J. Math. Soc. Japan
Vol. 43, No. 3, 1991

A decomposition theorem in a Banach $*$-algebra related to completely bounded maps on C^*-algebras

By Takashi ITOH

(Received Feb. 23, 1990)
(Revised Oct. 5, 1990)

1. Introduction.

As one of the fundamental theorems in C^*-algebras, it is well known that a self-adjoint element has the Jordan decomposition and a self-adjoint bounded linear functional has the Hahn decomposition: i.e. if x is a self-adjoint element in a C^*-algebra A, then there exist two positive elements $x_1, x_2 \in A$ such that $x=x_1-x_2$, $x_1x_2=0$ and $\|x\|=\|x_1+x_2\|$. If f is a self-adjoint bounded linear functional on A, then there exist two positive linear functionals $f_1, f_2 \in A^*$ such that $f=f_1-f_2$ and $\|f\|=\|f_1+f_2\|$.

As a generalized version of the Hahn decomposition, Loebl and Tsui considered independently whether the bounded self-adjoint map has the positive decomposition [**10**], [**16**]. The answer was negative except a few cases. Furthermore Huruya and Tomiyama obtained a non-existence theorem of the Hahn decomposition of bounded maps in the general situation [8]. However, it was Wittstock who showed the self-adjoint completely bounded map of a C^*-algebra to an injective C^*-algebra can be written as a difference of two completely positive maps with the norm condition [**17**]. This can be seen as a generalized Hahn decomposition, since the complete boundedness coincides with the boundedness and the complete positivity coincides with the positivity if the range algebra is commutative.

On the other hand, a completely bounded map can be regarded as an element in the dual space of a certain Banach space [6], [**9**], [**5**]. In this paper, especially motivated by the isomorphism which is obtained by Effros and Exel [**5**], we intend to get the Jordan decomposition and the Hahn decomposition in an advanced form. The main theorem is the following.

THEOREM B. *Let A be a C^*-algebra and $B(H)$ be all bounded operators on a Hilbert space H. Suppose that p is a finite dimensional projection.*

(1) *If V is a self-adjoint element in $pB(H)\otimes_h A\otimes_h B(H)p$ with the Haagerup norm $\| \ \|_h$, then*

$$\|V\|_h = \inf\{\|V_1+V_2\|_h \mid V = V_1-V_2,\ V_i \geqq 0\ (i=1, 2)\}.$$

(2) *If $\tilde{\Phi}$ is a self-adjoint bounded linear functional on $pB(H)\otimes_h A\otimes_h B(H)p$, then*

$$\|\tilde{\Phi}\| = \min\{\|\tilde{\Phi}_1+\tilde{\Phi}_2\| \mid \tilde{\Phi}=\tilde{\Phi}_1-\tilde{\Phi}_2,\ \tilde{\Phi}_i \geqq 0\ (i=1, 2)\}.$$

To see this, we introduce a $*$-operation and a new product to a triple tensor product space and get a new Banach $*$-algebra with the Haagerup norm (see Theorem A in Section 2). Next we investigate the order structure of the Banach $*$-algebra to get the Theorem B in Section 3. As an application, we can prove the Wittstock's theorem from the view point of the classical ordered Banach space theory in Section 4.

2. Banach $*$-algebra with the Haagerup norm.

Let $B(H)$ be all bounded operators on a Hilbert space H over the complex number field $\boldsymbol{C}$. We say that a subspace X in $B(H)$ is an operator space and denote the set $\{\xi^* \in B(H) \mid \xi \in X\}$ by X^*. Let A be a C^*-algebra and F be a bounded sesquilinear form on $X \times X$. We call F is positive and denote $F \geqq 0$ if $F(\xi, \xi) \geqq 0$ for any $\xi \in X$, and call F is strictly positive and denote $F > 0$ if $F \geqq 0$ and $F(\xi, \xi)=0$ implies $\xi=0$. We introduce a product and a $*$-operation to the algebraic tensor product space $X^*\otimes A\otimes X$ as follows.

DEFINITION 1. Let F be a fixed bounded sesquilinear form on $X \times X$. Given $\xi^*\otimes a\otimes \eta$, $\varphi^*\otimes b\otimes \psi \in X^*\otimes A\otimes X$, we define

$$(\xi^*\otimes a\otimes\eta)(\varphi^*\otimes b\otimes\psi) = \xi^*\otimes ab\otimes\psi F(\eta, \varphi),$$
$$(\xi^*\otimes a\otimes\eta)^* = \eta^*\otimes a^*\otimes\xi.$$

This product is different from the usual product cf. [3]. Let $M_{nm}(X)$ be the space of $n\times m$ matrices with the entries in X (abbreviated to $M_n(X)$ in case $n=m$), which is a subspace in $M_{nm}(B(H)) \cong B(H^m, H^n)$ with the operator norm. Throughout this paper, we introduce the Haagerup norm to $X^*\otimes A\otimes X$ and denote by $X^*\otimes_h A\otimes_h X$ the completion of $X^*\otimes A\otimes X$ with the Haagerup norm $\|\ \|_h$. The Haagerup norm is defined as follows [6], [12].

Given $V \in X^*\otimes A\otimes X$,

$$\|V\|_h = \inf\left\{\left\|\sum_{i=1}^n \xi_i^*\xi_i\right\|^{1/2} \|[a_{ij}]\| \left\|\sum_{j=1}^n \eta_j^*\eta_j\right\|^{1/2} \,\middle|\, V = \sum_{i,j=1}^n \xi_i^*\otimes a_{ij}\otimes\eta_j\right\},$$

where $[a_{ij}]$ is in $M_n(A)$. For convenience, we use the operation [3]

$$\underset{\sim}{\xi}^*\odot[a_{ij}]\odot\underset{\sim}{\eta} = \sum_{i,j=1}^n \xi_i^*\otimes a_{ij}\otimes\eta_j,$$

where $\underset{\sim}{\xi}^*=[\xi_1^*, \cdots, \xi_n^*] \in M_{1n}(X^*)$, and $\underset{\sim}{\eta}=[\eta_1, \cdots, \eta_n]^t \in M_{n1}(X)$. It follows that

$\|\xi^*\| = \|\sum_{i=1}^n \xi_i^* \xi_i\|^{1/2}$ and $\|\underset{\sim}{\eta}\| = \|\sum_{j=1}^n \eta_j^* \eta_j\|^{1/2}$. To see that the product on $X^* \otimes_h A \otimes_h X$ with $\| \ \|_h$ is well-defined and continuous, the following will be needed.

