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On regular subalgebras of Kac-Moody algebras and their associated invariant forms
—Symmetrizable case—

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Introduction.

In [3], Dynkin classified all the semi-simple subalgebras of finite dimensional complex semi-simple Lie algebras. There, a special kind of subalgebras with compatible root space decompositions, called *regular semi-simple subalgebras*, played an important role.

In this paper, we treat a Kac-Moody algebra with a symmetrizable generalized Cartan matrix (=GCM), and study its *regular subalgebras*, defined as a natural infinite dimensional analogue of Dynkin's ones. Though being no more isomorphic to Kac-Moody algebras in general, these regular subalgebras are isomorphic to generalized Kac-Moody algebras (=GKM algebras) introduced by Borcherds [1].

We now give a constructive definition of regular subalgebras. Let $\mathfrak{g}(A)$ be a Kac-Moody algebra with a symmetrizable GCM A, $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}(A)$, and Δ be the root system of $\mathfrak{g}(A)$. A subset $\{\beta_1, \cdots, \beta_m, \beta_{m+1}, \cdots, \beta_{m+k}\}\subset\Delta$ is called *fundamental* if it satisfies the following three conditions (see Definition 5.2):

(1) $\{\beta_r\}_{r=1}^{m+k}\subset\mathfrak{h}^*$ is a linearly independent subset;

(2) $\beta_s-\beta_t\notin\Delta\cup\{0\}$ $(1\leqq s\neq t\leqq m+k)$;

(3) β_i is a real root $(1\leqq i\leqq m)$ and β_j is a positive imaginary root $(m+1\leqq j\leqq m+k)$.

Let $\tilde{\mathfrak{g}}$ be a subalgebra of $\mathfrak{g}(A)$ generated by root vectors attached to each root $\pm\beta_r$ $(1\leqq r\leqq m+k)$ and a certain vector subspace $\mathfrak{h}_0$ of $\mathfrak{h}$. Then, $\tilde{\mathfrak{g}}$ is canonically isomorphic to a GKM algebra (see Theorem 5.1). We call this subalgebra $\tilde{\mathfrak{g}}$ a *regular subalgebra* of $\mathfrak{g}(A)$ after Dynkin.

The above definition of a fundamental subset and the construction of a subalgebra $\tilde{\mathfrak{g}}$ are generalizations of those by Morita in [8]. There, he considered only the case all β_r are real roots (i.e., $k=0$ in the above notation) and constructed a subalgebra $\hat{\mathfrak{g}}$, which coincides with the derived algebra $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]$ of the above $\tilde{\mathfrak{g}}$, in order to introduce certain subsystems of the root system Δ of

$\mathfrak{g}(A)$ (see Remarks 2.3 and 3.1).

In this paper, we obtain three main results.

Let us explain the first result. Let $\{\beta_r\}_{r=1}^{m}\subset\Delta^{re}$ be fundamental. And we take a "good" vector subspace $\mathfrak{h}_0$ of $\mathfrak{h}$ in the above construction of $\tilde{\mathfrak{g}}$. Then, we show in Theorem 3.6 that the resulting $\tilde{\mathfrak{g}}$ is isomorphic to a Kac-Moody algebra $\mathfrak{g}(\tilde{A})$ with some symmetrizable GCM $\tilde{A}$. (This is a special case ($k=0$) of Theorem 5.1. We give here a different proof from that of Theorem 5.1.) Moreover, we prove in Theorem 3.7 that the restriction of a *standard invariant form* (cf. §1) on $\mathfrak{g}(A)$ to its regular subalgebra $\tilde{\mathfrak{g}}$ coincides with a standard invariant form on $\tilde{\mathfrak{g}}$, considered canonically as a Kac-Moody algebra $\mathfrak{g}(\tilde{A})$. As a consequence of this, real (resp. imaginary) roots of $\mathfrak{g}(\tilde{A})\cong\tilde{\mathfrak{g}}$ can be regarded canonically as a subset of real (resp. imaginary) roots of $\mathfrak{g}(A)$ (Theorem 3.8).

The second result is as follows. As an application of Theorem 3.7, we obtain a fact that, if $\mathfrak{g}(A)$ is of affine type, then any regular subalgebra $\tilde{\mathfrak{g}}$ which is a Kac-Moody algebra is a direct sum of Kac-Moody algebras of finite or of affine type, and the number of direct summands of affine type is at most one (Theorem 4.1). Further, when $\mathfrak{g}(A)$ is of non-twisted affine type, we determine, using the results in [**3**], all the types of regular subalgebras which are Kac-Moody algebras (Theorem 4.2).

Contrary to this affine case, we see through an example (Example 4.1) that even if $\mathfrak{g}(A)$ is of hyperbolic type, there can be a regular subalgebra $\tilde{\mathfrak{g}}$ which is a Kac-Moody algebra, but whose direct summands do not remain inside the category of finite, affine, or hyperbolic types. This is mainly because a Kac-Moody algebra of hyperbolic type can not be characterized only by the signature of a standard invariant form on it (cf. Proposition A in §1).

For our third result, let us consider a general fundamental subset $\{\beta_r\}_{r=1}^{m+k}\subset\Delta$. We show in Theorem 5.1 that $\tilde{\mathfrak{g}}$ is canonically isomorphic to a GKM algebra $\mathfrak{g}(\tilde{A})$ with some symmetrizable GGCM $\tilde{A}$ (cf. Definition 5.1).

This paper is organized as follows. In §1, we recall some preliminary facts about Kac-Moody algebras. In §2, we review the notion of a fundamental subset of a Kac-Moody root system introduced by Morita [8] (the case $k=0$ in our general definition), and prove some elementary properties. In §3, we establish the first result stated above. In §4, as our second result, we study the types of regular subalgebras of a given Kac-Moody algebra. In §5, we deal with a general fundamental subset and obtain the third result.

NOTATIONS. We denote by $\boldsymbol{C}$ the complex number field, $\boldsymbol{R}$ the real number field, $\boldsymbol{Q}$ the rational number field, and $\boldsymbol{Z}$ the ring of integers. We define subsets $\boldsymbol{N}$ and $\boldsymbol{Z}_+$ of $\boldsymbol{Z}$ by

$$\boldsymbol{N}:=\{k\in\boldsymbol{Z}\,;\,k\geqq 1\},\qquad \boldsymbol{Z}_+:=\{k\in\boldsymbol{Z}\,;\,k\geqq 0\}.$$

And we denote by $\langle\cdot,\cdot\rangle$ a pairing between a vector space V over $\boldsymbol{R}$ or $\boldsymbol{C}$ and its algebraic dual V^*.

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§ 1. Preliminaries.

In this section, we recall some prelimiary facts about Kac-Moody algebras. For detailed accounts, see [**4**].

1.1. Kac-Moody algebras. Let n be a positive integer, and $I:=\{1, 2, \cdots, n\}$ be an index set. A matrix $A=(a_{ij})_{i,j\in I}$ is called a generalized Cartan matrix (=GCM) if it satisfies the following three conditions:

(C1) $a_{ii}=2$, for all $i\in I$;

(C2) a_{ij} are non-positive integers, for $i\neq j$;

(C3) $a_{ij}=0$ implies $a_{ji}=0$.

A GCM A is called indecomposable if there is no permutation σ on I such that $A^\sigma:=(a_{\sigma(i),\sigma(j)})_{i,j\in I}$ is a direct sum of its non-trivial diagonal blocks. We call an indecomposable GCM A of *finite type* if all its principal minors are positive, and of *affine type* if all its proper principal minors are positive and $\det A=0$. It is called of *hyperbolic type* if it is symmetrizable, neither of finite type nor of affine type, and possesses the property that a removal of any row and the corresponding column makes A a direct sum of GCM's of finite type or of affine type.

A triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ is called a *realization* of A if it satisfies the following three conditions:

(R1) $\mathfrak{h}$ is a finite dimensional complex vector space, and $\dim_C\mathfrak{h}=2n-\operatorname{rank} A$;

(R2) $\Pi=\{\alpha_i\}_{i\in I}$ is a linearly independent subset of $\mathfrak{h}^*$, and $\Pi^\vee=\{\alpha_i^\vee\}_{i\in I}$ is a linearly independent subset of $\mathfrak{h}$;

(R3) $\langle\alpha_j, \alpha_i^\vee\rangle=a_{ij}$, for $i, j\in I$.

REMARK 1.1. For any $n\times n$ GCM A, there exists an essentially unique realization.

For an arbitrary GCM $A=(a_{ij})_{i,j\in I}$, there is a unique, up to isomorphism, Lie algebra $\mathfrak{g}(A)$ which satisfies the following:

(L1) e_i, f_i $(i\in I)$ and $\mathfrak{h}$ generate $\mathfrak{g}(A)$ as a Lie algebra, and $\mathfrak{h}$ is a commutative subalgebra of $\mathfrak{g}(A)$;

(L2) $[e_i, f_j]=\delta_{ij}\alpha_i^\vee$ $(i, j\in I)$;

(L3) $[h, e_i]=<\alpha_i, h>e_i$, $[h, f_i]=-<\alpha_i, h>f_i$ $(i\in I, h\in\mathfrak{h})$;

(L4) $\mathfrak{g}(A)$ has no non-zero ideals which intersect $\mathfrak{h}$ trivially.

We call the Lie algebra $\mathfrak{g}(A)$ a Kac-Moody algebra associated with A, the subalgebra $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{g}(A)$, and the matrix A the Cartan matrix of $\mathfrak{g}(A)$. Elements of Π (resp. $\Pi^\vee$) are called simple roots (resp. simple coroots), and the elements e_i, f_i $(i\in I)$ are called the Chevalley generators of $\mathfrak{g}(A)$. We set

$$Q:=\sum_{i\in I}\boldsymbol{Z}\alpha_i, \qquad Q_+:=\sum_{i\in I}\boldsymbol{Z}_+\alpha_i, \qquad Q^\vee:=\sum_{i\in I}\boldsymbol{Z}\alpha_i^\vee.$$

With respect to $\mathfrak{h}$, we have a root space decomposition of $\mathfrak{g}(A)$:

$$\mathfrak{g}(A)=\sum_{\alpha\in Q}^{\oplus}\mathfrak{g}_\alpha ,$$

where $\mathfrak{g}_\alpha:=\{x\in\mathfrak{g}(A); [h, x]=<\alpha, h>x$, for all $h\in\mathfrak{h}\}$ for $\alpha\in Q$. Note that $\mathfrak{g}_0=\mathfrak{h}$, $\mathfrak{g}_{\alpha_i}=\boldsymbol{C}e_i$, and $\mathfrak{g}_{-\alpha_i}=\boldsymbol{C}f_i$ for $i\in I$. An element α in Q_+ (resp. in $-Q_+$) is called positive (resp. negative). Denote by Δ, Δ_+, and Δ_- the set of all roots, positive roots, and negative roots, respectively, and then $\Delta=\Delta_+\amalg\Delta_-$ (a disjoint union). Let $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) be the subalgebra of $\mathfrak{g}(A)$ generated by e_i, $i\in I$ (resp. f_i, $i\in I$), then $\mathfrak{n}_+=\sum_{\alpha\in\Delta_+}^{\oplus}\mathfrak{g}_\alpha$ and $\mathfrak{n}_-=\sum_{\alpha\in\Delta_+}^{\oplus}\mathfrak{g}_{-\alpha}$.

REMARK 1.2. Let $(\mathfrak{h}_{\boldsymbol{R}}, \Pi, \Pi^\vee)$ be a realization of a matrix A over $\boldsymbol{R}$, then $(\mathfrak{h}:=\boldsymbol{C}\otimes_{\boldsymbol{R}}\mathfrak{h}_{\boldsymbol{R}}, \Pi, \Pi^\vee)$ is a realization of A over $\boldsymbol{C}$. In this case, we can define an antilinear automorphism ω_0 of $\mathfrak{g}(A)$ by

$$\omega_0(e_i)=-f_i, \qquad \omega_0(f_i)=-e_i\ (i\in I), \qquad \omega_0(h)=-h\ (h\in\mathfrak{h}_{\boldsymbol{R}}).$$

We call ω_0 the compact involution of $\mathfrak{g}(A)$. Note that $\omega_0(\mathfrak{g}_\alpha)=\mathfrak{g}_{-\alpha}$ for all $\alpha\in\Delta\cup\{0\}$.

