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Classification of totally real 3-dimensional submanifolds of $S^6(1)$ with $K \geqq 1/16$

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1. Introduction.

It is well-known that a 6-dimensional sphere S^6 does not admit any Kaehler structure. However, using the Cayley algebra, a natural almost complex structure J can be defined on S^6 considered as a hypersurface in $\boldsymbol{R}^7$ which itself is viewed as the set of the purely imaginary Cayley numbers. And, together with the standard metric g on S^6, this almost complex structure J determines a *nearly Kaehler* structure in the sense of A. Gray **[G2]**. In Section 2, we recall the construction of this structure working with the 6-dimensional unit sphere $S^6(1)$, (of radius and constant curvature 1).

With respect to the almost complex structure J on $S^6(1)$, two natural particular types of submanifolds M can be investigated: those which are *almost complex* (i.e. for which the tangent space of M at each point is invariant under the action of J) and those which are *totally real* (i.e. for which the tangent space of M at each point is mapped into the normal space at that point by J). The almost complex submanifolds M of the nearly Kaehler $S^6(1)$ are, as the invariant submanifolds of Kaehlerian manifolds, automatically minimal and even dimensional, and therefore of dimension 2 or 4. Moreover, A. Gray **[G1]** showed that there do not exist 4-dimensional almost complex submanifolds in $S^6(1)$. So, for this case, only the almost complex surfaces of $S^6(1)$ need to be studied. Curvature properties for such surfaces were first obtained by K. Sekigawa **[Se]**. As follows at once from their definition, for the other case, only 2- and 3-dimensional totally real submanifolds can occur in $S^6(1)$. N. Ejiri **[E1]** proved that every 3-dimensional totally real submanifold of $S^6(1)$ is orientable and minimal, and he first investigated curvature conditions on such manifolds. The 3-dimensional totally real submanifolds of $S^6(1)$ were also considered, for instance, by H. Bl. Lawson Jr. and R. Harvey **[H-L]** in their study of calibrated geometries, and by K. Mashimo **[M2]** from the viewpoint of homogeneous manifolds.

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In our study of submanifolds of the nearly Kaehler 6-sphere, we concentrated on the following problems.

PROBLEM A. *Which real numbers can be realized as the constant sectional curvatures of almost complex or minimal totally real submanifolds* M *of* $S^6(1)$?

PROBLEM B. *Let* K_1 *and* K_2 *be two consecutive numbers as in Problem* A. *Then, do there exist compact submanifolds* M *of* $S^6(1)$ *whose sectional curvatures* K *satisfy* $K_1 \leqq K \leqq K_2$, *other than those for which* $K \equiv K_1$ *or* $K \equiv K_2$?

In the more general situation, when M is a minimal surface in a unit sphere $S^n(1)$ of arbitrary dimension n, one has a complete answer to Problem A and partial answers to Problem B. Namely, O. Boruvka [**Bo**] constructed full (i.e. not lying in a totally geodesic hypersurface of the ambient space) minimal immersions of 2-spheres $S^2(2/m(m+1))$ of constant Gauss curvature $K=2/m(m+1)$ into $S^{2m}(1)$ for every m, Later, E. Calabi [**Ca**] showed that, up to rigid motions, these Boruvka spheres are the only compact minimal surfaces with constant Gauss curvature >0 in $S^n(1)$ for any n. Moreover, N. Wallach [**Wa**] proved that any minimal surface with constant Gauss curvature $K>0$ in $S^n(1)$ is locally an open subset of a Boruvka sphere, and, recently, R. Bryant [**Br**] proved that there are no minimal surfaces of constant negative Gauss curvature in any sphere S^n (whether, in this last statement, the condition on the negative Gauss curvature to be constant can eventually be dropped, as far as we know, is still not settled [**Y**]). Concerning Problem B, U. Simon [**S-K**] conjectured the following.

