \in [16, 18]$,
- 9_n with $n \in \{29\} \cup [32, 35] \cup [37, 41] \cup [46, 49]$,
- 10_n with $n \in [61, 69] \cup [74, 75] \cup [79, 123] \cup [140, 144] \cup [146, 160] \cup [163, 166]$.

In the above, we use the numbering in the table of Rolfsen’s book. The result can be rephrased as follows: A prime knot K with ≤ 10 crossings has tunnel number one, if and only if it is equivalent to (1) a 2-bridge knot, (2) $M(b; (2, 1), (\alpha_2, \beta_2), (\alpha_3, \beta_3))$, (3) $M(b; (\alpha_1, \beta_1), (3, 1), (3, 1))$ with $e(K) = \pm 1/(3\alpha_1)$, (4) a torus knot, or (5) $10_{161} = 10_{162}$.

3. Criterion by means of Yamada’s invariants of spatial graphs.

First, we recall the *linear skeins* of planar surfaces developed in [L1, L2]. Let F be a compact, connected, 2-dimensional submanifold of S^2 . A *diagram* on F is immersed circles and arcs joining specified points on ∂F whose singular set consists of a finite number of double points equipped with over-under information. Two diagrams are regarded as the same if they differ by an isotopy of F relative to ∂F .

DEFINITION 3.1 [L2, Section 2]. Let A be a *unit* complex number. The *linear skein* $\mathcal{S}(F)$ of F is a complex vector space of formal linear sums of diagrams on F quotiented by relations

$$D \cup \bigcirc = -(A^2 + A^{-2}) \cdot D,$$

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = A \begin{array}{c} \rule{0.5cm}{0.4pt} \\ \rule{0.5cm}{0.4pt} \end{array} + A^{-1} \begin{array}{c} \rule{0.5cm}{0.4pt} \\ \rule{0.5cm}{0.4pt} \end{array} \quad (,$$

where \bigcirc stands for the boundary of a disk in F .

Then $\mathcal{S}(S^2)$ is a 1-dimensional vector space with the empty diagram ϕ as a natural base. In fact, for each element D of $\mathcal{S}(S^2)$, we have $D = \langle D \rangle \phi$ in $\mathcal{S}(S^2)$, where $\langle \cdot \rangle$ denotes the Kauffman bracket [Ka] normalized so that $\langle \phi \rangle = 1$.

In what follows, D_l denotes an oriented disk with $l \in 2\mathbf{Z}$ specified points on its boundary. Furthermore, for a partition $l = l_1 + \dots + l_n$, $l_i > 0$, $D_{l_1 + \dots + l_n}$ denotes D_l equipped with disjoint n arcs on ∂D_l which cover l_1, \dots, l_n points in order with respect to the orientation of ∂D_l . Then, there is an important element f_m in $\mathcal{S}(D_{m+m})$, which is called the *Jones-Wenzel idempotent* or the *magic knitting* (see [L2, Lemma 1]). In the following figures, f_m will be shown as a small blank square. Let Δ_m and $I(x, y, z)$ be the complex numbers corresponding to the elements of $\mathcal{S}(S^2)$ represented by the diagrams as illustrated in Figure 3.1(a) and (b) respectively. Here m, x, y , and z are non-negative integers, and the numbers in Figure 3.1 represent the numbers of parallel copies of arcs. Then we obtain the following [L1, Lemma 1]:

$$\Delta_m = (-1)^m \frac{A^{2(m+1)} - A^{-2(m+1)}}{A^2 - A^{-2}},$$

$$\Gamma(x, y, z) = \frac{\Delta_{x+y+z}! \Delta_{x-1}! \Delta_{y-1}! \Delta_{z-1}!}{\Delta_{y+z-1}! \Delta_{z+x-1}! \Delta_{x+y-1}!}.$$

Here, $\Delta_m! = \Delta_m \Delta_{m-1} \cdots \Delta_0$ and $\Delta_{-1}! = 1$.