1.2. Standard invariant forms.

Suppose that $A=(a_{ij})_{i,j\in I}$ is a symmetrizable GCM (i.e., there exist an invertible diagonal matrix D and a symmetric matrix B, such that $A=DB$). And fix a decomposition $A=DB$, where $D=\mathrm{diag}(\varepsilon_1, \cdots, \varepsilon_n)$ is a positive rational diagonal matrix, and $B=(b_{ij})_{i,j\in I}$ is a symmetric rational matrix. Fix a complementary subspace $\mathfrak{h}''$ to $\mathfrak{h}':=\sum_{i\in I}\boldsymbol{C}\alpha_i^\vee$ in $\mathfrak{h}$, then we can define uniquely a non-degenerate symmetric invariant bilinear form $(\cdot, \cdot)$ on $\mathfrak{g}(A)$ such that

(B1) $(\alpha_i^\vee, h)=<\alpha_i, h>\varepsilon_i$ $(i\in I, h\in\mathfrak{h})$,

(B2) $(h', h'')=0$ $(h', h''\in\mathfrak{h}'')$.

This form is called a *standard invariant form* on $\mathfrak{g}(A)$. This induces an isomorphism $\nu: \mathfrak{h}\to\mathfrak{h}^*$, defined by $\langle\nu(h), h'\rangle=(h, h')$ $(h, h'\in\mathfrak{h})$, as well as an induced bilinear form on $\mathfrak{h}^*$. Note that

(1) $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta)=0$ $(\alpha, \beta\in\Delta\cup\{0\}, \alpha+\beta\neq 0)$,

(2) For all $x\in\mathfrak{g}_\alpha$, $y\in\mathfrak{g}_{-\alpha}$, $[x, y]=(x, y)\cdot\nu^{-1}(\alpha)$ $(\alpha\in\Delta)$,

(3) $(\cdot, \cdot)|_{\mathfrak{h}\times\mathfrak{h}}$ is non-degenerate.

It is clear that $\nu(\alpha_i^\vee)=\varepsilon_i\alpha_i$ and $(\alpha_i, \alpha_j)=b_{ij}=a_{ij}/\varepsilon_i$ $(i, j\in I)$. So we have $\alpha_i^\vee=(2/(\alpha_i, \alpha_i))\cdot\nu^{-1}(\alpha_i)$ $(i\in I)$ and $A=(2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i))_{i,j\in I}$. Moreover, we can define a Hermitian form $(\cdot, \cdot)_0$ on $\mathfrak{g}(A)$ by

$$(x, y)_0:=-(x, \omega_0(y)) \qquad (x, y\in\mathfrak{g}(A)).$$

Note that $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta)_0=0$ $(\alpha, \beta\in\Delta\cup\{0\}, \alpha\neq\beta)$ and $(\cdot, \cdot)_0|_{\mathfrak{g}_\alpha\times\mathfrak{g}_\alpha}$ is positive definite for all $\alpha\in\Delta$ (cf. [**4**, Chap. 11]).

Let $A=DB$ be the decomposition of a symmetrizable GCM A, and denote by (a, b, c) the number of positive, zero, and negative eigenvalues of the "symmetrized" matrix B. This (a, b, c) is called the signature of the standard invariant form $(\cdot, \cdot)$ defined by (B1) and (B2). Then we know the following.

PROPOSITION A ([**4**]). *Let $A=(a_{ij})_{i,j\in I}$ be an indecomposable symmetrizable* GCM, *and $(\cdot, \cdot)$ be a fixed standard invariant form on $\mathfrak{g}(A)$. Then the following hold.*

(1) *A is of finite type if and only if $(\cdot, \cdot)$ is positive definite on $\sum_{i\in I}\boldsymbol{R}\alpha_i^\vee$.*

(2) *A is of affine type if and only if $(\cdot, \cdot)$ is positive semi-definite of rank $n-1$ on $\sum_{i\in I}\boldsymbol{R}\alpha_i^\vee$. In this case, there exists $\delta=(a_i)_{i\in I}$ such that $A\delta=0$ and $a_i\in\boldsymbol{N}$ for all $i\in I$. Moreover, $\{\alpha\in Q; (\alpha, \alpha)=0\}=\boldsymbol{Z}\cdot(\sum_{i\in I}a_i\alpha_i)$.*

(3) *If A is of hyperbolic type, then the signature of $(\cdot, \cdot)$ is $(n-1, 0, 1)$. Note that the converse is not true, in general.*

1.3. Real roots and imaginary roots.

Let $A=(a_{ij})_{i,j\in I}$ be a GCM and $(\mathfrak{h}, \Pi=\{\alpha_i\}_{i\in I}, \Pi^\vee=\{\alpha_i^\vee\}_{i\in I})$ be a realization of A. Then the transposed matrix ${}^tA$ is also a GCM and $(\mathfrak{h}^*, \Pi^\vee, \Pi)$ is a realization of ${}^tA$. In this case, the root system $\Delta^\vee$ of $\mathfrak{g}({}^tA)$ is a subset of $Q^\vee$, and is called the dual root system of $\mathfrak{g}(A)$. For each $i\in I$, we define a fundamental reflection r_i of the space $\mathfrak{h}^*$ by

$$r_i(\lambda):=\lambda-\langle\lambda, \alpha_i^\vee\rangle\alpha_i \quad (\lambda\in\mathfrak{h}^*).$$

The subgroup $W(A)$ of $GL(\mathfrak{h}^*)$ generated by r_i $(i\in I)$ is called the *Weyl group* of $\mathfrak{g}(A)$. Since the subgroup $W({}^tA)$ of $GL(\mathfrak{h})$ and the subgroup $W(A)$ of $GL(\mathfrak{h}^*)$ are contragradient linear groups, we can identify these groups.

We call a root $\alpha\in\Delta$ *real* if there exists a $w\in W$ such that $w(\alpha)=\alpha_i$ for some $i\in I$, and *imaginary* if it is not real. Denote by Δ^{re} (resp. Δ^{im}) the set of all real (resp. imaginary) roots. For each real root $\alpha=w(\alpha_i)$, define the dual real root $\alpha^\vee\in(\Delta^\vee)^{re}$ by $\alpha^\vee:=w(\alpha_i^\vee)$. Note that $\alpha^\vee$ is independent of the presentation of $\alpha=w(\alpha_i)$ ($w\in W$, $i\in I$).

We know the following.

PROPOSITION B ([4]). *Let α be a real root of a Kac-Moody algebra $\mathfrak{g}(A)$. Then,*

(a) $\dim_C\mathfrak{g}_\alpha=1$,

(b) *$k\alpha\in\Delta$ for $k\in\boldsymbol{C}$ if and only if $k=\pm1$,*

(c) *for $\beta\in\Delta$, there exist non-negative integers p and q with $p-q=\langle\beta, \alpha^\vee\rangle$, such that $\beta+k\alpha\in\Delta\cup\{0\}$ if and only if $-p\leqq k\leqq q$, $k\in\boldsymbol{Z}$.*

PROPOSITION C ([4]). *Let A be a symmetrizable* GCM, *and $(\cdot, \cdot)$ be a fixed standard invariant from on $\mathfrak{g}(A)$. Then,*

(a) *$(\cdot, \cdot)|_{\mathfrak{h}^*\times\mathfrak{h}^*}$ is W-invariant,*

(b) *$[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]=\boldsymbol{C}\nu^{-1}(\alpha)$, for $\alpha\in\Delta^{re}$,*

(c) *$\alpha^\vee=(2/(\alpha, \alpha))\cdot\nu^{-1}(\alpha)$, for $\alpha\in\Delta^{re}$,*

(d) *for $\alpha\in\Delta$, $\alpha\in\Delta^{re}$ if and only if $(\alpha, \alpha)>0$, and $\alpha\in\Delta^{im}$ if and only if $(\alpha, \alpha)\leqq0$,*

(e) *for all $\alpha\in\Delta^{im}$ and $m\in\boldsymbol{Z}\setminus\{0\}$, we have $m\alpha\in\Delta^{im}$.*

§2. Subroot systems of a Kac-Moody root system.

In this section, we review the notion of a *fundamental* subset of a Kac-Moody root system introduced by Morita [8], and prove some elementary properties. In the sequel, we always denote by $A=(a_{ij})_{i,j\in I}$ a symmetrizable GCM and by ($\mathfrak{h}$, $\Pi=\{\alpha_i\}_{i\in I}$, $\Pi^\vee=\{\alpha_i^\vee\}_{i\in I}$) a realization of the GCM A.

We owe the following definition and proposition to Morita.

DEFINITION 2.1 ([8]). A subset $\tilde{\Pi}=\{\beta_1, \beta_2, \cdots, \beta_m\}$ of Δ^{re} is called *fundamental* if

(1) $\tilde{\Pi}$ is linearly independent over $\boldsymbol{C}$, and

(2) $\beta_i-\beta_j\notin\Delta\cup\{0\}$ $(1\leqq i\neq j\leqq m)$.

Let $\tilde{\Pi}=\{\beta_1, \beta_2, \cdots, \beta_m\}$ be fundamental. Put $\tilde{A}:=(\tilde{a}_{ij})_{i,j=1}^m$ with $\tilde{a}_{ij}=\langle\beta_j, \beta_i^\vee\rangle$, where $\beta_i^\vee$ is the dual real root of β_i, $1\leqq i\leqq m$.

PROPOSITION D ([8]). *The matrix $\tilde{A}$ is a* GCM.

PROPOSITION 2.1. *In case $m=2$, there exists a $w\in W$ such that either $w(\beta_1)$, $w(\beta_2)\in\Delta_+$ or $w(\beta_1)$, $w(\beta_2)\in\Delta_-$.*

PROOF. Since $\beta_1 \in \Delta^{re}$, we can assume that $\beta_1 = \alpha_i$, for some $i \in I$. Then it is enough to assume $\beta_2 \in \Delta_-$. By Proposition B, there exist non-negative integers p and q with $p-q = \langle \beta_2, \alpha_i^\vee \rangle$, such that $\beta_2 + k\alpha_i \in \Delta \cup \{0\}$ if and only if $-p \leqq k \leqq q$, $k \in \boldsymbol{Z}$. As $\beta_2 - \alpha_i \notin \Delta \cup \{0\}$ from the assumption, we have $p=0$, and so $-q = \langle \beta_2, \alpha_i^\vee \rangle$. Therefore $r_i(\beta_2) = \beta_2 - \langle \beta_2, \alpha_i^\vee \rangle \alpha_i = \beta_2 + q\alpha_i \in \Delta$. Since $\beta_2 \in \Delta_-$, $\beta_2 + q\alpha_i \in \Delta$, and $\beta_2 \neq \pm \alpha_i$, we get $\beta_2 + q\alpha_i \in \Delta_-$. Hence $r_i(\beta_2) = \beta_2 + q\alpha_i \in \Delta_-$. On the other hand, $r_i(\alpha_i) = -\alpha_i \in \Delta_-$. Thus the assertion is proved. □

PROPOSITION 2.2. *The matrix $\tilde{A}$ is symmetrizable.*

PROOF. Let $(\cdot, \cdot)$ be a standard invariant form on $\mathfrak{g}(A)$, and $\nu : \mathfrak{h} \to \mathfrak{h}^*$ be an isomorphism induced by $(\cdot, \cdot)$. Then, by Proposition C, for any $\alpha \in \Delta^{re}$, we have $\alpha^\vee = (2/(\alpha, \alpha)) \cdot \nu^{-1}(\alpha)$. Therefore, $\tilde{a}_{ij} = \langle \beta_j, \beta_i^\vee \rangle = (2/(\beta_i, \beta_i)) \cdot \langle \beta_j, \nu^{-1}(\beta_i) \rangle = (2/(\beta_i, \beta_i)) \cdot (\beta_i, \beta_j)$, $1 \leqq i, j \leqq m$. So putting $\tilde{D} := \mathrm{diag}\ (2/(\beta_1, \beta_1), \cdots, 2/(\beta_m, \beta_m))$ and $\tilde{B} := ((\beta_i, \beta_j))_{1 \leq i, j \leq m}$, we have $\tilde{A} = \tilde{D}\tilde{B}$, and $\tilde{D}$ is a positive rational diagonal matrix and $\tilde{B}$ is a symmetric rational matrix. Hence the matrix $\tilde{A}$ is symmetrizable. □

REMARK 2.1. Since $\beta_i^\vee = (2/(\beta_i, \beta_i)) \cdot \nu^{-1}(\beta_i)$, $1 \leqq i \leqq m$, $\tilde{\Pi}^\vee := \{\beta_i^\vee\}_{i=1}^m \subset \mathfrak{h}$ is linearly independent over $\boldsymbol{C}$. But, it is an open problem whether or not $\tilde{\Pi}^\vee$ is still linearly independent if we do not assume the symmetrizability of the GCM A.