U. SIMON'S CONJECTURE. *Let* M *be a compact surface which is minimally immersed in* $S^n(1)$ *and whose Gauss curvature* K *satisfies* $2/m(m+1) \leqq K \leqq 2/m(m-1)$ *for some* $m \in \boldsymbol{N} \setminus \{0, 1\}$. *Then* $K \equiv 2/m(m+1)$ *or* $K \equiv 2/m(m-1)$, (*and hence* M *is a Boruvka sphere*).

For $m=2$ and $m=3$, this conjecture is known to be true, as was shown by H. Bl. Lawson [**L**], and by U. Simon and his coworkers [**B-K-S-S**], [**K-S**] essentially based on formulas for the Laplacian of certain functions of K. Recently, quite a number of people have been working on this conjecture, using various methods and sometimes adding some additional assumption, such as T. Ogata, S. Montiel, T. Itoh, G. Jensen, M. Rigoli, J. Bolton, L. Woodward, and U. Simon, A. Schwenk and B. Opozda together with the present authors. As far as we know however, in general, for $m>3$, this conjecture is still open.

In our work in this field, yielding amongst others an alternative proof of this conjecture in case $m=2$ and $m=3$ (see for instance [**D-V**]), a crucial role is played by the method which is based on some integral formulas of A. Ros, which he first published in his solution [**R**] of a conjecture of K. Ogiue on

Kähler submanifolds of complex projective spaces. Proposition 3.1 of the present paper is obtained using this method. As the Lemma of H. Hopf, we believe that these integral formulas of A. Ros, which are given below, provide a powerful tool for the study of problems in global Riemannian geometry.

LEMMA OF A. ROS. *Let M be a compact Riemannian manifold. Denote by UM the unit tangent bundle of M, and by UM_p, the fiber of UM over a point p of M. Let dp, du and du_p respectively be the canonical measures on M, UM and UM_p. Then, for any continuous function $f: UM \to \boldsymbol{R}$, one has*

$$\int_{UM} f\,du = \int_M \left(\int_{UM_p} f\,du_p \right) dp\,.$$

Now, let T be any k-covariant tensor field on M. Then

$$\int_{UM} (\nabla T)(u, u, \cdots, u)du = 0\,,$$

where ∇ is the Levi-Civita connection of M.

A. Almost complex surfaces in $S^6(1)$.

Concerning Problem A, K. Sekigawa [**Se**] obtained the following.

THEOREM A. *If an almost complex surface M in $S^6(1)$ has constant Gauss curvature K, then either $K=1$ (and M is totally geodesic) or $K=1/6$ or $K=0$.*

Moreover, for each of these possible cases, explicit examples are known (see, for instance, [**Se**]). The following results give a complete answer to Problem B, for almost complex submanifolds.

THEOREM B. *Let M be a compact almost complex surface in $S^6(1)$ which Gauss curvature K.*

(a) *If $1/6 \leqq K$ (or equivalently $1/6 \leqq K \leqq 1$), then either $K \equiv 1/6$ or $K \equiv 1$.*

(b) *If $0 \leqq K \leqq 1/6$, then either $K \equiv 0$ or $K \equiv 1/6$.*

We obtained these results in [**D-V-V1**], and in [**D-O-V-V1**] together with B. Opozda, and our method of proof consisted in applying the Lemma of A. Ros for some suitable tensors T constructed in terms of the second fundamental form of the submanifold M in $S^6(1)$ and its derivatives of van der Waerden-Bortolotti (see, for instance, [**Ch**]). We remark that (a) also follows from Theorem B of [**O**].

B. **Totally real minimal surfaces in** $S^6(1)$.

Whereas, as stated before, every 3-dimensional totally real submanifold of $S^6(1)$ is minimal, in general this is not so in dimension 2. This can be seen for instance as follows. As we will mention later on, $S^3(1)$ can be isometrically immersed in $S^6(1)$ as a totally real and totally geodesic submanifold. Consider a small hypersphere $S^2(1/r^2)$ of radius $r<1$ in $S^3(1)$. Under the above immersion of $S^3(1)$ in $S^6(1)$, this $S^2(1/r^2)$ then becomes a totally real surface in $S^6(1)$ with constant Gauss curvature $K=1/r^2$, which is not minimal.

The following results answer Problems A and B in the present case.