LEMMA 2. *Let X be an operator space. If F is a bounded sesquilinear form on $X \times X$ with $\|F\| \leqq 1$, then*

$$\|[F(\xi_i, \eta_j)]\| \leqq \left\| \sum_{i=1}^n \xi_i^* \xi_i \right\|^{1/2} \left\| \sum_{j=1}^n \eta_j^* \eta_j \right\|^{1/2}$$

for any $\{\xi_i\}_{i=1}^n$, $\{\eta_j\}_{j=1}^n$ in X.

The right hand side can be replaced by

$$\left\| \sum_{i=1}^n \xi_i^* \xi_i \right\|^{1/2} \left\| \sum_{j=1}^n \eta_j \eta_j^* \right\|^{1/2}, \ \left\| \sum_{i=1}^n \xi_i \xi_i^* \right\|^{1/2} \left\| \sum_{j=1}^n \eta_j^* \eta_j \right\|^{1/2}$$

$$\textit{and} \quad \left\| \sum_{i=1}^n \xi_i \xi_i^* \right\|^{1/2} \left\| \sum_{j=1}^n \eta_j \eta_j^* \right\|^{1/2}.$$

PROOF. Let $\{\lambda_i\}_{i=1}^n$, $\{\mu_j\}_{j=1}^n$ be in $\boldsymbol{C}$ such that $(\sum_{i=1}^n |\lambda_i|^2)^{1/2} = (\sum_{j=1}^n |\mu_j|^2)^{1/2} = 1$. Then we have

$$\begin{aligned}
\|[F(\xi_i, \eta_j)]\| &= \sup \left| \sum_{i,j=1}^n \lambda_i \bar{\mu}_j F(\xi_i, \eta_j) \right| \\
&= \sup \left| F\Big(\sum_{i=1}^n \lambda_i \xi_i, \sum_{j=1}^n \mu_j \eta_j \Big) \right| \leqq \sup \left\| \sum_{i=1}^n \lambda_i \xi_i \right\| \left\| \sum_{j=1}^n \mu_j \eta_j \right\| \\
&= \sup \left\| \begin{bmatrix} \lambda_1 & \cdots & \lambda_n \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \xi_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \xi_n & 0 & \cdots & 0 \end{bmatrix} \right\| \left\| \begin{bmatrix} \mu_1 & \cdots & \mu_n \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \eta_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \eta_n & 0 & \cdots & 0 \end{bmatrix} \right\| \\
&\leqq \left\| \begin{bmatrix} \xi_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \xi_n & 0 & \cdots & 0 \end{bmatrix} \right\| \left\| \begin{bmatrix} \eta_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \eta_n & 0 & \cdots & 0 \end{bmatrix} \right\| = \left\| \sum_{i=1}^n \xi_i^* \xi_i \right\|^{1/2} \left\| \sum_{j=1}^n \eta_j^* \eta_j \right\|^{1/2},
\end{aligned}$$

where the supremums are taken over all $\{\lambda_i\}_{i=1}^n$, $\{\mu_j\}_{j=1}^n$ as above.

The remainder can be obtained in the similar way. ■

THEOREM A. *Let X be an operator space in $B(H)$ and A be a C^*-algebra. If F is a bounded sesquilinear form on $X \times X$ with $\|F\| \leqq 1$, then $X^* \otimes_h A \otimes_h X$ is a Banach $*$-algebra with the product and the $*$-operation in Definition* 1.

PROOF. Given $V = \underset{\sim}{\xi}^* \odot [a_{ij}] \odot \underset{\sim}{\eta}$, $W = \underset{\sim}{\varphi}^* \odot [b_{ij}] \odot \underset{\sim}{\psi}$ in $X^* \otimes A \otimes X$, it is easy to see that

$$VW = \underset{\sim}{\xi}^* \odot [a_{ij}][F(\eta_i, \varphi_j)][b_{ij}] \odot \underset{\sim}{\psi},$$

where $[a_{ij}]$, $[b_{ij}] \in M_n(A)$ and $[F(\eta_i, \varphi_j)] \in M_n(\boldsymbol{C})$. By Lemma 1, we have

$$\begin{aligned}\|VW\|_h &\leqq \|\underset{\sim}{\xi}^*\|\|[a_{ij}]\|\|[F(\eta_i, \varphi_j)]\|\|[b_{ij}]\|\|\underset{\sim}{\psi}\| \\ &\leqq \|\underset{\sim}{\xi}^*\|\|[a_{ij}]\|\|\underset{\sim}{\eta}\|\|\underset{\sim}{\varphi}^*\|\|[b_{ij}]\|\|\underset{\sim}{\psi}\|.\end{aligned}$$

Hence the product is well-defined and satisfies $\|VW\|_h \leqq \|V\|_h\|W\|_h$.