REMARK 2.2. If $\tilde{\Pi}$ is a subset of Π, then $\tilde{\Pi}$ is clearly fundamental in the sense of Definition 2.1. In this case, the Kac-Moody algebra $\mathfrak{g}(\tilde{A})$ associated with the GCM $\tilde{A}$ is canonically embedded in the Kac-Moody algebra $\mathfrak{g}(A)$ (see [**5**, p. 86]). In § 3, we generalize this fact to an arbitrary fundamental subset $\tilde{\Pi} \subset \Delta^{re}$ in case the GCM A is symmetrizable.

REMARK 2.3. In [8], Morita defined a subalgebra $\hat{\mathfrak{g}}$ of $\mathfrak{g}(A)$ generated by $\mathfrak{g}_{\pm\beta_j}$ $(1 \leqq j \leqq m)$, where $\tilde{\Pi} = \{\beta_j\}_{j=1}^m \subset \Delta^{re}$ is fundamental. Then $\hat{\mathfrak{g}} = \sum_{\beta \in \tilde{Q}}^{\oplus} (\hat{\mathfrak{g}} \cap \mathfrak{g}_\beta)$ with $\tilde{Q} := \sum_{j=1}^m \boldsymbol{Z}\beta_j$. So he put $\hat{\Delta} := \{\beta \in \tilde{Q} \setminus \{0\}\,;\ \hat{\mathfrak{g}} \cap \mathfrak{g}_\beta \neq \{0\}\}$ and called it a subsystem of Δ, and then the matrix $\tilde{A} = (\langle \beta_j, \beta_i^\vee \rangle)_{i,j=1}^m$ is a GCM and he introduced the notation $\tilde{A} < A$ (cf. also Remark 3.1). Further he proved the following theorem.

THEOREM. *Let $a \in \boldsymbol{Z}$, $a \geqq 2$. Then* $\begin{bmatrix} 2 & -a \\ -1 & 2 \end{bmatrix} < A$ *if and only if* $a \in \Omega(A)$: $= \{ |a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{s-1} i_s}| \geqq 2\,;\ |a_{i_s i_{s-1}} \cdots a_{i_3 i_2} a_{i_2 i_1}| = 1,\ i_1, i_2, \cdots, i_s \in I \}$.

However the "if" part of the statement in the above theorem is not correct. In fact, after discussions with Morita, we found the following counter-example for the "if" part.

COUNTER-EXAMPLE. Let $A=\begin{bmatrix} 2 & -2 & -8 \\ -1 & 2 & -2 \\ -2 & -1 & 2 \end{bmatrix}$ and $a=4$. Then $a\in\Omega(A)$, since $a=|a_{12}a_{23}|$ and $|a_{32}a_{21}|=1$, whereas we can not find any fundamental subset $\tilde{\Pi}=\{\beta_1, \beta_2\}\subset\Delta^{re}$ such that $\langle\beta_2, \beta_1^\vee\rangle=-a$ and $\langle\beta_1, \beta_2^\vee\rangle=-1$.

We know also that, in case the Dynkin diagram of GCM A is a "tree", the "if" part remains to be true.

§3. Construction of regular subalgebras.

3.1. Construction of a realization of the GCM $\tilde{A}$.

Let the triple $(\mathfrak{h}_{\boldsymbol{R}}, \Pi=\{\alpha_i\}_{i\in I}, \Pi^\vee=\{\alpha_i^\vee\}_{i\in I})$ be a realization of the GCM A over $\boldsymbol{R}$. Then $(\mathfrak{h}=\boldsymbol{C}\otimes_{\boldsymbol{R}}\mathfrak{h}_{\boldsymbol{R}}, \Pi, \Pi^\vee)$ is a realization of A over $\boldsymbol{C}$. Let $(\cdot, \cdot)$ be a fixed standard invariant form on $\mathfrak{g}(A)$, and $\tilde{\Pi}=\{\beta_1, \beta_2, \cdots, \beta_m\}$ be a fundamental subset of Δ^{re}. We set $l=\operatorname{rank} A$ and $k=\operatorname{rank}\tilde{A}$, where $\tilde{A}=(\langle\beta_j, \beta_i^\vee\rangle)_{i,j=1}^m$. Clearly, we have $k\leqq l$.

PROPOSITION 3.1. *There exists a basis $\{h_i\}_{i=1}^{m+N}\cup\{v_i\}_{i=k+1}^m$ of $\mathfrak{h}_{\boldsymbol{R}}$ over $\boldsymbol{R}$, such that the presentation matrix R of $(\cdot, \cdot)$ with respect to this basis is of the form*

$$R=\begin{bmatrix} J_1 & O & O & O \\ O & O_{m-k} & O & I_{m-k} \\ O & O & J_2 & O \\ O & I_{m-k} & O & O_{m-k} \end{bmatrix},$$

where I_{m-k} is the identity matrix of degree $m-k$, O_{m-k} is the 0-matrix of degree $m-k$, $J_1=\operatorname{diag}(\pm1, \pm1, \cdots, \pm1)$: $k\times k$-matrix, and $J_2=\operatorname{diag}(\pm1, \pm1, \cdots, \pm1)$: $N\times N$-matrix, $N:=(2n-l)-(2m-k)\geqq 0$.

PROOF. STEP 1. We put $\tilde{H}:=\sum_{j=1}^m\boldsymbol{R}\beta_j^\vee$, and take $\{h_1, h_2, \cdots, h_m\}$ as a basis of $\tilde{H}$ over $\boldsymbol{R}$ such that the presentation matrix of $(\cdot, \cdot)$ with respect to this basis is as

$$\begin{bmatrix} J_1 & O \\ O & O_{m-k} \end{bmatrix}: m\times m\text{-matrix, with } J_1 \text{ and } O_{m-k} \text{ as above.}$$

To take such a basis, first take $h_1'\in\tilde{H}$ such that $(h_1', h_1')\neq 0$ (for instance, take $\beta_1^\vee$). Here since $(h_1', h_1')\in\boldsymbol{R}^\times$, by normalizing h_1', we can take $h_1\in\tilde{H}$ such that $(h_1, h_1)=\pm1$. Then we have $\tilde{H}=\boldsymbol{R}h_1\oplus(\boldsymbol{R}h_1)^\perp$, where $(\boldsymbol{R}h_1)^\perp:=\{h\in\tilde{H}; (h, h_1)=0\}$. Next, if there is $h_2'\in(\boldsymbol{R}h_1)^\perp$ such that $(h_2', h_2')\neq 0$, then we can get $h_2\in(\boldsymbol{R}h_1)^\perp$ such that $(h_2, h_2)=\pm1$ by normalizing h_2'. Then we have $\tilde{H}=\langle h_1, h_2\rangle_{\boldsymbol{R}}\oplus(\langle h_1, h_2\rangle_{\boldsymbol{R}})^\perp$, where $\langle h_1, h_2\rangle_{\boldsymbol{R}}:=\boldsymbol{R}h_1+\boldsymbol{R}h_2$. By repeating this procedure, we can finally get some $k'\in\boldsymbol{N}$, such that $\tilde{H}=\langle h_1, \cdots, h_{k'}\rangle_{\boldsymbol{R}}\oplus(\langle h_1, \cdots,$

$h_{k'}\rangle_{R})^{\perp}$ and $(h, h)=0$ for any $h\in(\langle h_1, \cdots, h_{k'}\rangle_{R})^{\perp}$. Note that $(h, h')=0$ for any $h, h'\in(\langle h_1, \cdots, h_{k'}\rangle_{R})^{\perp}$.

By taking a basis of $(\langle h_1, \cdots, h_{k'}\rangle_{R})^{\perp}$ and combining it with $\{h_1, \cdots, h_{k'}\}$, we get a basis of $\tilde{H}$ such that the presentation matrix of $(\cdot, \cdot)$ with respect to it is as follows, with $J_1'=\operatorname{diag}(\pm 1, \cdots, \pm 1)$: $k'\times k'$-matrix,

$$\begin{bmatrix} J_1' & O \\ O & O_{m-k'} \end{bmatrix}: m\times m\text{-matrix}.$$

On the other hand, the presentation matrix of $(\cdot, \cdot)$ with respect to the basis $\{\beta_j^{\vee}\}_{j=1}^{m}$ is $\tilde{D}^{-1}\tilde{A}$, where $\tilde{D}=\operatorname{diag}(2/(\beta_1, \beta_1), \cdots, 2/(\beta_m, \beta_m))$ and $\operatorname{rank}\tilde{D}^{-1}\tilde{A}$ $=\operatorname{rank}\tilde{A}=k$. Therefore we get $k'=k$. So if we take a basis $\{h_{k+1}, \cdots, h_m\}$ of $(\langle h_1, \cdots, h_k\rangle_{R})^{\perp}$, then $\{h_1, \cdots, h_k, h_{k+1}, \cdots, h_m\}$ is a desired basis of $\tilde{H}$ over $\boldsymbol{R}$.

STEP 2. Let $\tilde{H}_1:=\langle h_1, \cdots, h_k\rangle_{R}$. Then $(\cdot, \cdot)$ is non-degenerate on $\tilde{H}_1$ as well as on $\mathfrak{h}_{R}$. Therefore $\mathfrak{h}_{R}=\tilde{H}_1\oplus(\tilde{H}_1)^{\perp}$, and $(\cdot, \cdot)$ is non-degenerate also on $(\tilde{H}_1)^{\perp}$, where $(\tilde{H}_1)^{\perp}:=\{h\in\mathfrak{h}_{R}; (h, \tilde{H}_1)=0\}$.

Let $\tilde{H}_2:=(\tilde{H}_1)^{\perp}$. Note that $h_{k+1}, \cdots, h_m\in\tilde{H}_2$ by the definition of $\tilde{H}_2$. Further we recall $(h_i, h_j)=0$ $(k+1\leqq i, j\leqq m)$. So there exist $v_{k+1}, \cdots, v_m\in\tilde{H}_2$ such that $(v_i, h_j)=\delta_{ij}$ and $(v_i, v_j)=0$ $(k+1\leqq i, j\leqq m)$ by Cor. 1 of Prop. 2 in [2, §4].

Then $\{v_i, h_i\}_{i=k+1}^{m}$ is linearly independent over $\boldsymbol{R}$ and $(\cdot, \cdot)$ is non-degenerate on $\tilde{H}_3:=\langle v_i, h_i(k+1\leqq i\leqq m)\rangle_{R}$. Therefore $\tilde{H}_2=\tilde{H}_3\oplus\tilde{H}_3^{\perp}$ and $(\cdot, \cdot)$ is non-degenerate on $\tilde{H}_3^{\perp}$, where $\tilde{H}_3^{\perp}:=\{h\in\tilde{H}_2; (h, \tilde{H}_3)=0\}$. So if we set $\tilde{H}_4:=\tilde{H}_3^{\perp}$, then $\mathfrak{h}_{R}=\tilde{H}_1\oplus\tilde{H}_3\oplus\tilde{H}_4$, where the sum is an orthogonal direct sum.

Here $\dim_{R}\tilde{H}_4=\dim_{R}\mathfrak{h}_{R}-(\dim_{R}\tilde{H}_1+\dim_{R}\tilde{H}_3)=(2n-l)-(k+2(m-k))=(2n-l)-(2m-k)=N$. Since $(\cdot, \cdot)$ is non-degerate on $\tilde{H}_4$, we can construct, as in Step 1, a basis $\{h_{m+1}, \cdots, h_{m+N}\}$ of $\tilde{H}_4$ such that the presentation matrix of $(\cdot, \cdot)$ with respect to this basis is J_2. So $\{h_i\}_{i=1}^{m+N}\cup\{v_i\}_{i=k+1}^{m}$ is the basis of $\mathfrak{h}_{R}$ over $\boldsymbol{R}$, which is desired. Thus the proposition is now proved. □

Now let $\tilde{\mathfrak{h}}_{R}:=\langle h_1, \cdots, h_m, v_{k+1}, \cdots, v_m\rangle_{R}=\tilde{H}_1\oplus\tilde{H}_3$. Then we have the following proposition.

PROPOSITION 3.2. *The triple* $(\tilde{\mathfrak{h}}_{R}, \{\beta_j|_{\tilde{\mathfrak{h}}_{R}}\}_{j=1}^{m}, \{\beta_j^{\vee}\}_{j=1}^{m})$ *is a realization of the* GCM $\tilde{A}$ *over* $\boldsymbol{R}$.