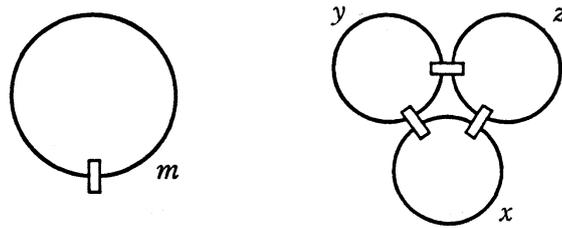


Figure 3.1.

A triple (a, b, c) of non-negative integers is said to be *admissible* if

$$|a-b| \leq c \leq a+b, \quad a+b+c \in 2\mathbf{Z}.$$

This condition is equivalent to the condition that there are non-negative integers $x, y,$ and z such that

$$x+y = a, \quad y+z = b, \quad z+x = c.$$

We put $\Delta_{a,b,c} = \Gamma(x, y, z)$ for each admissible triple (a, b, c) .

We now recall the definition of Yamada's invariant of spatial graphs [Ya]. Let G be a trivalent graph embedded in S^3 and D a diagram of G . A *weight* of G is a map from the set of edges of G to the set of non-negative integers. Let v be a vertex of G and α, β, γ the edges incident with v . A weight ω of G is said to be *admissible* at v if $(\omega(\alpha), \omega(\beta), \omega(\gamma))$ is admissible. If ω is admissible at each vertex of G , ω is said to be *admissible*. For an admissible weight ω , we define D^ω by decorating each edge e of D with $\omega(e)$ parallel curves equipped with the magic knitting $f_{\omega(e)}$ and by joining them at each vertex as shown in Figure 3.2.

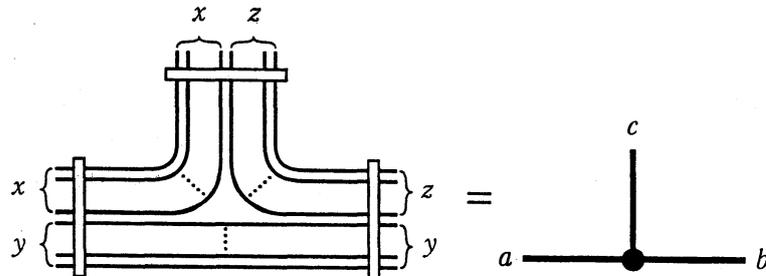


Figure 3.2.

Then Yamada's invariant of G is defined by $Y_{G,\omega}(A) = \langle D^\omega \rangle$. This is an invariant of G up to a multiplication of $\pm A^{\pm n}$ [Ya, p. 452]. (This invariant is generalized to an invariant of graphs in S^3 which are not necessarily trivalent in [Yo2].)

Suppose G is a θ -curve in S^3 with edges α, β , and γ . By $Y_{G,a,b,c}(A)$, we denote $Y_{G,\omega}(A)$ with $\omega(\alpha)=a, \omega(\beta)=b$ and $\omega(\gamma)=c$. The following theorem gives a criterion by means of Yamada's invariant to determine whether a θ -curve arising from a strong inversion of a knot satisfies the conditions in Theorem 1.2:

THEOREM 3.2. (1) Suppose G has a 3-bridge decomposition which induces 1- and 2-bridge decompositions of $\alpha \cup \beta$ and $\alpha \cup \gamma$ respectively. Then, we have

$$\sum_a \frac{\Delta_a}{\Delta_{a,b,c}^2 \Delta_b \Delta_c^3} |Y_{G,a,b,c}(e^{i\phi})|^2 \leq 1$$

for $|\phi| < \pi/2(b+2c+1)$, where a varies so that (a, b, c) is admissible.

(2) Suppose G has a 2-bridge decomposition, which induces a 1-bridge decomposition of $\alpha \cup \beta$. Then, we have

$$|Y_{G,a,b,c}(e^{i\phi})| \leq \Delta_{a,b,c} \Delta_c$$

for $|\phi| < \pi/(a+b+3c+2)$.