For the $*$-operation, we have

$$\begin{aligned}\|V^*\|_h &= \inf \|\underset{\sim}{\eta}^*\|\|[a_{ji}^*]\|\|\underset{\sim}{\xi}\| \\ &= \inf \|\underset{\sim}{\xi}^*\|\|[a_{ij}]\|\|\underset{\sim}{\eta}\| \\ &= \|V\|_h. \quad \blacksquare\end{aligned}$$

3. Order structure of Banach $*$-algebras.

Let p be a k-dimensional projection ($k \in \boldsymbol{N}$) in $B(H)$. $B(H)p = \{xp \in B(H) \mid x \in B(H)\}$ is an operator space and $(B(H)p)^* = pB(H)$. Let $\{e_{ij}\}$ be a matrix unit in $B(H)$ such that $p = \sum_{i=1}^k e_{ii}$ and put $P_\lambda = \sum_{i \in \lambda} \sum_{j=1}^k k e_{ij}^* \otimes 1 \otimes e_{ij}$, where λ is a finite subset in Λ whose cardinal number is the same as the dimension of H and which is directed by set inclusion. Let τ be the normalized trace on $M_k(\boldsymbol{C})$. We define a sesquilinear form F_0 on $B(H)p \times B(H)p$ by $F_0(x, y) = \tau(y^* x)$. From now on, we consider the product defined in Section 1 using F_0 to $pB(H) \otimes_h A \otimes_h B(H)p$. It is easy to see that

$$P_\lambda^2 = P_\lambda^* = P_\lambda.$$

If A is not unital, we regard P_λ as an operator acts on $pB(H) \otimes_h A \otimes_h B(H)p$ from the left and the right. We note that

$$\lim_{\lambda \in \Lambda} \left\| \xi - \sum_{i \in \lambda} \sum_{j=1}^k k F_0(\xi, e_{ij}) e_{ij} \right\| = 0$$

for any $\xi \in B(H)p$, since $\|\xi\| \leqq \sqrt{k}\,\tau(\xi^* \xi)^{1/2}$. We put

$$q_\lambda \xi = \sum_{i \in \lambda} \sum_{j=1}^k k F_0(\xi, e_{ij}) e_{ij}.$$

PROPOSITION 3. *For all $V \in pB(H) \otimes_h A \otimes_h B(H)p$, we have the following properties.*

(1) $\|P_\lambda V\|_h \leqq \|V\|_h$.
(2) $\|P_\lambda V P_\lambda\|_h \leqq \|V\|_h$.
(3) $\lim_{\lambda \in \Lambda} \|V - P_\lambda V\|_h = 0$.
(4) $\lim_{\lambda \in \Lambda} \|V - P_\lambda V P_\lambda\|_h = 0$.

PROOF. We show only (2) and (4). Given $V = \underset{\sim}{\xi}^* \odot [a_{ij}] \odot \underset{\sim}{\eta} \in pB(H) \otimes A \otimes B(H)p$ and fixed a $\lambda \in \Lambda$, it is easy to see that

$$P_\lambda V P_\lambda = \sum_{i,j=1}^{n} (q_\lambda \xi_i)^* \otimes a_{ij} \otimes q_\lambda \eta_j .$$

Since

$$\xi^*\xi \geqq (q_\lambda \xi)^* q_\lambda \xi$$

for any $\xi \in B(H)p$, given $\varepsilon > 0$, we have

$$\begin{aligned} \|V\|_h + \varepsilon &\geqq \|\underset{\sim}{\xi}^*\| \|[a_{ij}]\| \|\underset{\sim}{\eta}\| \\ &\geqq \left\| \sum_{i=1}^{n} (q_\lambda \xi_i)^* q_\lambda \xi_i \right\|^{1/2} \|[a_{ij}]\| \left\| \sum_{j=1}^{n} (q_\lambda \eta_j)^* q_\lambda \eta_j \right\|^{1/2} \geqq \|P_\lambda V P_\lambda\|_h . \end{aligned}$$

Since $\|P_\lambda\|$ is bounded for the fixed $\lambda \in \Lambda$, we then have

$$\|P_\lambda V P_\lambda\|_h \leqq \|V\|_h$$

for any $V \in pB(H) \otimes_h A \otimes_h B(H)p$.

Since $\|\underset{\sim}{\xi}\| \leqq \sum_{i=1}^{n} \|\xi_i\|$ and $\|[a_{ij}]\| \leqq \sum_{i,j=1}^{n} \|a_{ij}\|$,

$$\|V - P_\lambda V P_\lambda\|_h \leqq \sum_{i,j=1}^{n} (\|\xi_i - q_\lambda \xi_i\| \|[a_{ij}]\| \|\eta_j\| + \|q_\lambda \xi_i\| \|[a_{ij}]\| \|\eta_j - q_\lambda \eta_j\|)$$

and $\|P_\lambda W P_\lambda\|_h \leqq \|W\|_h$ for any $W \in pB(H) \otimes_h A \otimes_h B(H)p$, it turns out that

$$\lim_{\lambda \in \Lambda} \|V - P_\lambda V P_\lambda\|_h = 0$$

for any $V \in pB(H) \otimes_h A \otimes_h B(H)p$. ∎

Let $\tilde{\Phi} \in (X^* \otimes_h A \otimes_h X)^d$, where $(X^* \otimes_h A \otimes_h X)^d$ means the dual space of $X^* \otimes_h A \otimes_h X$ to avoid the confusion. We call $\tilde{\Phi}$ is positive if $\tilde{\Phi}(V^*V) \geqq 0$ for any $V \in X^* \otimes_h A \otimes_h X$. We define $\tilde{\Phi}^*$ by $\tilde{\Phi}^*(V) = \overline{\tilde{\Phi}(V^*)}$ and say $\tilde{\Phi}$ is self-adjoint if $\tilde{\Phi}^* = \tilde{\Phi}$. Let A_X^+ be the $\| \ \|_h$-closure of the convex combinations of the elements of the form V^*V. We call V is positive and denote $V \geqq 0$ if $V \in A_X^+$.