PROOF. First, note that $\sum_{i=1}^{m}\boldsymbol{R}\beta_i^{\vee}=\tilde{H}=\langle h_1, \cdots, h_k, h_{k+1}, \cdots, h_m\rangle_{R}\subset\tilde{\mathfrak{h}}_{R}$. Next, $\dim_{R}\tilde{\mathfrak{h}}_{R}=\dim_{R}\tilde{H}_1+\dim_{R}\tilde{H}_3=k+2(m-k)=2m-k$. Then we have only to show that $\{\beta_i|_{\tilde{\mathfrak{h}}_{R}}\}_{i=1}^{m}$ is linearly independent.

CLAIM. Let $\tilde{C}_{R}:=\{h\in\mathfrak{h}_{R}; \langle\beta_j, h\rangle=0, \text{ for } j=1, \cdots, m\}$. Then $\tilde{C}_{R}=\langle h_{k+1}, \cdots, h_m, h_{m+1}, \cdots, h_{m+N}\rangle_{R}$.

PROOF OF THE CLAIM. For each i $(1 \leqq i \leqq m)$ and $h \in \mathfrak{h}_{\boldsymbol{R}}$,

$$\langle \beta_i, h \rangle = (\nu^{-1}(\beta_i), h) = \frac{1}{(\beta_i, \beta_i)} \cdot (\beta_i^\vee, h).$$

Therefore for all j $(k+1 \leqq j \leqq m+N)$, $\langle \beta_i, h_j \rangle = 0$ as $(\beta_i^\vee, h_j)=0$ by the construction in Proposition 3.1. So we have

$$\langle h_{k+1}, \cdots, h_m, h_{m+1}, \cdots, h_{m+N} \rangle_{\boldsymbol{R}} \subset \tilde{C}_{\boldsymbol{R}}.$$

On the other hand, $\dim_{\boldsymbol{R}} \tilde{C}_{\boldsymbol{R}} = \dim_{\boldsymbol{R}} \mathfrak{h}_{\boldsymbol{R}} - m = (2n-l)-m$, and this is equal to the dimension of $\langle h_{k+1}, \cdots, h_m, h_{m+1}, \cdots, h_{m+N} \rangle_{\boldsymbol{R}}$. Thus we get the claim.

From the claim, we have $\mathfrak{h}_{\boldsymbol{R}} = \tilde{\mathfrak{h}}_{\boldsymbol{R}} + \tilde{C}_{\boldsymbol{R}}$ (not necessarily direct). Therefore $\{\beta_j|_{\tilde{\mathfrak{h}}_{\boldsymbol{R}}}\}_{j=1}^m$ is linearly independent, because $\{\beta_j\}_{j=1}^m$ is originally linearly independent. So the proposition is proved. □

Putting $\tilde{\mathfrak{h}} := \boldsymbol{C} \otimes_{\boldsymbol{R}} \tilde{\mathfrak{h}}_{\boldsymbol{R}}$, we thus get the triple $(\tilde{\mathfrak{h}}, \{\beta_j|_{\tilde{\mathfrak{h}}}\}_{j=1}^m, \{\beta_j^\vee\}_{j=1}^m)$ as a realization of the GCM $\tilde{A}$ over $\boldsymbol{C}$.

We note that if we merely want to construct the Cartan subalgebra $\tilde{\mathfrak{h}}$ of $\mathfrak{g}(\tilde{A})$ as a subspace of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}(A)$, in such a way that the triple $(\tilde{\mathfrak{h}}, \{\beta_j|_{\tilde{\mathfrak{h}}}\}_{j=1}^m, \{\beta_j^\vee\}_{j=1}^m)$ is a realization of the GCM $\tilde{A}$, it is sufficient to give a much simpler construction. But, we have taken into account the proof of Theorem 3.7, so we constructed $\tilde{\mathfrak{h}}$ in this detailed way.

Note further that $\omega_0(\tilde{\mathfrak{h}}) = \tilde{\mathfrak{h}}$, where ω_0 is the compact involution of $\mathfrak{g}(A)$.

3.2. Construction of a regular subalgebra of $\mathfrak{g}(A)$**.**

Let the triple $(\tilde{\mathfrak{h}}, \{\beta_j|_{\tilde{\mathfrak{h}}}\}_{j=1}^m, \{\beta_j^\vee\}_{j=1}^m)$ be the realization of $\tilde{A}$ constructed in 3.1. And let $\mathfrak{g}_{\beta_j}$ (resp. $\mathfrak{g}_{-\beta_j}$) be the root space attached to β_j (resp. $-\beta_j$), and $(\cdot, \cdot)$ be a fixed standard invariant form on $\mathfrak{g}(A)$. By Theorem 11.7 in [**4**], the restriction $(\cdot, \cdot)_0|_{\mathfrak{g}_\alpha \times \mathfrak{g}_\alpha}$ is positive definite for all $\alpha \in \Delta$, so we can choose $E_i \in \mathfrak{g}_{\beta_i}$ such that $(E_i, E_i)_0 = 2/(\beta_i, \beta_i)$ for each i $(1 \leqq i \leqq m)$. Set $F_i := -\omega_0(E_i)$, then $F_i \in \mathfrak{g}_{-\beta_i}$ and $[E_i, F_i] = (E_i, F_i) \cdot \nu^{-1}(\beta_i) = (2/(\beta_i, \beta_i)) \cdot ((\beta_i, \beta_i)/2) \cdot \beta_i^\vee = \beta_i^\vee$ $(1 \leqq i \leqq m)$. Note that $\mathfrak{g}_{\beta_i} = \boldsymbol{C} E_i$ and $\mathfrak{g}_{-\beta_i} = \boldsymbol{C} F_i$ since $\beta_i \in \Delta^{re}$ $(1 \leqq i \leqq m)$.

Let $\tilde{\mathfrak{n}}_+$ (resp. $\tilde{\mathfrak{n}}_-$) be a subalgebra of $\mathfrak{g}(A)$ generated by E_i, $1 \leqq i \leqq m$ (resp. F_i, $1 \leqq i \leqq m$), and $\tilde{\mathfrak{g}}$ be a subalgebra of $\mathfrak{g}(A)$ generated by E_i, F_i $(1 \leqq i \leqq m)$, and $\tilde{\mathfrak{h}}$.

LEMMA 3.3. *$\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}_+ \oplus \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}_-$. Put $\tilde{Q}_+ := \sum_{j=1}^m \boldsymbol{Z}_+ \beta_i$, then*

$$\tilde{\mathfrak{n}}_+ = \textstyle\sum^{\oplus}_{\beta \in \tilde{Q}_+ \setminus \{0\}} (\tilde{\mathfrak{g}} \cap \mathfrak{g}_\beta), \qquad \tilde{\mathfrak{n}}_- = \textstyle\sum^{\oplus}_{\beta \in \tilde{Q}_+ \setminus \{0\}} (\tilde{\mathfrak{g}} \cap \mathfrak{g}_{-\beta}).$$

PROOF. This follows easily from the assumption that $\beta_i - \beta_j \notin \Delta \cup \{0\}$ $(1 \leqq i \neq j \leqq m)$. □

LEMMA 3.4. *We have the following decompositions of $\tilde{\mathfrak{g}}$.*

(I) $\tilde{\mathfrak{g}}=\sum^{\oplus}_{\alpha\in Q_+\setminus\{0\}}(\tilde{\mathfrak{g}}\cap\mathfrak{g}_\alpha)\oplus(\tilde{\mathfrak{g}}\cap\mathfrak{h})\oplus\sum^{\oplus}_{\alpha\in Q_+\setminus\{0\}}(\tilde{\mathfrak{g}}\cap\mathfrak{g}_{-\alpha})$.

(II) $\tilde{\mathfrak{g}}=\sum^{\oplus}_{\beta\in\tilde{Q}_+\setminus\{0\}}(\tilde{\mathfrak{g}}\cap\mathfrak{g}_\beta)\oplus\tilde{\mathfrak{h}}\oplus\sum^{\oplus}_{\beta\in\tilde{Q}_+\setminus\{0\}}(\tilde{\mathfrak{g}}\cap\mathfrak{g}_{-\beta})$.

PROOF. Taking Lemma 3.3 into account, we have to prove only the decomposition (I). Again by Lemma 3.3, we see that $\tilde{\mathfrak{g}}=\tilde{\mathfrak{n}}_+\oplus\tilde{\mathfrak{h}}\oplus\tilde{\mathfrak{n}}_-$ is $\mathfrak{h}$-invariant. Therefore by Proposition 1.5 in [4], the decomposition (I) follows from the root space decomposition: $\mathfrak{g}(A)=\sum^{\oplus}_{\alpha\in\Delta}\mathfrak{g}_\alpha\oplus\mathfrak{h}$. □

Now we set for $\beta\in\tilde{Q}=\sum_{j=1}^{m}\boldsymbol{Z}\beta_j$,

$$\tilde{\mathfrak{g}}_\beta:=\{x\in\tilde{\mathfrak{g}}\,;\,[h,x]=\langle\beta,h\rangle x\,,\quad\text{for all }h\in\tilde{\mathfrak{h}}\}.$$

Since $\tilde{\mathfrak{h}}$ is a subspace of $\mathfrak{h}$, we can define a surjective map $P\colon\mathfrak{h}^*\to\tilde{\mathfrak{h}}^*$ by restricting a linear form on $\mathfrak{h}$ to $\tilde{\mathfrak{h}}$. Because $\{\beta_j|_{\tilde{\mathfrak{h}}}\}_{j=1}^{m}$ is linearly independent, the restriction of the map P to $\tilde{Q}=\sum_{j=1}^{m}\boldsymbol{Z}\beta_j$ is injective. So by identifying $\tilde{Q}$ with $P(\tilde{Q})=\sum_{j=1}^{m}\boldsymbol{Z}(\beta_j|_{\tilde{\mathfrak{h}}})$ through P, we can regard $\tilde{Q}$ as a subset of $\tilde{\mathfrak{h}}^*$.

LEMMA 3.5. *For all $\beta\in\tilde{Q}\subset\tilde{\mathfrak{h}}^*$, we have $\tilde{\mathfrak{g}}\cap\mathfrak{g}_\beta=\tilde{\mathfrak{g}}_\beta$.*

PROOF. Clearly $\tilde{\mathfrak{g}}\cap\mathfrak{g}_\beta\subset\tilde{\mathfrak{g}}_\beta$ and $\tilde{\mathfrak{g}}=\sum_{\beta\in\tilde{Q}}\tilde{\mathfrak{g}}_\beta$ is a direct sum, so that $\tilde{\mathfrak{g}}\cap\mathfrak{g}_\beta=\tilde{\mathfrak{g}}_\beta$ for all $\beta\in\tilde{Q}$. □

THEOREM 3.6. *The subalgebra $\tilde{\mathfrak{g}}$ of $\mathfrak{g}(A)$ is canonically isomorphic to $\mathfrak{g}(\tilde{A})$, where $\mathfrak{g}(\tilde{A})$ is a Kac-Moody algebra associated with $\tilde{A}$.*

PROOF. We already have the following relations:

$$[E_i,F_j]=\delta_{ij}\beta_i^\vee\qquad(1\leqq i,\,j\leqq m),$$

$$[h,E_j]=\langle\beta_j,h\rangle E_j,\quad[h,F_j]=-\langle\beta_j,h\rangle F_j\quad(1\leqq j\leqq m,\,h\in\tilde{\mathfrak{h}}),$$

$$[h,h']=0\qquad(h,h'\in\tilde{\mathfrak{h}}).$$

So we have only to prove the following claim.

CLAIM. $\tilde{\mathfrak{g}}$ has no non-zero ideals which intersect $\tilde{\mathfrak{h}}$ trivially.

PROOF OF THE CLAIM. Let ι be an ideal of $\tilde{\mathfrak{g}}$ which intersects $\tilde{\mathfrak{h}}$ trivially. By Lemmas 3.4 and 3.5, we have the decomposition $\tilde{\mathfrak{g}}=\sum^{\oplus}_{\beta\in\tilde{Q}_+\setminus\{0\}}\tilde{\mathfrak{g}}_\beta\oplus\tilde{\mathfrak{h}}\oplus\sum^{\oplus}_{\beta\in\tilde{Q}_+\setminus\{0\}}\tilde{\mathfrak{g}}_{-\beta}$. Since ι is $\tilde{\mathfrak{h}}$-invariant, we get $\iota=\sum^{\oplus}_{\beta\in\tilde{Q}_+\setminus\{0\}}(\iota\cap\tilde{\mathfrak{g}}_\beta)\oplus\sum^{\oplus}_{\beta\in\tilde{Q}_+\setminus\{0\}}(\iota\cap\tilde{\mathfrak{g}}_{-\beta})$. (Note that $\iota\cap\tilde{\mathfrak{h}}=\{0\}$ by assumption.)