To prove this theorem, we use the approach of [Yo2] to the invariants and use the idea due to [Wk, G, Ko, Yo1]. For $v \in \mathcal{S}(D_{l_1+\dots+l_n})$, we define $f(v) \in \mathcal{S}(D_{l_1+\dots+l_n})$ by glueing v and f_{l_1}, \dots, f_{l_n} at n arcs. Put $\mathcal{H}_{l_1+\dots+l_n} = f(\mathcal{S}(D_{l_1+\dots+l_n}))$. Let $-D_{l_1+\dots+l_n}$ denote $D_{l_1+\dots+l_n}$ with the opposite orientation. Note that the orientation reversing map $D_{l_1+\dots+l_n} \rightarrow -D_{l_1+\dots+l_n}$ induces an isomorphism

$$* : \mathcal{S}(D_{l_1+l_2+\dots+l_n}) \longrightarrow \mathcal{S}(D_{l_n+\dots+l_2+l_1})$$

which takes complex conjugate for coefficients and mirror images for diagrams. Since f_m is an \mathbf{R} -linear sum of diagrams without crossings [L1, Section 2], we see $f_m^* = f_m$. So, $*$ restricts to a map

$$* : \mathcal{H}_{l_1+l_2+\dots+l_n} \longrightarrow \mathcal{H}_{l_n+\dots+l_2+l_1}.$$

For $u, v \in \mathcal{H}_{l_1+\dots+l_n}$, we obtain an element of $\mathcal{S}(S^2)$ by glueing u and v^* at their boundary, which identified with some complex number through Kauffman's bracket. Let $\theta(u, v) \in \mathbf{C}$ be this complex number multiplied by $\sqrt{-1}^l$, which defines a Hermitian form

$$\theta : \mathcal{H}_{l_1+\dots+l_n} \times \mathcal{H}_{l_1+\dots+l_n} \longrightarrow \mathbf{C}.$$

From now onward, we choose A so that $(-1)^m \Delta_m > 0$ for each m up to the half of l . This assumption is satisfied if

$$(3.3) \quad A = e^{i\psi}, \quad |\psi| < \frac{\pi}{l+2}.$$

In what follows, for admissible (a, b, c) , we use an abbreviation shown in Figure 3.2 to describe a vector of \mathcal{H}_{a+b+c} , and note that $\sqrt{-1}^{a+b+c}\Delta_{a,b,c}$ is positive real if $a+b+c \leq l$. For an $(n-3)$ -tuple of integers $\mathbf{j}=(j_1, \dots, j_{n-3})$, $\hat{u}_{l_1+\dots+l_n}^{\mathbf{j}}$ denotes a vector of $\mathcal{H}_{l_1+\dots+l_n}$ depicted in Figure 3.3, where $(l_1, l_2, j_1), (j_1, j_2, l_3), \dots, (j_{n-3}, l_{n-1}, l_n)$ are admissible.

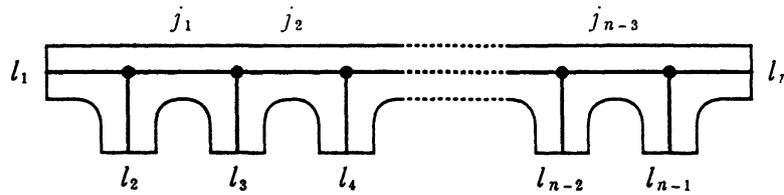


Figure 3.3.

Then, by using the identity [L2, Figure 2.7], we obtain the following :

$$(3.4) \quad \theta(\hat{u}_{l_1+\dots+l_n}^{\mathbf{j}}, \hat{u}_{l_1+\dots+l_n}^{\mathbf{j}}) = \sqrt{-1}^l \frac{\Delta_{l_1, l_2, j_1} \Delta_{j_1, j_2, l_3} \cdots \Delta_{j_{n-3}, l_{n-1}, l_n}}{\Delta_{j_1} \Delta_{j_2} \cdots \Delta_{j_{n-3}}},$$

which is *positive real* in our setting. Let $\mathcal{B}_{l_1+\dots+l_n}$ denote the set of vectors