As above, $\{P_\lambda\}_{\lambda \in \Lambda}$ behaves like an approximate identity even if A is not unital. However, $\lim_{\lambda \in \Lambda} \|P_\lambda\|_h = \infty$ if $\dim H = \infty$. If a Banach $*$-algebra B has a bounded approximate identity $\{P_\lambda\}_{\lambda \in \Lambda}$ with a constant $K \geqq 0$ such that $\|P_\lambda\| \leqq K$ for any $\lambda \in \Lambda$, and if $\tilde{\Phi}$ is positive on B, then $\tilde{\Phi}$ satisfies the Schwarz inequality

$$|\tilde{\Phi}(V)|^2 \leqq K \|\tilde{\Phi}\| \tilde{\Phi}(V^*V)$$

for any $V \in B$ [**15**, Lemma 9.11]. By the following proposition, $pB(H) \otimes_h A \otimes_h B(H)p$ is quite different from a C^*-algebra.

PROPOSITION 4. *Suppose that* $\dim H = \infty$, *then* $pB(H) \otimes_h A \otimes_h B(H)p$ *does not have any bounded approximate identity.*

PROOF. Let f be a fixed state on A. Define that

$$\tilde{f}(\underset{\sim}{\xi}^*\odot[a_{ij}]\odot\underset{\sim}{\eta}) = \sum_{i,j=1}^{n}\tau(\xi_i^*\eta_j)f(a_{ij})$$

for any $\underset{\sim}{\xi}^*\odot[a_{ij}]\odot\underset{\sim}{\eta}\in pB(H)\otimes A\otimes B(H)p$. It is easy to see that $\|\tilde{f}\|\leqq 1$ and $\tilde{f}\geqq 0$. Given $\varepsilon>0$, there exists $a\in A$ such that $a\geqq 0$, $\|a\|\leqq 1$ and $f(a)>1-\varepsilon$. Put $V_m=\sum_{i=1}^{m}\sum_{j=1}^{k}e_{ij}^*\otimes a\otimes e_{ij}$, then

$$\tilde{f}(V_m) = \sum_{i=1}^{m}\sum_{j=1}^{k}\tau(e_{ij}^*e_{ij})f(a) = m(1-\varepsilon)\,.$$

Since $V_m=V_m^*\geqq V_m^2$, it follows that

$$\frac{|\tilde{f}(V_m)|^2}{\tilde{f}(V_m^*V_m)} \geqq \frac{\tilde{f}(V_m)^2}{\tilde{f}(V_m)} = m(1-\varepsilon)\,.$$

Hence, $\tilde{f}$ does not satisfy the Schwarz inequality. ■

The following is the main theorem in this paper. We can regard this as an extension of Wittstock's theorem in Section 4.

THEOREM B. *Let A be a C^*-algebra and $B(H)$ be all bounded operators on a Hilbert space H. Suppose that p is a finite dimensional projection.*

(1) *Let $V\in pB(H)\otimes_h A\otimes_h B(H)p$ such that $V=V^*$, then*

$$\|V\|_h = \inf\{\|V_1+V_2\|_h \mid V=V_1-V_2,\ V_i\geqq 0\ (i=1,2)\}\,.$$

(2) *Let $\tilde{\Phi}\in(pB(H)\otimes_h A\otimes_h B(H)p)^d$ such that $\tilde{\Phi}=\tilde{\Phi}^*$, then*

$$\|\tilde{\Phi}\| = \min\{\|\tilde{\Phi}_1+\tilde{\Phi}_2\| \mid \tilde{\Phi}=\tilde{\Phi}_1-\tilde{\Phi}_2,\ \tilde{\Phi}_i\geqq 0\ (i=1,2)\}\,.$$

To see this, we have only to show two facts [**4**], [**1**, Theorem 1.3.1]: namely,

(1)′ Let $V, W\in pB(H)\otimes_h A\otimes_h B(H)p$ such that $V=V^*$, $W=W^*$ and $-W\leqq V\leqq W$, then $\|V\|_h\leqq\|W\|_h$.

(2)′ Let $\tilde{\Phi}, \tilde{\Psi}\in(pB(H)\otimes_h A\otimes_h B(H)p)^d$ such that $\tilde{\Phi}=\tilde{\Phi}^*$, $\tilde{\Psi}=\tilde{\Psi}^*$ and $-\tilde{\Psi}\leqq\tilde{\Phi}\leqq\tilde{\Psi}$, then $\|\tilde{\Phi}\|\leqq\|\tilde{\Psi}\|$.

In fact, by the argument followed Theorem 1.3.1 in [**1**], (1)′ implies that

$$\|V\|_h \leqq \inf\|V_1+V_2\|_h\,, \qquad \|\tilde{\Phi}\| \geqq \min\|\tilde{\Phi}_1+\tilde{\Phi}_2\|$$

and (2)′ implies that

$$\|V\|_h \geqq \inf\|V_1+V_2\|_h\,, \qquad \|\tilde{\Phi}\| \leqq \min\|\tilde{\Phi}_1+\tilde{\Phi}_2\|\,.$$

We provide some lemmas.

LEMMA 5.

$$A_{B(H)p}^{+} = \overline{\{V^*V \mid V\in pB(H)\otimes A\otimes B(H)p\}}^{\|\ \|_h}\,.$$

PROOF. Let $V_1, V_2\in pB(H)\otimes A\otimes B(H)p$. Then there exist two positive elements $[a_{ij}]$, $[b_{ij}]\in M_n(A)$ for some n such that

$$P_\lambda V_1^*V_1P_\lambda=\sum_{i,j=1}^{n}e_i^*\otimes a_{ij}\otimes e_j\,,\qquad P_\lambda V_2^*V_2P_\lambda=\sum_{i,j=1}^{n}e_i^*\otimes b_{ij}\otimes e_j\,,$$

for some suitable $\{e_i\}\subset\{e_{ij}\}$. It is easy to see that

$$P_\lambda(V_1^*V_1+V_2^*V_2)P_\lambda\in\{V^*V\,|\,V\in pB(H)\otimes A\otimes B(H)p\}\,.$$