Now let us prove that $\iota\cap\tilde{\mathfrak{g}}_\beta=\{0\}$, for all $\beta\in\tilde{Q}_+\setminus\{0\}$. (For $\beta\in-\tilde{Q}_+\setminus\{0\}$, the proof is similar.) First we recall that $\omega_0(\tilde{\mathfrak{h}})=\tilde{\mathfrak{h}}$, $\omega_0(\mathfrak{g}_{\beta_j})=\mathfrak{g}_{-\beta_j}$, $\omega_0(\mathfrak{g}_{-\beta_j})=\mathfrak{g}_{\beta_j}$ $(1\leqq j\leqq m)$. Therefore we have $\omega_0(\tilde{\mathfrak{g}})=\tilde{\mathfrak{g}}$ by the definition of $\tilde{\mathfrak{g}}$. Suppose that there exist $\beta\in\tilde{Q}_+\setminus\{0\}$ and $X\in\iota\cap\tilde{\mathfrak{g}}_\beta$, $X\neq0$. Then $X\in\tilde{\mathfrak{g}}_\beta=\tilde{\mathfrak{g}}\cap\mathfrak{g}_\beta\subset\mathfrak{g}_\beta$, so $\mathfrak{g}_\beta\neq\{0\}$ and $\beta\in\Delta$, where Δ is the root system of $\mathfrak{g}(A)$. Since $(\cdot,\cdot)_0|_{\mathfrak{g}_\beta\times\mathfrak{g}_\beta}$ is positive definite, we have $0<(X,X)_0=-(X,\omega_0(X))$. Here $\omega_0(X)\in\omega_0(\tilde{\mathfrak{g}})\cap\omega_0(\mathfrak{g}_\beta)=\tilde{\mathfrak{g}}\cap\mathfrak{g}_{-\beta}$

$=\tilde{\mathfrak{g}}_{-\beta}$, because $X\in\tilde{\mathfrak{g}}_{\beta}=\tilde{\mathfrak{g}}\cap\mathfrak{g}_{\beta}$.

On the other hand, as $X\in\mathfrak{g}_{\beta}$ and $\omega_0(X)\in\mathfrak{g}_{-\beta}$, we have

$$[X, \omega_0(X)] = (X, \omega_0(X))\cdot\nu^{-1}(\beta) = -(X, X)_0\cdot\nu^{-1}(\beta)\in\mathfrak{h}\,.$$

Since $\beta\neq 0$, we get $\nu^{-1}(\beta)\neq 0$ and $[X, \omega_0(X)]\neq 0$. However, since $X\in\iota$, an ideal of $\tilde{\mathfrak{g}}$, and $\omega_0(X)\in\tilde{\mathfrak{g}}$, we get $[X, \omega_0(X)]\in\iota$. Therefore, $0\neq[X, \omega_0(X)]\in\iota$ and $[X, \omega_0(X)]\in\tilde{\mathfrak{g}}\cap\mathfrak{h}=\tilde{\mathfrak{h}}$. This contradicts the assumption $\iota\cap\tilde{\mathfrak{h}}=\{0\}$. Thus the claim has been proved.

This completes the proof of the theorem. □

DEFINITION 3.1. The subalgebra $\tilde{\mathfrak{g}}$ of $\mathfrak{g}(A)$ in Theorem 3.6, which is generated by $\mathfrak{g}_{\beta_j}$, $\mathfrak{g}_{-\beta_j}$ $(1\leq j\leq m)$, and $\tilde{\mathfrak{h}}$, is called a *regular subalgebra* of $\mathfrak{g}(A)$.

REMARK 3.1. It is easy to see that $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]=\hat{\mathfrak{g}}$, where $\hat{\mathfrak{g}}$ is Morita's subalgebra defined in [8] (see Remark 2.3). So we have a canonical isomorphism: $\hat{\mathfrak{g}}\cong[\mathfrak{g}(\tilde{A}), \mathfrak{g}(\tilde{A})]$. It is an open problem whether or not this isomorphism still exists if we do not assume the symmetrizability of the GCM A.

3.3. The inheritance of a standard invariant form.

Since we have constructed the Cartan subalgebra $\tilde{\mathfrak{h}}$ of $\mathfrak{g}(\tilde{A})$ as in 3.1, we can easily prove the following theorem.

THEOREM 3.7. *Let A be a symmetrizable* GCM, *and $\{\beta_j\}_{j=1}^m\subset\Delta^{re}$ be a fundamental subset. Let the triple $(\tilde{\mathfrak{h}}, \{\beta_j|_{\tilde{\mathfrak{h}}}\}_{j=1}^m, \{\beta_j^\vee\}_{j=1}^m)$ be a realization of the* GCM *$\tilde{A}=(\langle\beta_j, \beta_i^\vee\rangle)_{i,j=1}^m$ in Proposition* 3.2. *Further let $\tilde{\mathfrak{g}}$ be the regular subalgebra of $\mathfrak{g}(A)$ corresponding to β_j $(1\leq j\leq m)$, $\tilde{\mathfrak{h}}$. Set*

$$\tilde{B}:=((\beta_i, \beta_j))_{i,j=1}^m, \qquad \tilde{D}:=\operatorname{diag}\left(\frac{2}{(\beta_1, \beta_1)}, \cdots, \frac{2}{(\beta_m, \beta_m)}\right),$$

where $(\cdot, \cdot)$ is a fixed standard invariant form on $\mathfrak{g}(A)$. Then the restriction of $(\cdot, \cdot)$ to $\tilde{\mathfrak{g}}\subset\mathfrak{g}(A)$ coincides with a standard invariant from on $\tilde{\mathfrak{g}}$, canonically identified with $\mathfrak{g}(\tilde{A})$, which corresponds, according to (B1)-(B2), *to the decomposition $\tilde{A}=\tilde{D}\tilde{B}$.*

PROOF. We see from the construction in Propositions 3.1 and 3.2 that $\tilde{\mathfrak{h}}=\sum_{j=1}^m\boldsymbol{C}\beta_j^\vee\oplus\langle v_{k+1}, \cdots, v_m\rangle_{\boldsymbol{C}}$. So we take $\tilde{\mathfrak{h}}'':=\langle v_{k+1}, \cdots, v_m\rangle_{\boldsymbol{C}}$ as a complementary subspace to $\tilde{\mathfrak{h}}':=\sum_{j=1}^m\boldsymbol{C}\beta_j^\vee$ in $\tilde{\mathfrak{h}}$. Then we have the following from the construction in Propositions 3.1 and 3.2.

(1) $(\beta_j^\vee, h)=\langle\beta_j, h\rangle\cdot\dfrac{2}{(\beta_j, \beta_j)} \qquad (h\in\tilde{\mathfrak{h}},\ 1\leq j\leq m)$,

(2) $(h', h'')=0 \qquad (h', h''\in\langle v_{k+1}, \cdots, v_m\rangle_{\boldsymbol{C}})$.

Thus the theorem has been proved. □

Corresponding to the notations Δ^{re} and Δ^{im} for the root system Δ of $\mathfrak{g}(A)$, we denote by $\tilde{\Delta}^{re}$ (resp. $\tilde{\Delta}^{im}$) the set of all real (resp. imaginary) roots for the root system $\tilde{\Delta}$ of $\tilde{\mathfrak{g}}\cong\mathfrak{g}(\tilde{A})$. We saw in 3.2 that $\tilde{\mathfrak{g}}=\sum^{\oplus}_{\beta\in\tilde{Q}_+\setminus\{0\}}\tilde{\mathfrak{g}}_{\beta}\oplus\tilde{\mathfrak{h}}\oplus\sum^{\oplus}_{\beta\in\tilde{Q}_+\setminus\{0\}}\tilde{\mathfrak{g}}_{-\beta}$, and $\tilde{\mathfrak{g}}_{\beta}=\tilde{\mathfrak{g}}\cap\mathfrak{g}_{\beta}$ for $\beta\in\tilde{Q}$. So we can regard $\tilde{\Delta}$ as a subset of Δ by identifying $\beta_j|_{\tilde{\mathfrak{h}}}$ with β_j, $1\leqq j\leqq m$. Then we have the following as a direct consequence of Theorem 3.7.

THEOREM 3.8. *The notations are the same as in Theorem* 3.7. *Then we have,*

$$\tilde{\Delta}^{re}=\tilde{\Delta}\cap\Delta^{re}, \qquad \tilde{\Delta}^{im}=\tilde{\Delta}\cap\Delta^{im}.$$

PROOF. For $\alpha\in\Delta$ (resp. $\tilde{\Delta}$), we know that $\alpha\in\Delta^{re}$ (resp. $\tilde{\Delta}^{re}$) if and only if $(\alpha, \alpha)>0$, and that $\alpha\in\Delta^{im}$ (resp. $\tilde{\Delta}^{im}$) if and only if $(\alpha, \alpha)\leqq 0$, by Proposition C. Here $(\cdot, \cdot)$ is a standard invariant form on $\mathfrak{g}(A)$ (resp. $\mathfrak{g}(\tilde{A})$). Therefore the assertion immediately follows from Theorem 3.7. □

REMARK 3.2. As we have seen above, $\tilde{\Delta}\subset\Delta\cap\tilde{Q}$. In case $\tilde{\Pi}=\{\beta_j\}_{j=1}^{m}$ is a subset of $\Pi=\{\alpha_i\}_{i\in I}$, the equality $\tilde{\Delta}=\Delta\cap\tilde{Q}$ holds. So we have the following question, which is discussed in §4.

QUESTION 3.1. *Does this equality always hold with an arbitrary fundamental subset $\tilde{\Pi}$ of Δ^{re}?*

§4. Type of the GCM $\tilde{A}=(\langle\beta_j, \beta_i^{\vee}\rangle)_{i,j=1}^{m}$.

4.1. General results.

THEOREM 4.1. *Let $A=(a_{ij})_{i,j\in I}$ be a* GCM *of affine type, and $\tilde{\Pi}=\{\beta_i\}_{i=1}^{m}\subset\Delta^{re}$ be fundamental. Put $\tilde{A}:=(\langle\beta_j, \beta_i^{\vee}\rangle)_{i,j=1}^{m}$. Then, $\tilde{A}$ is a direct sum of generalized Cartan matrices of finite type or of affine type. Moreover, the number of direct summands of affine type is at most one.*

PROOF. This will follow immediately from Theorem 3.7 and Proposition A. □

Assume the GCM A is of hyperbolic type, then the signature of the restriction to $\sum_{j=1}^{m}\boldsymbol{R}\beta_j^{\vee}$ of a standard invariant form $(\cdot, \cdot)$ on $\mathfrak{g}(A)$ is $(m, 0, 0)$, $(m-1, 1, 0)$, or $(m-1, 0, 1)$ (cf. Moody [**6**]). Therefore, in view of Proposition A, the following question naturally arises.

QUESTION 4.1. *Do the direct summands of $\tilde{A}$ remain inside of three types, finite, affine, or hyperbolic if A is of hyperbolic type?*

We have the following example, which answers Questions 3.1 and 4.1.

EXAMPLE 4.1 (cf. Moody and Pianzola [**7**]).

Let A be a 3×3-matrix given below. Then A is a GCM of hyperbolic type with the Dynkin diagram below.

$$A=\begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix},$$

Put $\beta_1:=(r_3r_2)(\alpha_1)$, $\beta_2:=r_1(\beta_1)$, and $\beta_3:=r_2(\beta_2)$, where r_i $(1\leqq i\leqq 3)\in W$ are fundamental reflections. Here W denotes $W(A)\cong W({}^tA)$ identified as contragradient linear groups. Then

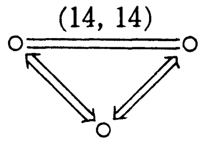
$$\beta_1=\alpha_1+2\alpha_2+2\alpha_3,\quad \beta_2=3\alpha_1+2\alpha_2+2\alpha_3,\quad \beta_3=3\alpha_1+6\alpha_2+2\alpha_3\,,$$

$$\beta_1^\vee=\alpha_1^\vee+2\alpha_2^\vee+2\alpha_3^\vee,\quad \beta_2^\vee=3\alpha_1^\vee+2\alpha_2^\vee+2\alpha_3^\vee,\quad \beta_3^\vee=3\alpha_1^\vee+6\alpha_2^\vee+2\alpha_3^\vee.$$

In this case $\{\beta_1, \beta_2, \beta_3\}\subset\Delta^{re}$ is fundamental because $\beta_2-\beta_1=2\alpha_1$, $\beta_3-\beta_2=4\alpha_2$, $\beta_3-\beta_1=2(\alpha_1+2\alpha_2)$, where $\alpha_1+2\alpha_2=r_2(\alpha_1)\in\Delta^{re}$ (cf. Proposition B).