$$(3.5) \quad u_{l_1+\dots+l_n}^{\mathbf{j}} = \frac{\hat{u}_{l_1+\dots+l_n}^{\mathbf{j}}}{\sqrt{\theta(\hat{u}_{l_1+\dots+l_n}^{\mathbf{j}}, \hat{u}_{l_1+\dots+l_n}^{\mathbf{j}})}},$$

where j varies so that $(l_1, l_2, j_1), (j_1, j_2, l_3), \dots, (j_{n-3}, l_{n-1}, l_n)$ are admissible. Then we have

LEMMA 3.6 [Yo2, Proposition 3.2]. $\mathcal{B}_{l_1+\dots+l_n}$ is an orthonormal basis of $\mathcal{H}_{l_1+\dots+l_n}$. In particular, θ is positive definite.

For each element ξ of the braid group B_n on n strings, we can define a natural isomorphism $\xi: \mathcal{H}_{l_1+\dots+l_n} \rightarrow \mathcal{H}_{l_{\xi(1)}+\dots+l_{\xi(n)}}$. (See Figure 3.4, where the isomorphism corresponding to a generator $\sigma^{\pm 1}$ is illustrated.) Then we obtain the following :

LEMMA 3.7 [Yo2, Proposition 3.1]. ξ is unitary with respect to θ , that is,

$$\theta(\xi(u), \xi(v)) = \theta(u, v)$$

for any $u, v \in \mathcal{H}_{l_1+\dots+l_n}$.

PROOF OF THEOREM 3.2. Suppose G has a 3-bridge decomposition as shown in Figure 3.5, where ξ denotes some element of the braid group on 6 strings. By Lemma 3.6, ξ determines a unitary operator

$$\xi: \mathcal{H}_{b+c+b+c+c+c} \longrightarrow \mathcal{H}_{b+b+c+c+c+c}.$$

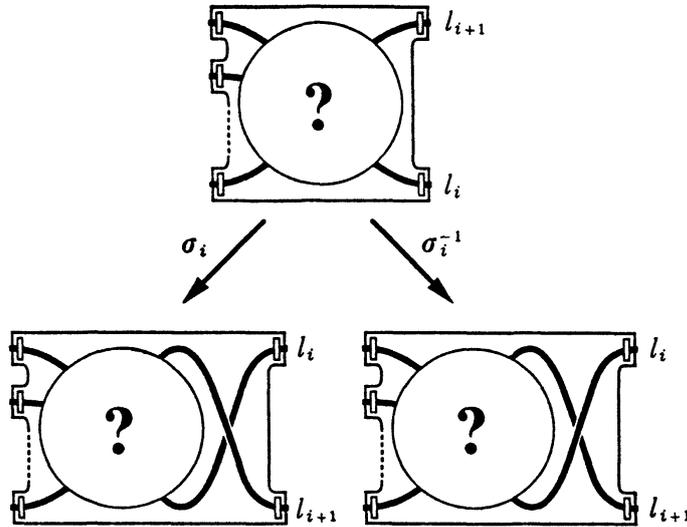


Figure 3.4.

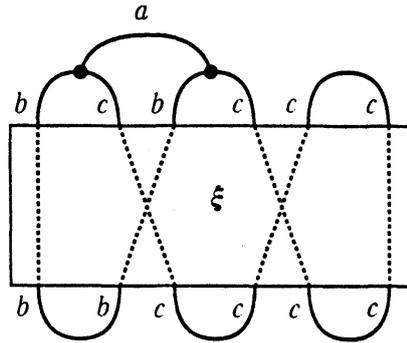


Figure 3.5.