Hence we have

$$\begin{aligned}V_1^*V_1+V_2^*V_2&=\lim_{\lambda\in\Lambda}P_\lambda(V_1^*V_1+V_2^*V_2)P_\lambda\\&\in\overline{\{V^*V\,|\,V\in pB(H)\otimes A\otimes B(H)p\}}^{\|\|_h}\,.\quad\blacksquare\end{aligned}$$

LEMMA 6. *Let X be an n-dimensional operator space, $F>0$ and $\{e_i\}_{i=1}^n$ be an orthonormal basis with respect to F. If $V\in X^*\otimes_h A\otimes_h X$, then there exists a unique $[a_{ij}]\in M_n(A)$ such that $V=\sum_{i,j=1}^n e_i^*\otimes a_{ij}\otimes e_j$.*

Moreover

(1) *V is self-adjoint if and only if $[a_{ij}]=[a_{ij}]^*$.*

(2) *V is positive if and only if $[a_{ij}]\geqq 0$.*

PROOF. Since X is finite dimensional, the first assertion is trivial.

(1) is clear.

If V is positive, as in the proof of Lemma 5, we may assume that V is of the form W^*W for some $W\in X^*\otimes_h A\otimes_h X$ such that $W=\sum_{i,j=1}^n e_i^*\otimes[b_{ij}]\otimes e_j$, since X is finite dimensional. Then we have

$$V=\underset{\sim}{e}^*\odot[b_{ij}]^*[F(e_i,e_j)][b_{ij}]\odot\underset{\sim}{e}=\underset{\sim}{e}^*\odot[b_{ij}]^*[b_{ij}]\odot e\,.$$

The converse is trivial. $\blacksquare$

We note that the operation $\odot$ satisfies the following property. Let $L\underset{\sim}{\xi}=[\sum_{j=1}^n\alpha_{1j}\xi_j,\cdots,\sum_{j=1}^n\alpha_{nj}\xi_j]^t$ for $L=[\alpha_{ij}]\in M_n(\boldsymbol{C})$ and $\underset{\sim}{\xi}=[\xi_1,\cdots,\xi_n]^t\in M_{n1}(X)$. Then we have

$$(L\underset{\sim}{\xi})^*\odot[a_{ij}]\odot M\underset{\sim}{\eta}=\underset{\sim}{\xi}^*\odot L^*[a_{ij}]M\odot\underset{\sim}{\eta}\,,$$

for any L, $M\in M_n(\boldsymbol{C})$.

LEMMA 7. *Let $V\in X^*\otimes A\otimes X$ such that $V^*=V$. Then*

$$\|V\|_h=\inf\|\underset{\sim}{\eta}\|^2\|[a_{ij}]\|\,,$$

where the infimum is taken over all representations of V such that $V=\underset{\sim}{\eta}^*\odot[a_{ij}]\odot\underset{\sim}{\eta}\in X^*\otimes A\otimes X$, $[a_{ij}]\in M_n(A)$ is self-adjoint and $\{\eta_i\}_{i=1}^n\subset X$ is linearly independent.

PROOF. First we show that the infimum can be taken over all representations of V such that $V=\underset{\sim}{\eta}^*\odot[a_{ij}]\odot\underset{\sim}{\eta}\in X^*\otimes A\otimes X$ and $[a_{ij}]=[a_{ij}]^*\in M_n(A)$.

To see this, given $\varepsilon>0$, then there exist $\underset{\sim}{\xi}, \underset{\sim}{\eta}\in M_{n1}(X)$ and $[a_{ij}]\in M_n(A)$ such that

$$V=\underset{\sim}{\xi}^*\odot[a_{ij}]\odot\underset{\sim}{\eta}, \qquad \|V\|_h+\varepsilon>\|\underset{\sim}{\xi}^*\|\|[a_{ij}]\|\|\underset{\sim}{\eta}\|.$$

Then we have, for any $\lambda>0$,

$$\begin{aligned} V&=\frac{1}{2}(V+V^*)\\ &=\frac{1}{2}(\lambda\underset{\sim}{\xi}^*\odot[a_{ij}]\odot\lambda^{-1}\underset{\sim}{\eta}+\lambda^{-1}\underset{\sim}{\eta}^*\odot[a_{ij}]^*\odot\lambda\underset{\sim}{\xi})\\ &=\frac{1}{\sqrt{2}}\begin{bmatrix}\lambda\underset{\sim}{\xi}\\ \lambda^{-1}\underset{\sim}{\eta}\end{bmatrix}^*\odot\begin{bmatrix}0 & [a_{ij}]\\ [a_{ij}]^* & 0\end{bmatrix}\odot\frac{1}{\sqrt{2}}\begin{bmatrix}\lambda\underset{\sim}{\xi}\\ \lambda^{-1}\underset{\sim}{\eta}\end{bmatrix}. \end{aligned}$$

Since

$$\left\|\frac{1}{\sqrt{2}}\begin{bmatrix}\lambda\underset{\sim}{\xi}\\ \lambda^{-1}\underset{\sim}{\eta}\end{bmatrix}^*\right\|^2\left\|\begin{bmatrix}0 & [a_{ij}]\\ [a_{ij}]^* & 0\end{bmatrix}\right\|\leqq\frac{1}{2}(\lambda^2\|\underset{\sim}{\xi}^*\|^2+\lambda^{-2}\|\underset{\sim}{\eta}\|^2)\|[a_{ij}]\|$$

and

$$\min_{\lambda>0}\frac{1}{2}(\lambda^2\|\underset{\sim}{\xi}^*\|^2+\lambda^{-2}\|\underset{\sim}{\eta}\|^2)=\|\underset{\sim}{\xi}^*\|\|\underset{\sim}{\eta}\|,$$

there exists $\lambda_0>0$ such that

$$\|V\|_h+\varepsilon>\left\|\frac{1}{\sqrt{2}}\begin{bmatrix}\lambda_0\underset{\sim}{\xi}\\ \lambda_0^{-1}\underset{\sim}{\eta}\end{bmatrix}^*\right\|^2\left\|\begin{bmatrix}0 & [a_{ij}]\\ [a_{ij}]^* & 0\end{bmatrix}\right\|.$$