We get the GCM $\tilde{A}$ and its Dynkin diagram as follows.

$$\tilde{A}=\begin{bmatrix} 2 & -2 & -2 \\ -2 & 2 & -14 \\ -2 & -14 & 2 \end{bmatrix},$$



Obviously $\tilde{A}$ is neither of finite type, of affine type, nor of hyperbolic type. This is mainly because a GCM of hyperbolic type can not be characterized only by the signature of a standard invariant form on the Kac-Moody algebra associated with that GCM (cf. Proposition A).

Further in this example, we have the inequality $\tilde{Q}\cap\Delta\supsetneqq\tilde{\Delta}$. In fact,

$$2\beta_1-\beta_2-\beta_3=-4(\alpha_1+\alpha_2)\in\tilde{Q}\,,\qquad (\alpha_1+\alpha_2, \alpha_1+\alpha_2)=0\,.$$

Therefore, $\alpha_1+\alpha_2\in\Delta^{im}$ and $-4(\alpha_1+\alpha_2)\in\Delta^{im}$ since A is of hyperbolic type (see [**4**, Chap. 5]). Hence $2\beta_1-\beta_2-\beta_3\in\tilde{Q}\cap\Delta^{im}$. But, since it does not belong to either $\tilde{Q}_+$ or $-\tilde{Q}_+$, we have $2\beta_1-\beta_2-\beta_3\notin\tilde{\Delta}$.

4.2. Case of affine type GCM. In this subsection, we assume that the GCM $A=(a_{ij})_{0\leq i,j\leq l}$ is of non-twisted affine type (cf. [**4**, Chap. 4]). So, there exists $\delta=(a_i)_{i=0}^l$ such that $A\delta=0$ and $a_i\in\boldsymbol{N}$ for all i $(0\leqq i\leqq l)$. Such a δ is unique under the condition that the a_i's are relatively prime integers. So we take this δ, and also denote $\sum_{i=0}^l a_i\alpha_i$ by δ. Then, we know the following facts (cf. [**4**, Chap. 6]):

$$\Delta^{im} = \{k\delta;\ k\in \boldsymbol{Z}\setminus\{0\}\}, \qquad \Delta^{re} = \{\gamma+k\delta;\ \gamma\in\mathring{\Delta},\ k\in\boldsymbol{Z}\},$$

where $\mathring{\Delta}$ is the root system of the finite type Kac-Moody algebra $\mathfrak{g}(\mathring{A})\subset\mathfrak{g}(A)$ associated with the principal submatrix $\mathring{A}=(a_{ij})_{1\leq i,j\leq l}$ of A. Note that the choice of the removed vertex 0 of the Dynkin diagram of A is so made that $a_0=1$ and the type of $\mathring{A}$ is X_l when the type of the GCM A is $X_l^{(1)}$. Therefore, we have

$$\mathring{\Delta}\subset\mathring{Q}:=\textstyle\sum_{i=1}^{l}\boldsymbol{Z}\alpha_i, \qquad \delta=\alpha_0+\sum_{i=1}^{l}a_i\alpha_i.$$

Hence, if we take $\beta_j\in\Delta^{re}$ $(1\leqq j\leqq m)$ and express them as $\beta_j=\gamma_j+k_j\delta$ $(\gamma_j\in\mathring{\Delta},$ $k_j\in\boldsymbol{Z})$, then the following are equivalent:

(1) $\beta_i-\beta_j\notin\Delta\cup\{0\}$ $(1\leqq i\neq j\leqq m)$;

($\mathring{1}$) $\gamma_i-\gamma_j\notin\mathring{\Delta}\cup\{0\}$ $(1\leqq i\neq j\leqq m)$.

Since $\mathring{A}$ is of finite type, we know that the Dynkin diagram of $\{\gamma_j\}_{j=1}^m\subset\mathring{\Delta}$ is of finite type or of non-twisted affine type if it is connected and $\{\gamma_j\}_{j=1}^m\subset\mathring{\Delta}$ satisfies the condition ($\mathring{1}$) (cf. [**3**, Chap. II, §5]). So, if $\{\gamma_j\}_{j=1}^m\subset\mathring{\Delta}$ satisfies the condition ($\mathring{1}$), its Dynkin diagram is a disjoint union of those of finite type or of non-twisted affine type. Note that if all the connected components are of finite type, then $\{\gamma_j\}_{j=1}^m\subset(\mathring{\mathfrak{h}})^*$ is linearly independent, where $\mathring{\mathfrak{h}}:=\sum_{i=1}^{l}\boldsymbol{C}\alpha_i^\vee$ is a Cartan subalgebra of $\mathfrak{g}(\mathring{A})\subset\mathfrak{g}(A)$.

Now, let $(\cdot,\cdot)$ be a standard invariant form on $\mathfrak{g}(A)$. Then the restriction of $(\cdot,\cdot)$ to $\mathfrak{g}(\mathring{A})\subset\mathfrak{g}(A)$ is again a standard invariant form on $\mathfrak{g}(\mathring{A})$, and

$$\langle\beta_j,\beta_i^\vee\rangle=\frac{2(\beta_i,\beta_j)}{(\beta_i,\beta_i)}, \qquad \langle\gamma_j,\gamma_i^\vee\rangle=\frac{2(\gamma_i,\gamma_j)}{(\gamma_i,\gamma_i)} \quad (1\leqq i,\ j\leqq m).$$

On the other hand, we have $(\delta,\alpha_i)=0$ for all i $(0\leqq i\leqq l)$ since $A\delta=0$. Therefore, we have $(\beta_i,\beta_j)=(\gamma_i+k_i\delta,\gamma_j+k_j\delta)=(\gamma_i,\gamma_j)$, so that $\tilde{a}_{ij}=\langle\beta_j,\beta_i^\vee\rangle=(2(\beta_i,\beta_j)/(\beta_i,\beta_i))=(2(\gamma_i,\gamma_j)/(\gamma_i,\gamma_i))=\langle\gamma_j,\gamma_i^\vee\rangle$ $(1\leqq i,\ j\leqq m)$ and $\tilde{A}:=(\tilde{a}_{ij})_{i,j=1}^m=(\langle\gamma_j,\gamma_i^\vee\rangle)_{1\leq i,j\leq m}$.

From now on, we assume that $\{\beta_j\}_{j=1}^m\subset\Delta^{re}$ is fundamental. Then $\{\beta_j\}_{j=1}^m\subset\Delta^{re}$ satisfies the condition (1), and so $\{\gamma_j\}_{j=1}^m\subset\mathring{\Delta}$ satisfies the condition ($\mathring{1}$). Therefore, from the above argument and Theorem 4.1, we can deduce that the GCM $\tilde{A}$ is a direct sum of generalized Cartan matrices of finite type or of non-twisted affine type, and that the number of direct summands of non-twisted affine type is at most one. So we have the following two cases.

CASE (a). If the number of direct summands of non-twisted affine type is zero, then $\{\gamma_i\}_{i=1}^m\subset(\mathring{\mathfrak{h}})^*$ is linearly independent. Hence $\{\gamma_i\}_{i=1}^m\subset\mathring{\Delta}$ is a fundamental subset of $\mathring{\Delta}$. Types of the Dynkin diagrams of such subsets of $\mathring{\Delta}$ are completely determined by Dynkin [**3**, Chap. II, §5].

CASE (b). Let the number of direct summands of non-twisted affine type be exactly one. Then by the removal of an appropriate root (say γ_m for definiteness) the Dynkin diagram of $\{\gamma_i\}_{i=1}^{m-1}$ is a direct sum of those of finite type. Therefore $\{\gamma_i\}_{i=1}^{m-1}\subset(\mathring{\mathfrak{h}})^*$ is linearly independent, and so it is a fundamental subset of $\mathring{\Delta}$. For the type of such a subset of $\mathring{\Delta}$, cf. Dynkin [3]. Further we know that the removed vertex corresponding to γ_m is the "extended" vertex of the Dynkin diagram of the connected component which is of non-twisted affine type $X_t^{(1)}$, that is to say, γ_m is the lowest root of the root system of type X_t. So, suppose that the Dynkin diagram of $\{\gamma_i\}_{i=1}^{m-1}\subset\mathring{\Delta}$ is of type $X_{t_1}+X_{t_2}+\cdots+X_{t_r}$ with $t_1+t_2+\cdots+t_r=m-1$, then clearly the Dynkin diagram of $\{\gamma_i\}_{i=1}^{m}$ is of type either $X_{t_1}^{(1)}+X_{t_2}+\cdots+X_{t_r}$, $X_{t_1}+X_{t_2}^{(1)}+\cdots+X_{t_r}$, $\cdots$, or $X_{t_1}+X_{t_2}+\cdots+X_{t_r}^{(1)}$, where X_{t_i} is a Dynkin diagram of finite type of rank t_i.

Combining these two cases, we can conclude that, if $\{\beta_j=\gamma_j+k_j\delta\}_{j=1}^{m}\subset\Delta^{re}$ is fundamental, then its Dynkin diagram is necessarily of type either $X_{t_1}+X_{t_2}+\cdots+X_{t_r}$, $X_{t_1}^{(1)}+X_{t_2}+\cdots+X_{t_r}$, $X_{t_1}+X_{t_2}^{(1)}+\cdots+X_{t_r}$, $\cdots$, or $X_{t_1}+X_{t_2}+\cdots+X_{t_r}^{(1)}$, where $X_{t_1}+X_{t_2}+\cdots+X_{t_r}$ is a Dynkin diagram of some fundamental subset of $\mathring{\Delta}$, and X_{t_i} is a Dynkin diagram of finite type of rank t_i.

Conversely we now prove the actual existence of a fundamental subset $\{\beta_j\}_{j=0}^{m}$ of Δ^{re} whose Dynkin diagram is of type $X_{t_1}^{(1)}+X_{t_2}+\cdots+X_{t_r}$, where $X_{t_1}+X_{t_2}+\cdots+X_{t_r}$ is a Dynkin diagram of an arbitrary fundamental subset $\{\gamma_j\}_{j=1}^{m}$ of $\mathring{\Delta}$. For $X_{t_1}+X_{t_2}^{(1)}+\cdots+X_{t_r}$, $\cdots$, or $X_{t_1}+X_{t_2}+\cdots+X_{t_r}^{(1)}$, the proof is similar.