It should be noted that condition (3.3) corresponds to the condition that $A=e^{i\psi}$, $|\psi|<\pi/2(b+2c+1)$ in this case; and under this condition, $\mathcal{B}_{b+c+b+c+c+c}$ and $\mathcal{B}_{b+b+c+c+c+c}$ exist. From the definition of the invariant Y , we have

$$Y_{G,a,b,c}(e^{i\psi}) = \theta(\xi(\hat{u}_{b+c+b+c+c+c}^{(a,c,0)}, \hat{u}_{b+b+c+c+c+c}^{(0,c,0)})).$$

On the other hand, by (3.4),

$$\theta(\hat{u}_{b+c+b+c+c+c}^{(a,c,0)}, \hat{u}_{b+b+c+c+c+c}^{(a,c,0)}) = \sqrt{-1}^l \frac{\Delta_{a,b,c}^2 \Delta_c^2 \Delta_0}{\Delta_a \Delta_c \Delta_0} = \sqrt{-1}^l \frac{\Delta_{a,b,c}^2 \Delta_c}{\Delta_a},$$

where $l=2(b+2c)$. So, by (3.5),

$$\hat{u}_{b+c+b+c+c+c}^{(a,c,0)} = \sqrt{\sqrt{-1}^l \frac{\Delta_{a,b,c}^2 \Delta_c}{\Delta_a}} u_{b+c+b+c+c+c}^{(a,c,0)}.$$

Similarly,

$$\hat{u}_{b+b+c+c+c+c}^{(0,c,0)} = \sqrt{\sqrt{-1}^l \Delta_b \Delta_c^2} u_{b+b+c+c+c+c}^{(0,c,0)}.$$

Hence, we have

$$Y_{G,a,b,c}(e^{i\psi}) = \sqrt{\frac{\Delta_{a,b,c}^2 \Delta_b \Delta_c^3}{\Delta_a}} \theta(\xi(u_{b+c+b+c+c+c}^{(a,c,0)}, u_{b+b+c+c+c+c}^{(0,c,0)})).$$

Since ξ is unitary,

$$\begin{aligned} 1 &= \sum_j |\theta(\xi u_{b+c+b+c+c+c}^j, u_{b+b+c+c+c+c}^{(0,c,0)})|^2 \\ &\geq \sum_a |\theta(\xi(u_{b+c+b+c+c+c}^{(a,c,0)}), u_{b+b+c+c+c+c}^{(0,c,0)})|^2 \\ &= \sum_a \frac{\Delta_a}{\Delta_{a,b,c}^2 \Delta_b \Delta_c^3} |Y_{G,a,b,c}(e^{i\psi})|^2. \end{aligned}$$

This proves Theorem 3.1 (1).

Next, we prove Theorem 3.2 (2). Suppose G has a 2-bridge decomposition as shown in Figure 3.6, where ξ denotes some element of the braid group on 5 strings.

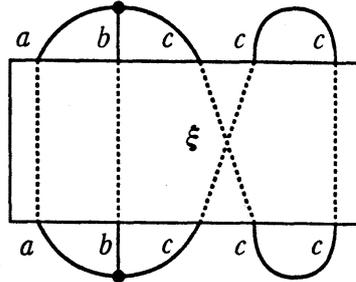


Figure 3.6.

By Lemma 3.6, ξ determines a unitary operator

$$\xi: \mathcal{H}_{a+b+c+c+c} \longrightarrow \mathcal{H}_{a+b+c+c+c}.$$

It should be noted that condition (3.3) corresponds to the condition that $A=e^{i\psi}$, $|\psi| < \pi/(a+b+3c+2)$. From the definition together with (3.4) and (3.5), we have

$$\begin{aligned} Y_{G,a,b,c}(e^{i\psi}) &= \theta(\xi(\hat{u}_{a+b+c+c+c}^{(c,0)}), \hat{u}_{a+b+c+c+c}^{(c,0)}) \\ &= \Delta_{a,b,c} \Delta_c \theta(\xi(u_{a+b+c+c+c}^{(c,0)}), u_{a+b+c+c+c}^{(c,0)}). \end{aligned}$$

Since ξ is unitary,

$$\begin{aligned} 1 &= \sum_j |\theta(\xi(u_{a+b+c+c+c}^j), u_{a+b+c+c+c}^{(c,0)})|^2 \\ &\geq |\theta(\xi(u_{a+b+c+c+c}^{(c,0)}), u_{a+b+c+c+c}^{(c,0)})|^2 \\ &= \frac{1}{\Delta_{a,b,c}^2 \Delta_c^2} |Y_{G,a,b,c}(e^{i\psi})|^2. \end{aligned}$$

This proves Theorem 3.2 (2).