Next we show that one can choose $\{\eta_i\}_{i=1}^n$ is linearly independent. Let $V=\underset{\sim}{\xi}^*\odot[a_{ij}]\odot\underset{\sim}{\xi}$ such that $[a_{ij}]^*=[a_{ij}]$, $\|V\|+\varepsilon>\|\underset{\sim}{\xi}\|^2\|[a_{ij}]\|$. We may assume that $\{\xi_1,\cdots,\xi_k\}_{k\leqq n}$ is linearly independent. Then there exists $L\in M_{nk}(\boldsymbol{C})$ such that $\underset{\sim}{\xi}=L[\xi_1,\cdots,\xi_k]^t$. Let $U|L|$ be the polar decomposition of L, where $U\in M_{nk}(\boldsymbol{C})$ and $|L|\in M_k(\boldsymbol{C})$. Put $|L|[\xi_1,\cdots,\xi_k]^t=[\eta_1,\cdots,\eta_k]^t$, then $\{\eta_1,\cdots,\eta_k\}$ is linearly independent. By the property of $\odot$, it follows that

$$V=\underset{\sim}{\xi}^*\odot[a_{ij}]\odot\underset{\sim}{\xi}=(U\underset{\sim}{\eta})^*\odot[a_{ij}]\odot U\underset{\sim}{\eta}=\underset{\sim}{\eta}^*\odot U^*[a_{ij}]U\odot\underset{\sim}{\eta}.$$

Moreover we obtain

$$\|\underset{\sim}{\eta}\|^2\|U^*[a_{ij}]U\|=\|\underset{\sim}{\xi}\|^2\|[a_{ij}]\|<\|V\|_h+\varepsilon. \quad \blacksquare$$

LEMMA 8. *Let X be an n-dimensional operator space and $F\geqq0$. If $V, W\in X^*\otimes_h A\otimes_h X$ such that $V=V^*$, $W=W^*$ and $-W\leqq V\leqq W$, then $\|V\|_h\leqq\|W\|_h$.*

PROOF. Given $\varepsilon>0$, by Lemma 7, there exist $\underset{\sim}{\eta}\in M_{1k}(X)$ and $[b_{ij}]\in M_k(A)$, $(k\leqq n)$ such that $W=\underset{\sim}{\eta}^*\odot[b_{ij}]\odot\underset{\sim}{\eta}$, $[b_{ij}]^*=[b_{ij}]$, $\{\eta_1,\cdots,\eta_k\}$ is linearly independent and $\|W\|_k+\varepsilon>\|\underset{\sim}{\eta}\|^2\|[b_{ij}]\|$. Then we can choose an invertible matrix $L\in M_k(\boldsymbol{C})$ such that $\underset{\sim}{\eta}=L\underset{\sim}{e}'$ where $\underset{\sim}{e}'=[e_1,\cdots,e_k]^t$ and $\{e_1,\cdots,e_n\}$ is a basis in X. Because

$$W=\left(\begin{bmatrix} L & 0 & \cdots & 0 \\ 0 & \delta & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & \delta \end{bmatrix}\underset{\sim}{e}\right)^{*}\odot\begin{bmatrix}[b_{ij}] & 0 \\ 0 & 0\end{bmatrix}\odot\begin{bmatrix} L & 0 & \cdots & 0 \\ 0 & \delta & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & \delta \end{bmatrix}\underset{\sim}{e},$$

for any $\delta>0$, where $\underset{\sim}{e}=[e_1, \cdots, e_n]^t$ and

$$\left\|\begin{bmatrix} L & 0 & \cdots & 0 \\ 0 & \delta & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & \delta \end{bmatrix}\underset{\sim}{e}\right\|^2\left\|\begin{bmatrix}[b_{ij}] & 0 \\ 0 & 0\end{bmatrix}\right\| < \|W\|_k+\varepsilon ,$$

for some $\delta>0$, we may assume that $L\in M_n(\boldsymbol{C})$ is invertible, $\underset{\sim}{\eta}\in M_{n1}(X)$, $[b_{ij}]\in M_n(X)$, and $W=\underset{\sim}{e}^*\odot L^*[b_{ij}]L\odot\underset{\sim}{e}$. Let $V=\underset{\sim}{e}^*\odot[a_{ij}]\odot\underset{\sim}{e}$. By Lemma 5, we have

$$-L^*[b_{ij}]L \leqq [a_{ij}] \leqq L^*[b_{ij}]L .$$

Hence

$$-[b_{ij}] \leqq L^{-1*}[a_{ij}]L^{-1} \leqq [b_{ij}] .$$

Since $V=\underset{\sim}{\eta}^*\odot L^{-1*}[a_{ij}]L^{-1}\odot\underset{\sim}{\eta}$, we have

$$\begin{aligned} \|W\|_h+\varepsilon &> \|\underset{\sim}{\eta}\|^2\|[b_{ij}]\| \\ &\geqq \|\underset{\sim}{\eta}\|^2\|L^{-1*}[a_{ij}]L^{-1}\| \\ &\geqq \|V\|_h . \quad \blacksquare \end{aligned}$$

It is now possible to prove Theorem B.

PROOF OF THEOREM B. First given $V, W\in pB(H)\otimes_h A\otimes_h B(H)p$ such that $V=V^*$, $W=W^*$ and $-W\leqq V\leqq W$, it is easy to see that

$$-P_\lambda WP_\lambda \leqq P_\lambda VP_\lambda \leqq P_\lambda WP_\lambda ,$$

where $P_\lambda=\sum_{i\in\lambda}\sum_{j=1}^{k} ke_{ij}^*\otimes 1\otimes e_{ij}$ in Proposition 3. From Lemma 8, it follows that

$$\|P_\lambda VP_\lambda\|_h \leqq \|P_\lambda WP_\lambda\|_h .$$

Hence we obtain that $\|V\|_h\leqq\|W\|_h$ by Proposition 3.