Let us divide $\{\gamma_j\}_{j=1}^{m}$ into the disjoint union of $\{\gamma_i^{(p)}\}_{i=1}^{t_p}$ $(1\leqq p\leqq r)$ such that the Dynkin diagram of $\{\gamma_i^{(p)}\}_{i=1}^{t_p}$ is of type X_{t_p} for each p $(1\leqq p\leqq r)$. And let $(\mathring{\mathfrak{g}}_p)^{\sim}$ be a regular subalgebra of $\mathfrak{g}(\mathring{A})\subset\mathfrak{g}(A)$ corresponding to the fundamental subset $\{\gamma_i^{(p)}\}_{i=1}^{t_p}\subset\mathring{\Delta}$ $(1\leqq p\leqq r)$, and $(\mathring{\mathfrak{g}})^{\sim}$ be a regular subalgebra of $\mathfrak{g}(\mathring{A})$ corresponding to $\{\gamma_j\}_{j=1}^{m}\subset\mathring{\Delta}$. Then we have

$$(\mathring{\mathfrak{g}})^{\sim}=\textstyle\sum^{\oplus}{}_{p=1}^{r}(\mathring{\mathfrak{g}}_p)^{\sim}\qquad\text{(direct sum of ideals)}.$$

And $\sum^{\oplus}{}_{p=1}^{r}(\mathring{\mathfrak{h}}_p)^{\sim}$ is a Cartan subalgebra of $(\mathring{\mathfrak{g}})^{\sim}$, where $(\mathring{\mathfrak{h}}_p)^{\sim}:=\sum_{i=1}^{t_p}\boldsymbol{C}(\gamma_i^{(p)})^{\vee}$ $(1\leqq p\leqq r)$ is a Cartan subalgebra of $(\mathring{\mathfrak{g}}_p)^{\sim}$. Moreover, the root system $(\mathring{\Delta})^{\sim}\subset\sum^{\oplus}{}_{p=1}^{r}(\mathring{\mathfrak{h}}_p)^{\sim *}$ of $(\mathring{\mathfrak{g}})^{\sim}$ is a disjoint union of the root system $(\mathring{\Delta}_p)^{\sim}\subset(\mathring{\mathfrak{h}}_p)^{\sim *}$ of $(\mathring{\mathfrak{g}}_p)^{\sim}$. Therefore, by adding the lowest root $\gamma_0^{(1)}$ of the root system $(\mathring{\Delta}_1)^{\sim}$ of $(\mathring{\mathfrak{g}}_1)^{\sim}$ to $\{\gamma_i^{(1)}\}_{i=1}^{t_1}$, we have a subset $\{\gamma_i^{(1)}\}_{i=0}^{t_1}$ of $(\mathring{\Delta}_1)^{\sim}$ whose Dynkin diagram is of type $X_{t_1}^{(1)}$. Hence the Dynkin diagram of $\{\gamma_0^{(1)}\}\cup\{\gamma_j\}_{j=1}^{m}$ is of type $X_{t_1}^{(1)}+X_{t_2}+\cdots+X_{t_r}$.

Since $X_{t_1}^{(1)}$ is of affine type, there exists $\delta_1=(a_i)_{i=0}^{t_1}$ such that $\{z\in\boldsymbol{C}^{t_1+1};\ X_{t_1}^{(1)}z=0\}=\boldsymbol{C}\delta_1$, and $a_i\in\boldsymbol{N}$ for all i $(0\leqq i\leqq t_1)$, where the last symbol $X_{t_1}^{(1)}$ denotes the corresponding GCM. Therefore, if we set $\beta_i^{(1)}:=\gamma_i^{(1)}+a_i\delta_1$ $(0\leqq i\leqq t_1)$ and

$\beta_i^{(p)} := \gamma_i^{(p)}$ $(2 \leqq p \leqq r,\ 1 \leqq i \leqq t_p)$, then we have the following claim.

CLAIM. $\{\beta_j\}_{j=0}^m := \{\beta_i^{(1)}\}_{i=0}^{t_1} \cup \bigcup_{p=2}^r \{\beta_i^{(p)}\}_{i=1}^{t_p} \subset \Delta^{re}$ is a fundamental subset of Δ^{re}.

PROOF OF THE CLAIM. (1) $\beta_i - \beta_j \notin \Delta \cup \{0\}$ $(0 \leqq i \neq j \leqq m)$. To show this, we have only to prove that $\gamma_0^{(1)} - \gamma_i^{(1)} \notin \mathring{\Delta} \cup \{0\}$ $(1 \leqq i \leqq t_1)$. But this follows from the fact that $(\mathring{\Delta})^\sim = \amalg_{p=1}^r (\mathring{\Delta}_p)^\sim$ is a closed (with respect to addition) subset of $\mathring{\Delta}$ (cf. [3, Chap. II, §5, p. 42]), and the fact that $\gamma_0^{(1)}$ is the lowest root of $(\mathring{\Delta}_1)^\sim$.

(2) $\{\beta_j\}_{j=0}^m \subset \mathfrak{h}^*$ is linearly independent. In fact, this follows from the above fact that $\{z \in \boldsymbol{C}^{t_1+1};\ X_{t_1}^{(1)} z = 0\} = \boldsymbol{C}\delta_1$, and the fact that $\bigcup_{p=2}^r \{\beta_i^{(p)}\}_{i=1}^{t_p}$ is linearly independent. □

Thus we get a fundamental subset $\{\beta_j\}_{j=0}^m$ of Δ^{re} whose Dynkin diagram is of type $X_{t_1}^{(1)} + X_{t_2} + \cdots + X_{t_r}$.

Summing up the above argument, we get the following theorem.

THEOREM 4.2. *Let $A = (a_{ij})_{0 \leq i, j \leq l}$ be a* GCM *of non-twisted affine type. Then the Dynkin diagram of any fundamental subset of Δ^{re} is of type either $X_{t_1} + X_{t_2} + \cdots + X_{t_r}$, $X_{t_1}^{(1)} + X_{t_2} + \cdots + X_{t_r}$, $X_{t_1} + X_{t_2}^{(1)} + \cdots + X_{t_r}$, $\cdots$, or $X_{t_1} + X_{t_2} + \cdots + X_{t_r}^{(1)}$, where $X_{t_1} + X_{t_2} + \cdots + X_{t_r}$ is a Dynkin diagram of some fundamental subset of the root sysyem $\mathring{\Delta}$ of $\mathfrak{g}(\mathring{A})$. Moreover for each of the above diagrams, there exists a fundamental subset of Δ^{re} whose Dynkin diagram is of that type.*

Owing to the above theorem, we can determine all the types of regular subalgebras of the non-twisted affine Lie algebra $\mathfrak{g}(A)$, because those of the finite dimensional simple Lie algebra $\mathfrak{g}(\mathring{A})$ are completely determined (see [3, Chap. II, §5]).

REMARK 4.1. Also in the case of twisted affine type GCM (but not of type $A_{2l}^{(2)}$ $(l \geqq 1)$), the sufficiency part (the second part) of Theorem 4.2 is true. Its proof is almost the same as that for Theorem 4.2, but note that for the GCM $A = (a_{ij})_{0 \leq i, j \leq l}$ of type $A_{2l-1}^{(2)}$ $(l \geqq 3)$, $D_{l+1}^{(2)}$ $(l \geqq 2)$, $E_6^{(2)}$, or $D_4^{(3)}$, the type of $\mathring{A} = (a_{ij})_{1 \leq i, j \leq l}$ is C_l, B_l, F_4, or G_2, respectively. (See [4, Chaps. 4 and 6].)

§5. A generalization of regular subalgebras.

In this section, we generalize the notion of regular subalgebras of a Kac-Moody algebra.

5.1. Generalized Kac-Moody algebras (GKM algebras).

For that purpose, we utilize the notion of generalized Kac-Moody algebras (GKM algebras for short) introduced by Borcherds in [1]. Here we abopt the definition in [4, Chap. 11] of GKM algebras, which is a little different from that in [1], as a definition of GKM algebras.

DEFINITION 5.1 ([4]). *A real $n'\times n'$ matrix $A'=(a'_{ij})_{1\leq i,j\leq n'}$ is called a* GGCM *if it satisfies the following three conditions*:

(C′1) either $a'_{ii}=2$ or $a'_{ii}\leqq 0$;

(C′2) $a'_{ij}\leqq 0$ if $i\neq j$, and $a'_{ij}\in \boldsymbol{Z}$ if $a'_{ii}=2$;

(C′3) $a'_{ij}=0$ implies $a'_{ji}=0$.

Note that when $a'_{ii}=2$ for every i, A' is a GCM.

A triple $(\mathfrak{h}, \Pi=\{\alpha_i\}_{i=1}^{n'}, \Pi^{\vee}=\{\alpha_i^{\vee}\}_{i=1}^{n'})$ is called a *realization* of the GGCM A' if it satisfies the conditions (R1), (R2), and (R3) in 1.1.

Let $\tilde{\mathfrak{g}}(A')$ be a Lie algebra with the generators e_i, f_i $(1\leqq i\leqq n')$, and $\mathfrak{h}$, and the following defining relations:

$$(\mathrm{I})\quad \begin{aligned}&[e_i, f_j]=\delta_{ij}\alpha_i^{\vee}\quad (1\leqq i, j\leqq n'),\\ &[h, h']=0\quad (h, h'\in\mathfrak{h}),\\ &[h, e_i]=\langle\alpha_i, h\rangle e_i,\quad [h, f_i]=-\langle\alpha_i, h\rangle f_i\quad (1\leqq i\leqq n', h\in\mathfrak{h}).\end{aligned}$$

We define $\mathfrak{g}(A'):=\tilde{\mathfrak{g}}(A')/\mathfrak{r}$, where $\mathfrak{r}$ is a unique maximal ideal among the ideals of $\tilde{\mathfrak{g}}(A')$ intersecting $\mathfrak{h}$ trivially. This $\mathfrak{g}(A')$ is called a *generalized Kac-Moody algebra* (GKM algebra). Especially when A' is a GCM, $\mathfrak{g}(A')$ is a Kac-Moody algebra (cf. [4, Chap. 1]).

It is also shown in [4, Chap. 11] that, when the GGCM A' is symmetrizable, the GKM algebra $\mathfrak{g}(A')$ is a Lie algebra with the generators e_i, f_i $(1\leqq i\leqq n')$ (*Chevalley generators* of $\mathfrak{g}(A')$), and $\mathfrak{h}$, and the defining relations (I) and (II):

$$(\mathrm{II})\quad \begin{aligned}&(ad\, e_i)^{1-a'_{ij}}e_j=0,\quad (ad\, f_i)^{1-a'_{ij}}f_j=0,\quad \text{if } a'_{ii}=2,\ i\neq j,\\ &[e_i, e_j]=0,\quad [f_i, f_j]=0\quad \text{if } a'_{ij}=0.\end{aligned}$$

Here A' is symmetrizable if there exist an invertible diagonal matrix D' and a symmetric matrix B' such that $A'=D'B'$.

5.2. A generalization of fundamental subsets.

Let $A=(a_{ij})_{1\leq i,j\leq n}$ be a symmetrizable GCM, $\mathfrak{g}(A)$ be a Kac-Moody algebra associated with A, and $(\cdot,\cdot)$ be a fixed standard invariant form on $\mathfrak{g}(A)$. Then, we have the following definition of a fundamental subset, which is a generalization of Definition 2.1.

DEFINITION 5.2. *A subset $\tilde{\Pi}=\{\beta_1,\cdots,\beta_m,\beta_{m+1},\cdots,\beta_{m+k}\}$ of the root system Δ of $\mathfrak{g}(A)$ is called fundamental if it satisfies the following*:

(1) $\tilde{\Pi}=\{\beta_r\}_{r=1}^{m+k}\subset\mathfrak{h}^*$ is linearly independent;

(2) $\beta_s-\beta_t\notin\Delta\cup\{0\}$ $(1\leqq s\neq t\leqq m+k)$;

(3) $\beta_i\in\Delta^{re}$ $(1\leqq i\leqq m)$ and $\beta_j\in\Delta_+^{im}$ $(m+1\leqq j\leqq m+k)$.

Note that $\tilde{\Pi}$ is fundamental in the sense of Definition 2.1 if and only if $k=0$.

Now for each imaginary root β_j $(m+1\leqq j\leqq m+k)$, we define $\beta_j^\vee:=\nu^{-1}(\beta_j)\in\mathfrak{h}$, where $\nu:\mathfrak{h}\to\mathfrak{h}^*$ is a linear isomorphism determined by $\langle\nu(h), h'\rangle=(h, h')$ $(h, h'\in\mathfrak{h})$. For real root β_i $(1\leqq i\leqq m)$, $\beta_i^\vee\in\mathfrak{h}$ has been defined as a dual real root of β_i, and we know $\beta_i^\vee=(2/(\beta_i, \beta_i))\cdot\nu^{-1}(\beta_i)$. We take and fix non-zero vectors $E_r\in\mathfrak{g}_{\beta_r}$ and $F_r\in\mathfrak{g}_{-\beta_r}$ such that $[E_r, F_r]=\beta_r^\vee$ $(1\leqq r\leqq m+k)$. Such vectors always exist since $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]=\boldsymbol{C}\nu^{-1}(\alpha)$ for all $\alpha\in\Delta$. Then we have the following propositions.

PROPOSITION 5.1. *If we set* $\tilde{A}:=(\tilde{a}_{ij})_{1\leqq i,j\leqq m+k}$, where $\tilde{a}_{ij}=\langle\beta_j, \beta_i^\vee\rangle$, *then* $\tilde{A}$ *is a symmetrizable* GGCM.