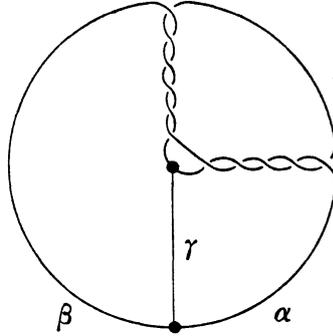


Figure 3.7.

EXAMPLE 3.8. Let $G = \alpha \cup \beta \cup \gamma$ be the θ -curve as illustrated in Figure 3.7. This does not have a 3-bridge decomposition where $\alpha \cup \beta$ has a one-bridge decomposition. To show this, put $b=3$ and $c=1$ in Theorem 3.2 (1). If G admits such a decomposition, then we have

$$y(\phi) := \frac{\Delta_4}{\Delta_{4,3,1}^2 \Delta_3 \Delta_1^3} |Y_{G,4,3,1}(e^{i\phi})|^2 + \frac{\Delta_2}{\Delta_{2,3,1}^2 \Delta_3 \Delta_1^3} |Y_{G,2,3,1}(e^{i\phi})|^2 \leq 1$$

for $|\phi| < \pi/12$. To calculate this, we use the identity [Ya, p. 452] shown in Figure 3.8 and quantum $6j$ -symbols $\begin{Bmatrix} a & b & i \\ c & d & j \end{Bmatrix} \in \mathbb{C}$ defined by Figure 3.9 which can be computed from the definition of magic knittings. Then, for example, we can compute $Y_{G,4,3,1}$ as shown in Figure 3.10, where the last ingredient can be computed by using [L2, Figure 27].

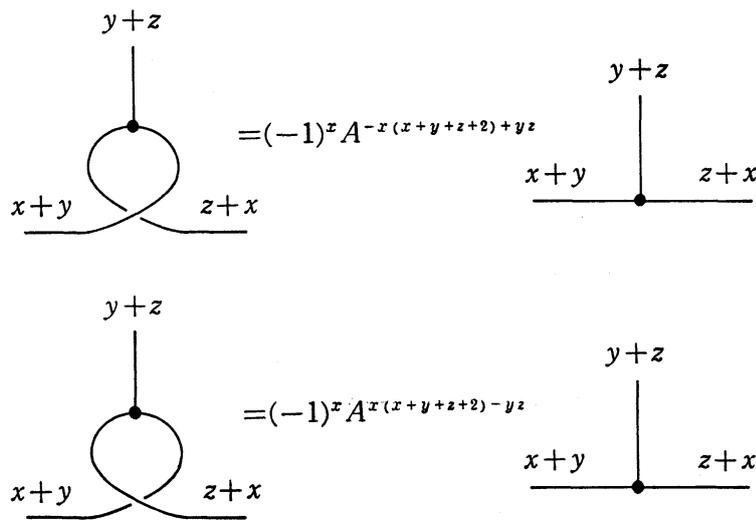


Figure 3.8.

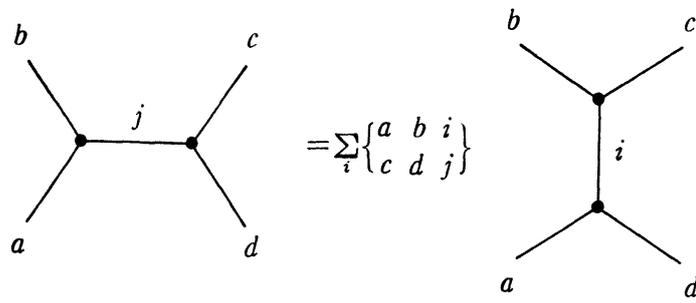


Figure 3.9.