THEOREM C. *If a minimal totally real surface M in $S^6(1)$ has constant Gauss curvature K, then either $K=1$ (and M is totally geodesic) or $K=0$.*

THEOREM D. *For a compact minimal totally real surface M in $S^6(1)$ with nonnegative Gauss curvature K (or equivalently, for which $0\leqq K\leqq 1$), either $K\equiv 0$ or $K\equiv 1$.*

The main point in our proofs of those results is to show that a minimal totally real surface in $S^6(1)$ which is homeomorphic to a sphere is totally geodesic. We did this in [**D-O-V-V3**], together with B. Opozda, where we used some formulas of S.S. Chern [**Chr**] and N. Ejiri [**E2**] for the second fundamental form of a surface of genus 0 in a sphere. For examples of the surfaces in $S^6(1)$ appearing in Theorems C and D, see [**D-O-V-V2**].

C. **Totally real 3-dimensional submanifolds of** $S^6(1)$.

In 1981, making use of a special choice of local orthonormal frames, N. Ejiri [**E1**] solved Problem A for totally real 3-dimensional submanifolds of $S^6(1)$ as follows.

THEOREM E. *If a 3-dimensional totally real submanifold M of $S^6(1)$ has constant curvature K, then either $K=1$ (and M is totally geodesic) or $K=1/16$.*

The Main Theorem of the present paper is given in Section 5; it gives a detailed classification of all totally real 3-dimensional submanifolds of the nearly Kaehler 6-sphere $S^6(1)$ of which the sectional curvatures K satisfy the condition $K\geqq 1/16$. Along proving this Main Theorem, in Section 4 we obtain, in Corollary 4.1, the solution of Problem B for the present situation. In Section 2 we recall the construction of the natural nearly Kaehler structure on $S^6(1)$, and in Section 3 we give some basic formulas concerning totally real submanifolds in $S^6(1)$, in particular some formulas on 3-dimensional totally real submanifolds in $S^6(1)$ which were obtained first in our paper [**D-O-V-V2**] with B. Opozda which gave a partial solution of Problem B in this case. The Main Theorem of this paper was announced in [**D-V-V2**].

2. The nearly Kaehler $S^6(1)$.

Let $e_0, e_1, \cdots, e_7$ be the standard basis of $\boldsymbol{R}^8$. Then each point α of $\boldsymbol{R}^8$ can be written in a unique way as

$$\alpha = Ae_0 + x,$$

where $A \in \boldsymbol{R}$ and x is a linear combination of $e_1, \cdots, e_7$. α can be viewed as a Cayley number, and is called purely imaginary when $A=0$. For any pair of purely imaginary x and y, we consider the multiplication $\cdot$ given by

$$x \cdot y = -\langle x, y \rangle e_0 + x \times y,$$

where $\langle , \rangle$ is the standard scalar product on $\boldsymbol{R}^8$ and $x \times y$ is defined by the following multiplication table for $e_j \times e_k$,

j/k	1	2	3	4	5	6	7
1	0	e_3	$-e_2$	e_5	$-e_4$	e_7	$-e_6$
2	$-e_3$	0	e_1	e_6	$-e_7$	$-e_4$	e_5
3	e_2	$-e_1$	0	$-e_7$	$-e_6$	e_5	e_4
4	$-e_5$	$-e_6$	e_7	0	e_1	e_2	$-e_3$
5	e_4	e_7	e_6	$-e_1$	0	$-e_3$	$-e_2$
6	$-e_7$	e_4	$-e_5$	$-e_2$	e_3	0	e_1
7	e_6	$-e_5$	$-e_4$	e_3	e_2	$-e_1$	0 .

For two Cayley numbers $\alpha = Ae_0 + x$ and $\beta = Be_0 + y$, the Cayley multiplication $\cdot$, which makes $\boldsymbol{R}^8$ the Cayley algebra $\mathcal{C}$, is defined by

$$\alpha \cdot \beta = ABe_0 + Ay + Bx + x \cdot y.$$

We recall that the multiplication$\cdot$ of $\mathcal{C}$ is neither commutative nor associative.