Next given $\tilde{\Phi}, \tilde{\Psi}\in(pB(H)\otimes_h A\otimes_h B(H)p)^d$ such that $\tilde{\Phi}^*=\tilde{\Phi}$, $\tilde{\Psi}^*=\tilde{\Psi}$ and $-\tilde{\Psi}\leqq\tilde{\Phi}\leqq\tilde{\Psi}$. Let $V=V^*\in pB(H)\otimes A\otimes B(H)p$. Then by Lemma 7, for given $\varepsilon>0$, V is represented as $V=\underset{\sim}{\eta}^*\odot[a_{ij}]\odot\underset{\sim}{\eta}$ such that $[a_{ij}]^*=[a_{ij}]$ and $\|V\|_h+\varepsilon>\|\underset{\sim}{\eta}\|^2\|[a_{ij}]\|$. We put that $W=\underset{\sim}{\eta}^*\odot|[a_{ij}]|\odot\underset{\sim}{\eta}$. It follows that $-W\leqq V\leqq W$ and $\|W\|_h\leqq\|V\|_h+\varepsilon$ by Lemma 7. Since $(\tilde{\Psi}+\tilde{\Phi})(W-V)\geqq 0$ and $(\tilde{\Psi}-\tilde{\Phi})(W+V)\geqq 0$, we get $\tilde{\Psi}(W)\geqq\tilde{\Phi}(V)$. Similarly we have $\tilde{\Psi}(W)\geqq-\tilde{\Phi}(V)$. Hence

$$|\tilde{\Phi}(V)| \leqq |\tilde{\Psi}(W)| \leqq \|\tilde{\Psi}\|\|W\|_h = \|\tilde{\Psi}\|(\|V\|_h+\varepsilon) .$$

Therefore we obtain that $\|\tilde{\Phi}\|\leqq\|\tilde{\Psi}\|$. $\blacksquare$

4. An application.

Recall that, if Φ is a linear map of an operator space A to $B(H)$, then linear maps Φ_n of $M_n(A)$ to $M_n(B(H))$ can be defined by $\Phi_n[a_{ij}]=[\Phi(a_{ij})]$ for $[a_{ij}]\in M_n(A)$. We say that Φ is completely positive if Φ_n is positive for any n and Φ is completely bounded if $\sup_n\|\Phi_n\|$ is finite and denote the supremum by $\|\Phi\|_{cb}$. It is well known that a completely positive map Φ is completely bounded and $\|\Phi\|_{cb}=\|\Phi\|$.

In [5], Effros and Exel introduced a norm $\|\ \|_{\sim}$ to $H^*\otimes A\otimes H$ as follows: for any $V=\underset{\sim}{\xi}^*\odot[a_{ij}]\odot\underset{\sim}{\eta}\in H^*\otimes A\otimes H$,

$$\|V\|_{\sim}=\inf\Big(\sum_{i=1}^n\|\xi_i\|^2\Big)^{1/2}\|[a_{ij}]\|\Big(\sum_{j=1}^n\|\eta_j\|^2\Big)^{1/2},$$

where the infimum is taken over all representations of V. They showed that the space of all completely bounded maps from an operator space A to $B(H)$ with $\|\ \|_{cb}$, which is denoted by $CB(A, B(H))$, is isomorphic onto $(H^*\otimes_{\sim}A\otimes_{\sim}H)^d$. The correspondence is defined as follows:

$$\tilde{\Phi}(\underset{\sim}{\xi}^*\odot[a_{ij}]\odot\underset{\sim}{\eta})=\sum_{i,j=1}^n(\Phi(a_{ij})\eta_j|\xi_i)$$

for $\Phi\in CB(A, B(H))$.

Suppose that p is a 1-dimensional projection in $B(H)$, then $B(H)p\cong B(\boldsymbol{C}, H)\cong H$ and $pB(H)\cong B(H, \boldsymbol{C})=H^*$. By this identification, we notice that $\|\ \|_{\sim}$ is nothing but the Haagerup norm $\|\ \|_h$. In this situation, it is clear that the product $\xi^*\eta$ for $\xi, \eta\in B(H)p$ is just the inner product $(\eta|\xi)$ and $F_0(\eta, \xi)=(\eta|\xi)$. The following means that a completely positive map is a positive linear functional on a Banach $*$-algebra.

PROPOSITION 9. *Let A be a C^*-algebra and Φ is a completely bounded map of A to $B(H)$.*

(1) *Φ is self-adjoint if and only if $\tilde{\Phi}$ is self-adjoint.*

(2) *Φ is completely positive if and only if $\tilde{\Phi}(V^*V)\geqq 0$ for any $V\in H^*\otimes_h A\otimes_h H$.*

PROOF. (1) If $\Phi=\Phi^*$, then

$$\begin{aligned}\tilde{\Phi}((\underset{\sim}{\xi}^*\odot[a_{ij}]\odot\underset{\sim}{\eta})^*)&=\sum_{i,j=1}^n(\Phi(a_{ji}^*)\xi_j|\eta_i)\\&=\sum_{i,j=1}^n\overline{(\Phi(a_{ij})\eta_j|\xi_i)}=\overline{\tilde{\Phi}(\underset{\sim}{\xi}^*\odot[a_{ij}]\odot\underset{\sim}{\eta})}\end{aligned}$$

for any $\underset{\sim}{\xi}^*\odot[a_{ij}]\odot\underset{\sim}{\eta}\in H^*\otimes A\otimes H$.