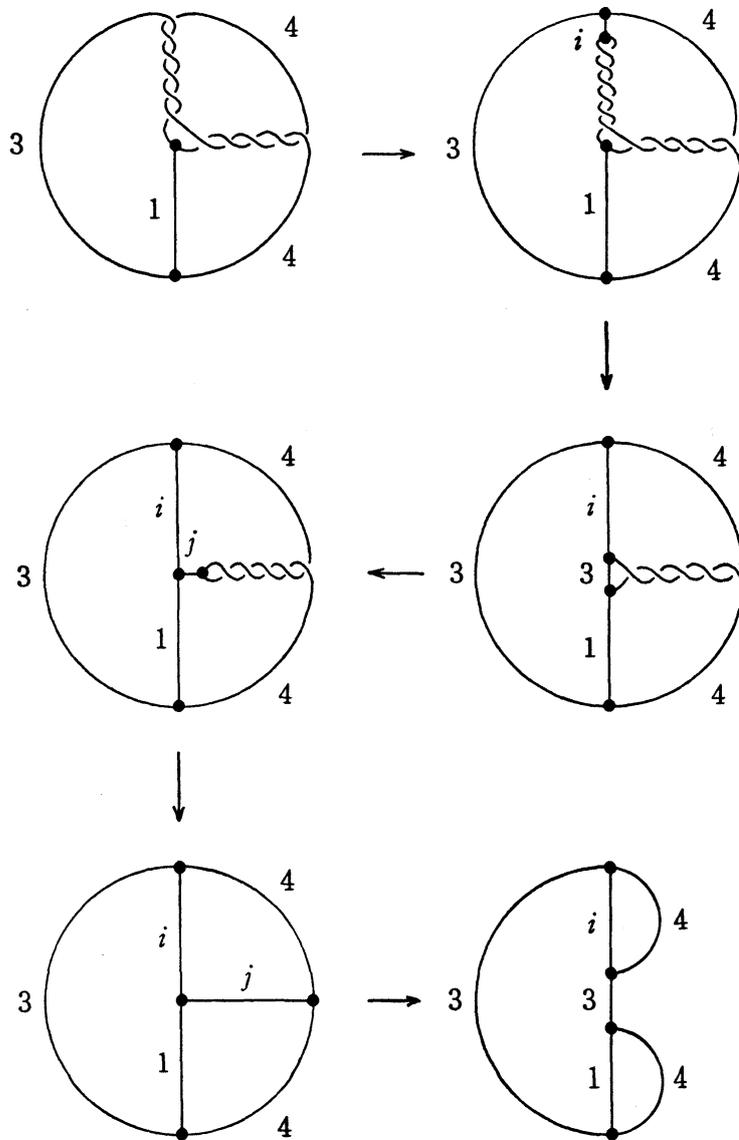


Figure 3.10.

Thus we have

$$\begin{aligned}
 Y_{G,2,3,1}(A) &= -A^{-26} - A^{-22} - A^{-18} - A^{-14} - A^{-10} - A^{-6} + A^{-2} \\
 &\quad + A^2 + A^{10} + A^{14} - 2A^{18} + A^{26} - 2A^{30} + A^{34} + A^{38} - A^{42}, \\
 Y_{G,4,3,1}(A) &= -A^{-72} + A^{-48} + A^{-44} - A^{-36} - A^{-24} - A^{-20} + A^{-16} + 2A^{-12} \\
 &\quad + 1 + A^4 + A^{16} + A^{20}.
 \end{aligned}$$

The practical graph of $\nu(\psi)$ is given by Figure 3.11, which contradicts the above inequality.