The set $\mathcal{C}_+$ of all purely imaginary Cayley numbers can clearly be viewed as a 7-dimensional linear subspace $\boldsymbol{R}^7$ of $\boldsymbol{R}^8$. In $\mathcal{C}_+$ we consider the unit hypersphere which is centered at the origin:

$$S^6(1) = \{x \in \mathcal{C}_+ \mid \langle x, x \rangle = 1\}.$$

Then the tangent space $T_x S^6$ of $S^6(1)$ at a point x may be identified with the affine subspace of $\mathcal{C}_+$ which is orthogonal to x.

On $S^6(1)$ we now define a (1, 1)-tensor field J by putting

$$J_x U = x \times U,$$

where $x \in S^6(1)$ and $U \in T_x S^6$. This tensor field is well-defined (i.e., $J_x U \in T_x S^6$) and determines an almost complex structure on $S^6(1)$, i.e.

$$J^2 = -\mathrm{Id}\,,$$

where Id is the identity transformation ([**F**]). The compact simple Lie group G_2 is the group of automorphisms of C and acts transitively on $S^6(1)$ and preserves both J and the standard metric on $S^6(1)$ ([**F-I**]).

Further, let G be the (2, 1)-tensor field on $S^6(1)$ defined by

$$G(X, Y) = (\tilde{\nabla}_X J)Y\,, \tag{2.1}$$

where $X, Y \in \mathfrak{X}(S^6)$ and where $\tilde{\nabla}$ is the Levi-Civita connection on $S^6(1)$. This tensor field has the following properties:

$$G(X, X) = 0\,, \tag{2.2}$$

$$G(X, Y) + G(Y, X) = 0\,, \tag{2.3}$$

$$G(X, JY) + JG(X, Y) = 0\,, \tag{2.4}$$

$$(\tilde{\nabla}_X G)(Y, Z) = \langle Y, JZ\rangle X + \langle X, Z\rangle JY - \langle X, Y\rangle JZ\,, \tag{2.5}$$

$$\langle G(X, Y), Z\rangle + \langle G(X, Z), Y\rangle = 0\,, \tag{2.6}$$

$$\begin{aligned}\langle G(X, Y), G(Z, W)\rangle = {} & \langle X, Z\rangle\langle Y, W\rangle - \langle X, W\rangle\langle Z, Y\rangle \\ & + \langle JX, Z\rangle\langle Y, JW\rangle - \langle JX, W\rangle\langle Y, JZ\rangle\,,\end{aligned} \tag{2.7}$$

$$G(X, Y) = X \times Y + \langle X, JY\rangle x \tag{2.8}$$

where $X, Y, Z, W \in \mathfrak{X}(S^6)$ ([**Se**], [**G3**]). We recall that (2.2) means that the structure J is *nearly Kaehler*, i.e. $\forall X \in \mathfrak{X}(S^6)$: $(\tilde{\nabla}_X J)X = 0$.

3. Totally real submanifolds of S^6.

A Riemannian manifold M isometrically immersed in S^6, is called a totally real submanifold of S^6 if $J(TM) \subseteq T^{\perp}M$, where $T^{\perp}M$ is the normal bundle of M in S^6. Then, we have $\dim M \leqq 3$. In this paper we consider the case $\dim M = 3$. In [**E1**] Ejiri proved that a 3-dimensional totally real submanifold of S^6 is orientable and minimal, and that $G(X, Y)$ is orthogonal to M for $X, Y \in \mathfrak{X}(M)$. We denote the Levi-Civita connection of M by ∇. The formulas of Gauss and Weingarten are then given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{3.1}$$

and

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi\,, \tag{3.2}$$

where X and Y are vector fields on M and ξ is a normal vector field on M. The second fundamental form h is related to A_ξ by

$$\langle h(X, Y), \xi\rangle = \langle A_\xi X, Y\rangle\,. \tag{3.3}$$

From (4.1) and (4.2) we find

$$(3.4) \qquad D_X(JY) = G(X, Y) + J\nabla_X Y$$

and

$$(3.5) \qquad A_{JX}Y = -Jh(X, Y).$$

If we denote the curvature tensors of ∇ and D by R and R^D, respectively, then the equations of Gauss, Codazzi and Ricci are given by