If $\tilde{\Phi}=\tilde{\Phi}^*$, then

$$(\Phi(a^*)\xi \mid \eta) = \overline{\tilde{\Phi}(\xi^*\otimes a\otimes\eta)} = (\tilde{\Phi}(a)^*\xi \mid \eta)$$

for any $a\in A$, ξ, $\eta\in H$.

(2) Suppose that Φ is completely positive. Let $V=\underset{\sim}{\xi}^*\odot[a_{ii}]\odot\underset{\sim}{\eta}\in H^*\otimes A\otimes H$. It follows that $V^*V=\underset{\sim}{\eta}^*\odot[a_{ij}]^*[(\xi_i|\xi_j)][a_{ij}]\odot\underset{\sim}{\eta}$. Then we have

$$\tilde{\Phi}(V^*V) = \left(\Phi_n([a_{ij}]^*[(\xi_i|\xi_j)][a_{ij}])\begin{bmatrix}\eta_1\\ \vdots\\ \eta_n\end{bmatrix}\middle|\begin{bmatrix}\eta_1\\ \vdots\\ \eta_n\end{bmatrix}\right) \geqq 0\,.$$

Conversely, let $a_1, \cdots, a_n\in A$ and $\xi_1, \cdots, \xi_n$, $\eta\in H$ with $\|\eta\|=1$. Then we have

$$\left([\Phi(a_i^*a_j)]\begin{bmatrix}\xi_1\\ \vdots\\ \xi_n\end{bmatrix}\middle|\begin{bmatrix}\xi_1\\ \vdots\\ \xi_n\end{bmatrix}\right) = \tilde{\Phi}(\underset{\sim}{\xi}^*\odot[a_i^*a_j]\odot\underset{\sim}{\xi})$$

$$= \tilde{\Phi}\left(\left(\sum_{i=1}^n \eta^*\otimes a_i\otimes\xi_i\right)^*\left(\sum_{j=1}^n \eta^*\otimes a_j\otimes\xi_j\right)\right) \geqq 0. \qquad \blacksquare$$

The following was obtained in [**17**], [**7**], [**11**]. We prove it as a Corollary of Theorem B.

COROLLARY 10 (Wittstock). *Let A be an operator system and Φ be a self-adjoint completely bounded map of A to $B(H)$. Then there exist two completely positive maps Φ_1, Φ_2 such that $\Phi=\Phi_1-\Phi_2$ and $\|\Phi\|_{cb}=\|\Phi_1+\Phi_2\|$.*

PROOF. We may assume that A is a C^*-algebra, because the Haagerup norm has the injectivity [**12**], [**2**]. From Proposition 9, $\tilde{\Phi}$ is self-adjoint in $(H^*\otimes_h A\otimes_h H)^d$. Then there exist two positive linear functionals $\tilde{\Phi}_1$, $\tilde{\Phi}_2$ on $H^*\otimes_h A\otimes_h H$ such that $\tilde{\Phi}=\tilde{\Phi}_1-\tilde{\Phi}_2$ and $\|\tilde{\Phi}\|=\|\tilde{\Phi}_1+\tilde{\Phi}_2\|$ by Theorem B(2). Hence, by the correspondence of Effros and Exel and Proposition 9, we get Φ_1, Φ_2 which satisfy the conditions. ■

ACKNOWLEDGEMENTS.

The main part of this work was carried out while the author was visiting the Mittag-Leffler Institute. The author would like to express his thanks to A. Kishimoto, E.G. Effros and Z-J. Ruan for fruitful discussions and also to the referee for his comment.

References

[1] C.J.K. Batty and D.W. Robinson, Positive one parameter semigroups on ordered Banach space, Acta Applicandae Mathematicae, 2 (1984), 221-296, D. Reidel Publishing Company.

[2] D.P. Blecher and V.I. Paulsen, Tensor products of operator spaces, preprint.

[3] E. Christensen, E.G. Effros and A.M. Sinclair, Completely bounded maps and C^*-algebraic cohomology, Inventiones Math., 90 (1987), 279-296.

[4] E. B. Davies, The structure and ideal theory of the predual of a Banach lattice, Trans. Amer. Math. Soc., **131** (1968), 544-555.
[5] E. G. Effros and R. Exel, On multilinear double commutant theorems, London Math. Soc., Lecture Note Seris, **135** (1988), 81-94, Cambridge Univ. Press.
[6] E. G. Effros and A. Kishimoto, Module maps and Hochschild-Johnson Cohomology, Indiana Univ. Math. J., **36** (1987), 257-276.
[7] U. Haagerup, Decomposition of completely bounded maps on operator algebras, unpublished manuscript 1980.
[8] T. Huruya and J. Tomiyama, Completely bounded maps of C^*-algebras, J. Operator theory, **10** (1983), 141-152.
[9] T. Itoh, Completely bounded maps and a certain cross norm, Math. Japonica, **32** (1987), 239-249.
[10] R. I. Loebl, A Hahn decomposition for linear maps, Pacific J. Math., **65** (1976), 119-133.
[11] V. I. Paulsen, Completely bounded maps on C^*-algebras and invariant operator ranges, Proc. Amer. Math. Soc., **86** (1982), 91-96.
[12] V. I. Paulsen and R. R. Smith, Multilinear maps and tensor norms on operator systems, J. Funct. Anal., **73** (1987), 258-276.
[13] Z-J. Ruan, Injectivity of operator space, Trans. Amer. Math. Soc. (to appear).
[14] E. Størmer, Extension of positive maps into $B(H)$, J. Funct. Anal., **66** (1986), 235-254.
[15] M. Takesaki, Theory of operator algebras 1, Springer, 1979.
[16] S-K. J. Tsui, Decompositions of linear maps, Trans. Amer. Math. Soc., **230** (1977), 87-112.
[17] G. Wittstock, Ein operatorwertiger Hahn-Banach Satz, J. Funct. Anal., **40** (1981), 127-150.

Takashi ITOH
Department of Mathematics
Faculty of Education
Gunma University
Maebashi, Gunma 371
Japan