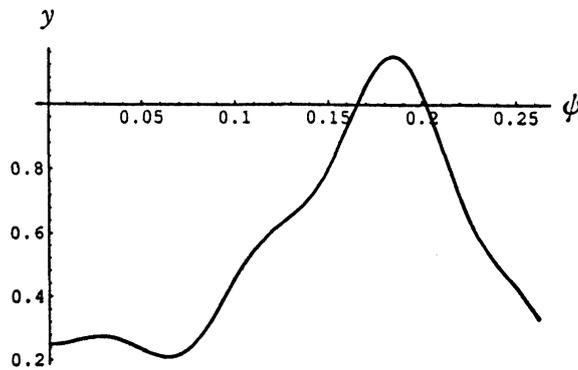


Figure 3.11.

Hence G does not have such a decomposition. This, together with the arguments in Section 2, proves that the Montesinos knot $M(0; (5, 1), (5, 1), (3, -1))$ does not have tunnel number one. Thus the condition in Theorem 2.2 is not sufficient.

4. Relation with reflection groups.

If a knot has tunnel number one, then its knot group is generated by two elements. On the other hand, the knot group of a Montesinos knot has a natural epimorphism to a reflection group (cf. [BuZ, Chapter 12]). Thus, Theorem 2.3 enables us to find two generator reflection groups.

PROPOSITION 4.1. *Consider the reflection group*

$$\Gamma(\alpha_1, \alpha_2, \alpha_3) = \langle x, y, z \mid x^2 = y^2 = z^2 = (xy)^{\alpha_1} = (yz)^{\alpha_2} = (zx)^{\alpha_3} = 1 \rangle.$$

Then it is generated by two elements if one of the following conditions are satisfied up to permutation of the indices:

- (1) $\alpha_1=2$ and $\alpha_2 \equiv 1 \pmod{2}$. In this case, $\{y, (yz)^p x\}$, where $p=(\alpha_2-1)/2$, forms a generator system.
- (2) $\alpha_2=\alpha_3=3$ and $\alpha_1 \not\equiv 0 \pmod{3}$. In this case, $\{y, yxyz\}$ forms a generator system.

PROOF. This follows from the latter half of Theorem 2.2 except the case where $\alpha_3 \equiv 0 \pmod{2}$ in (1). In this case, $M(b; (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3))$ is a 2-component link; but, we can also see that the core τ_3 of the rational tangle $\boxed{\beta_3/\alpha_3}$ forms its “unknotting tunnel”, and we obtain the desired result. The generator systems are obtained from Figures 2.2(a) and 2.3(a).

REMARK 4.2. (1) If $\alpha_1=2$ and $\alpha_2 \equiv \alpha_3 \equiv 1 \pmod{2}$, then the corresponding Montesinos knot has two unknotting tunnels τ_2 and τ_3 . We can see that the generator systems of the reflection group coming from these tunnels are Nielsen equivalent if and only if $\alpha_2 = \alpha_3 = 3$ by using the commutator invariant (cf. [Mo]). Thus it follows that these two tunnels are isotopic only if $\alpha_2 = \alpha_3 = 3$.

(2) The two unknotting tunnels τ and τ' for $M(0; (3, 1), (3, 1), (3n+2, -2n-1))$ in Figure 2.3(a) determine the same generator system of the reflection group.

After having done this work, the authors knew the work of Klimenko [Kl], which determines a certain class of 2-generator discrete subgroups of $PSL(2, \mathbb{C})$, through discussion with A. Mednykh. From this article, the authors learned that the results and methods of Matelski [Ma] are useful for our problem. In fact, we can see that if a reflection group is generated by a reflection and another element, then it belongs to the list given by Proposition 4.1. Thus, together with Theorem 2.2, this implies that a Montesinos knot admits a (1, 1)-decomposition if and only if it satisfies the condition (1), (2) or (3') of Theorem 2.2. We hope to discuss this approach in another paper.

ACKNOWLEDGEMENT. We would like to thank W.B.R. Lickorish and A. Mednykh for their valuable comments.

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Added in Proof.

E. Klimenko and the second author proved that the converse of Proposition 4.1 also holds. Thus it follows that a Montesinos knot has tunnel number 1 if and only if one of the conditions (1), (2), and (3)' in Theorem 2.2 holds. Furthermore, Y. Nakagawa determined the two-component Montesinos links with tunnel number 1.