$$(3.6) \qquad \begin{aligned} R(X, Y, Z, W) = &\langle X, W\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle \\ &+\langle h(X, W), h(Y, Z)\rangle - \langle h(X, Z), h(Y, W)\rangle, \end{aligned}$$

$$(3.7) \qquad (\nabla h)(X, Y, Z) = (\nabla h)(Y, X, Z),$$

$$(3.8) \qquad \langle R^D(X, Y)\xi, \mu\rangle = \langle [A_\xi, A_\mu]X, Y\rangle,$$

where $X, Y, Z, W \in \mathfrak{X}(M)$, ξ and μ are normal vector fields and ∇h is defined by $(\nabla h)(X, Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$.

From (3.5), (3.6) and (3.8) we obtain

$$(3.9) \qquad \langle R^D(X, Y)JZ, JW\rangle = \langle R(X, Y)Z, W\rangle + \langle Z, X\rangle\langle Y, W\rangle - \langle Z, Y\rangle\langle X, W\rangle.$$

From [**D-O-V-V2**], we will also need the following proposition.

PROPOSITION 3.1. *If M is a 3-dimensional, compact, totally real submanifold of $S^6(1)$ and if all sectional curvatures K of M satisfy $K \geqq 1/16$, then*

$$(3.10) \qquad \begin{aligned} &(1) \quad \langle(\nabla h)(v, v, v), Jv\rangle = 0, \quad \text{and} \\ &(2) \quad R(v, A_{Jv}v, A_{Jv}v, v) = \frac{1}{16}(\|A_{Jv}v\|^2 - \langle A_{Jv}v, v\rangle^2), \end{aligned}$$

for all $p \in M$ and $v \in T_pM$.

4. The condition $R(v, A_{Jv}v, A_{Jv}v, v) = 1/16(\|A_{Jv}v\|^2 - \langle A_{Jv}v, v\rangle^2)$.

Let $p \in M$. In this section, we will always use an orthonormal basis of T_pM constructed in the following way. Consider the function f_1 on UM_p defined by $f_1(v) = \langle h(v, v), Jv\rangle$. If f_1 attains an absolute maximum at u, then $\langle h(u, u), Jw\rangle = 0$, for w orthogonal to u. Choose e_1 to be an absolute maximum of f_1. Then, we consider the restriction of f_1 to $\{v \in UM_p \mid \langle v, e_1\rangle = 0\}$. We call this restriction f_2. If f_2 is identically zero, we choose e_2 as an eigenvector of A_{Je_1}. If f_2 is not identically zero, we take e_2 as an absolute maximum point of f_2. Finally, we choose e_3 such that $G(e_1, e_2) = Je_3$. Then, the second fundamental form can be written in the following way

$$h(e_1, e_1) = aJe_1, \quad h(e_2, e_2) = bJe_1 + dJe_2, \quad h(e_3, e_3) = -(a+b)Je_1 - dJe_2$$

$$h(e_1, e_2) = bJe_2 + cJe_3 \quad h(e_1, e_3) = -(a+b)Je_3 + cJe_2, \quad h(e_2, e_3) = cJe_1 - dJe_3,$$

where $a \geqq d \geqq 0$ and $b, c \in \boldsymbol{R}$. Since $\{e_1, e_2, e_3\}$ is an orthonormal basis of T_pM, any vector $v \in T_pM$ can be written as $v = v_1e_1 + v_2e_2 + v_3e_3$, where $v_1, v_2, v_3 \in \boldsymbol{R}$.