PROOF. First, note that we have the following equalities.

$$\langle\beta_j, \beta_i^\vee\rangle=\left\langle\beta_j, \frac{2}{(\beta_i, \beta_i)}\cdot\nu^{-1}(\beta_i)\right\rangle=\frac{2}{(\beta_i, \beta_i)}\cdot(\beta_i, \beta_j)\qquad(1\leqq i\leqq m).$$

$$\langle\beta_j, \beta_i^\vee\rangle=\langle\beta_j, \nu^{-1}(\beta_i)\rangle=(\beta_i, \beta_j)\qquad(m+1\leqq i\leqq m+k).$$

PROOF OF (C′1). For i $(1\leqq i\leqq m)$, $\tilde{a}_{ii}=(2/(\beta_i, \beta_i))\cdot(\beta_i, \beta_i)=2$ since $\beta_i\in\Delta^{re}$. For i $(m+1\leqq i\leqq m+k)$, $\tilde{a}_{ii}=(\beta_i, \beta_i)\leqq 0$ since $\beta_i\in\Delta^{im}$.

PROOF OF (C′2). For i $(1\leqq i\leqq m)$, there exist non-negative integers p and q related by the equality $p-q=\langle\beta_j, \beta_i^\vee\rangle$ such that $\beta_j+t\beta_i\in\Delta\cup\{0\}$ if and only if $-p\leqq t\leqq q$, $t\in\boldsymbol{Z}$. Since $\beta_j-\beta_i\notin\Delta\cup\{0\}$ from the assumption, we have $p=0$ so that $\tilde{a}_{ij}=\langle\beta_j, \beta_i^\vee\rangle=-q\leqq 0$. For i $(m+1\leqq i\leqq m+k)$ and j $(1\leqq j\leqq m)$, we have $\langle\beta_j, \beta_i^\vee\rangle=(\beta_j, \beta_i)$. But we had $0\geqq\tilde{a}_{ji}=\langle\beta_i, \beta_j^\vee\rangle=(2/(\beta_j, \beta_j))\cdot(\beta_i, \beta_j)$, and so $\tilde{a}_{ij}=\langle\beta_j, \beta_i^\vee\rangle=(\beta_i, \beta_j)\leqq 0$ since $\beta_j\in\Delta^{re}$ implies $(\beta_j, \beta_j)>0$. And for i and j $(m+1\leqq i, j\leqq m+k)$, we have $\tilde{a}_{ij}=\langle\beta_j, \beta_i^\vee\rangle=(\beta_i, \beta_j)$. Therefore we have $\tilde{a}_{ij}\leqq 0$ since $\beta_i, \beta_j\in\Delta_+^{im}$ from the assumption (cf. [**4**, Chap. 5, Ex. 5.16]).

PROOF OF (C′3). This is obvious since $(\cdot, \cdot)$ is symmetric and $(\beta_i, \beta_i)>0$ for i $(1\leqq i\leqq m)$.

Finally we prove the symmetrizability of $\tilde{A}$. Put $\tilde{B}:=((\beta_i, \beta_j))_{1\leqq i,j\leqq m+k}$ and $\tilde{D}:=\operatorname{diag}(2/(\beta_1, \beta_1), \cdots, 2/(\beta_m, \beta_m), 1, \cdots, 1)$. Then clearly $\tilde{A}=\tilde{D}\tilde{B}$, and $\tilde{B}$ is a symmetric matrix. □

PROPOSITION 5.2. *There exists a vector subspace* $\mathfrak{h}_0$ *of* $\mathfrak{h}$, *such that the triple* $(\mathfrak{h}_0, \{\beta_j|_{\mathfrak{h}_0}\}_{j=1}^{m+k}, \{\beta_j^\vee\}_{j=1}^{m+k})$ *is a realization of the* GGCM $\tilde{A}$.

PROOF. Since $\{\beta_j\}_{j=1}^{m+k}\subset\mathfrak{h}^*$ is linearly independent and $\nu:\mathfrak{h}\to\mathfrak{h}^*$ is a linear isomorphism, $\{\beta_j^\vee\}_{j=1}^{m+k}\subset\mathfrak{h}$ is also linearly independent by the definition of $\beta_j^\vee$ for imaginary root β_j $(m+1\leqq j\leqq m+k)$. And so, the assertion follows easily from the argument, which is exactly the same as that used in [**5**, p. 86] when $\tilde{\Pi}$ is a subset of Π. □

Then, we get the following result, which is a generalization of Theorem 3.6.

THEOREM 5.1. *Let* $\tilde{\mathfrak{g}}$ *be a subalgebra of* $\mathfrak{g}(A)$ *generated by* E_r, F_r $(1\leqq r\leqq m+k)$, *and* $\mathfrak{h}_0$. *Then we have the canonical isomorphism* $\Phi: \mathfrak{g}(\tilde{A})\cong\tilde{\mathfrak{g}}$, *such that* $\Phi(\tilde{e}_r)=E_r$, $\Phi(\tilde{f}_r)=F_r$ $(1\leqq r\leqq m+k)$, *and* $\Phi(\tilde{\mathfrak{h}})=\mathfrak{h}_0$. *Here* $(\tilde{\mathfrak{h}}, \tilde{\Pi}=\{\tilde{\alpha}_r\}_{r=1}^{m+k}, \tilde{\Pi}^\vee=\{\tilde{\alpha}_r^\vee\}_{r=1}^{m+k})$ *is a realization of the* GGCM $\tilde{A}$, *and* $\tilde{e}_r$, $\tilde{f}_r$ $(1\leqq r\leqq m+k)$ *are the Chevalley generators of the* GKM *algebra* $\mathfrak{g}(\tilde{A})$.

PROOF. We have to check that E_r, F_r $(1\leqq r\leqq m+k)$, and $\mathfrak{h}_0$ satisfy the defining relations for the GKM algebra $\mathfrak{g}(\tilde{A})$.

PROOF OF (I). We check that $[E_i, F_j]=\delta_{ij}\beta_i^\vee$ $(1\leqq i, j\leqq m+k)$. Note that $[E_i, F_j]\in[\mathfrak{g}_{\beta_i}, \mathfrak{g}_{-\beta_j}]\subset\mathfrak{g}_{\beta_i-\beta_j}$. Since $\beta_i-\beta_j\notin\Delta\cup\{0\}$ $(1\leqq i\neq j\leqq m+k)$ from the assumption, we have $\mathfrak{g}_{\beta_i-\beta_j}=\{0\}$ so that $[E_i, F_j]=0$ $(1\leqq i\neq j\leqq m+k)$. And $[E_i, F_i]=\beta_i^\vee$ $(1\leqq i\leqq m+k)$ from the definition of E_i, F_i. The other relations of (I) are obvious since $E_i\in\mathfrak{g}_{\beta_i}$, $F_i\in\mathfrak{g}_{-\beta_i}$ $(1\leqq i\leqq m+k)$, and $\mathfrak{h}_0$ is a vector subspace of $\mathfrak{h}$.

PROOF OF (II). We first check that $(ad\,E_i)^{1-\tilde{a}_{ij}}E_j=0$ and $(ad\,F_i)^{1-\tilde{a}_{ij}}F_j=0$ if $1\leqq i\leqq m$ and $j\neq i$. Note that $(ad\,E_i)^{1-\tilde{a}_{ij}}E_j\in(ad\,\mathfrak{g}_{\beta_i})^{1-\tilde{a}_{ij}}\mathfrak{g}_{\beta_j}\subset\mathfrak{g}_\beta$, for $\beta=\beta_j+(1-\tilde{a}_{ij})\beta_i$. But we have $r_{\beta_i}(\beta_j-\beta_i)=\beta_j-\langle\beta_j, \beta_i^\vee\rangle\beta_i+\beta_i=\beta$. Here r_{β_i} is a fundamental reflection defined by a real root β_i and preserves the root system Δ of $\mathfrak{g}(A)$ (cf. [**4**]). Therefore, we deduce that $\beta\notin\Delta\cup\{0\}$ since $\beta_j-\beta_i\notin\Delta\cup\{0\}$ from the assumption. Hence we have $\mathfrak{g}_\beta=\{0\}$ so that $(ad\,E_i)^{1-\tilde{a}_{ij}}E_j=0$ $(1\leqq i\leqq m, j\neq i)$. The equality $(ad\,F_i)^{1-\tilde{a}_{ij}}F_j=0$ can be proved similarly.

Finally we check $[E_i, E_j]=0$ and $[F_i, F_j]=0$ if $\tilde{a}_{ij}=0$. We have only to prove these relations for i and j $(m+1\leqq i, j\leqq m+k)$. Note that $[E_i, E_j]\in[\mathfrak{g}_{\beta_i}, \mathfrak{g}_{\beta_j}]\subset\mathfrak{g}_{\beta_i+\beta_j}$. Since $\beta_i, \beta_j\in\Delta_+^{im}$ and $\tilde{a}_{ij}=(\beta_i, \beta_j)=0$ from the assumption, we have the following claim.

CLAIM. $\beta_i+\beta_j\notin\Delta\cup\{0\}$.

PROOF OF THE CLAIM. We have $(\beta_i+\beta_j, \beta_i+\beta_j)=(\beta_i, \beta_i)+(\beta_j, \beta_j)\leqq 0$, since $\beta_i, \beta_j\in\Delta_+^{im}$. Therefore, $\beta_i+\beta_j\in\Delta\cup\{0\}$ implies $\beta_i+\beta_j\in\Delta_+^{im}$. But this contradicts the fact that $(\beta_i, \beta_j)=0$ (cf. [**4**, Chap. 5, Ex. 5.18]). Thus the claim has been proved.

Hence we have $\mathfrak{g}_{\beta_i+\beta_j}=\{0\}$, so that $[E_i, E_j]=0$. The proof is similar for the equality $[F_i, F_j]=0$.

Thus we have checked all the defining relations for the GKM algebra $\mathfrak{g}(\tilde{A})$. Therefore, we get the surjective homomorphism $\Phi: \mathfrak{g}(\tilde{A}) \to \tilde{\mathfrak{g}}$, such that $\Phi(\tilde{e}_r) = E_r$, $\Phi(\tilde{f}_r) = F_r$ $(1 \leqq r \leqq m+k)$, and $\Phi(\tilde{\mathfrak{h}}) = \mathfrak{h}_0$. But, since both $\tilde{\mathfrak{h}}$ and $\mathfrak{h}_0$ are realizations of the GGCM $\tilde{A}$, they have the same finite dimension. So, $\Phi(\tilde{\mathfrak{h}}) = \mathfrak{h}_0$ implies $\operatorname{Ker} \Phi \cap \tilde{\mathfrak{h}} = \{0\}$. Hence $\operatorname{Ker} \Phi = \{0\}$ from the above definition of $\mathfrak{g}(\tilde{A})$. This completes the proof of the theorem. □

DEFINITION 5.3. The subalgebra $\tilde{\mathfrak{g}}$ of $\mathfrak{g}(A)$ in Theorem 5.1 is called a *regular subalgebra* of $\mathfrak{g}(A)$.

Note that when $k=0$, this $\tilde{\mathfrak{g}}$ is a regular subalgebra of $\mathfrak{g}(A)$ in the sense of Definition 3.1.

REMARK 5.1. In Theorem 5.1, we have chosen, as generators of $\tilde{\mathfrak{g}}$, one element from each root subspace, that is, $E_r \in \mathfrak{g}_{\beta_r}$ and $F_r \in \mathfrak{g}_{-\beta_r}$ $(1 \leqq r \leqq m+k)$. For real root β_r $(1 \leqq r \leqq m)$, this is equivalent to taking root subspaces $\mathfrak{g}_{\beta_r}$ and $\mathfrak{g}_{-\beta_r}$ in place of E_r and F_r, since $\dim_{\boldsymbol{C}} \mathfrak{g}_{\beta_r} = 1$. But, if we consider a subalgebra $\bar{\mathfrak{g}}$ generated by the whole root subspaces $\mathfrak{g}_{\pm\beta_r}$ $(1 \leqq r \leqq m+k)$, then $\bar{\mathfrak{g}}$ is a homomorphic image of the derived algebra of a certain GKM algebra, and this homomorphism is not necessarily an isomorphism since $[\mathfrak{g}_{\beta_r}, \mathfrak{g}_{-\beta_r}] = \boldsymbol{C}\nu^{-1}(\beta_r)$ and $\dim_{\boldsymbol{C}} \mathfrak{g}_{\beta_r}$ can be greater than 1 $(m+1 \leqq r \leqq m+k)$ in general. In this connection, we have chosen, for imaginary root β_r $(m+1 \leqq r \leqq m+k)$, one non-zero vector E_r from $\mathfrak{g}_{\beta_r}$, and F_r from $\mathfrak{g}_{-\beta_r}$ for each r.

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