Therefore, using the Gauss equation, we find in a straightforward way that (3.10) is equivalent to the following equations in a, b, c and d.

$$\begin{aligned}(4.1)\quad & -32a^4-112a^3b-144a^2b^2-16a^2c^2-32a^2d^2+15a^2-80ab^3-32abc^2-112abd^2\\ & +30ab-16b^4-16b^2c^2-80b^2d^2+15b^2-16c^2d^2-32d^4+15d^2=0,\end{aligned}$$

$$(4.2)\quad c(48a^3+112a^2b+80ab^2+16ac^2-15a+16b^3+16bc^2+16bd^2-15b)=0,$$

$$\begin{aligned}(4.3)\quad & 80a^3b+176a^2b^2-400a^2c^2+96a^2d^2+15a^2+112ab^3-448abc^2+416abd^2+16b^4\\ & -48b^2c^2+352b^2d^2-15b^2-64c^4+224c^2d^2+60c^2+192d^4-90d^2=0,\end{aligned}$$

$$(4.4)\quad c(-80a^2b-80ab^2+80ac^2-15a-32bd^2)=0,$$

$$\begin{aligned}(4.5)\quad & -64a^2b^2-48ab^3+352abc^2-240abd^2-30ab+16b^4-48b^2c^2-336b^2d^2\\ & -15b^2-64c^4-528c^2d^2+60c^2-288d^4+135d^2=0,\end{aligned}$$

$$(4.6)\quad bc(32ab-16b^2-16c^2-48d^2+15)=0,$$

$$(4.7)\quad b^2(16ab-16b^2-16c^2+15)=0,$$

$$(4.8)\quad cd(48ab+48b^2+16c^2+48d^2-15)=0,$$

$$(4.9)\quad d(32a^3+112a^2b+160ab^2+48ac^2+32ad^2-25a+80b^3+16bc^2+64bd^2-35b)=0,$$

$$(4.10)\quad cd(-96a^2-240ab-240b^2-176c^2-192d^2+105)=0$$

$$\begin{aligned}(4.11)\quad & d(-176a^2b-496ab^2-176ac^2-288ad^2+105a-384b^3-512bc^2\\ & -576bd^2+240b)=0,\end{aligned}$$

$$(4.12)\quad cd(8ab^2+8b^2+32c^2+36d^2-15)=0,$$

$$(4.13)\quad bd(-16ab+16b^2+80c^2-15)=0,$$

$$\begin{aligned}(4.14)\quad & 192a^4+608a^3b+704a^2b^2+224a^2c^2+96a^2d^2-90a^2+352ab^3+224abc^2+272abd^2\\ & -150ab+64b^4+112b^2d^2-60b^2-64c^4-400c^2d^2+60c^2+15d^2=0,\end{aligned}$$

$$\begin{aligned}(4.15)\quad & c(-192a^3-368a^2b-304ab^2-176ac^2-96ad^2+105a-128b^3-128bc^2\\ & -320bd^2+120b)=0,\end{aligned}$$

$$\begin{aligned}(4.16)\quad & -48a^3b-112a^2b^2+144a^2c^2-112a^2d^2+5a^2-128ab^3-336abd^2+60ab\\ & -64b^4-240b^2d^2+60b^2+64c^4+144c^2d^2-60c^2+5d^2=0,\end{aligned}$$

$$(4.17)\quad c(144a^2b+80ab^2-48ac^2+96ad^2-15a+128b^3+128bc^2+448bd^2-120b)=0,$$

$$\begin{aligned}(4.18)\quad & 32a^2b^2-96ab^3-224abc^2+16abd^2+30ab+64b^4-16b^2d^2-60b^2-64c^4\\ & -272c^2d^2+60c^2+15d^2=0,\end{aligned}$$

$$(4.19)\quad cd(-112ab-48b^2+80c^2-15)=0,$$

$$(4.20)\quad d(-288a^3-848a^2b-976ab^2-176ac^2+105a-288b^3+480bc^2+30b)=0,$$

(4.21) $cd(96a^2+272ab+336b^2-48c^2-15)=0$,

(4.22) $d(-16a^2b+48ab^2+80ac^2-15a-32b^3-288bc^2+30b)=0$,

(4.23) $-288a^4-816a^3b-848a^2b^2-528a^2c^2+135a^2-384ab^3-384abc^2+180ab$
$-64b^4-128b^2c^2+60b^2-64c^4+60c^2=0$,

(4.24) $ac(36a^2+32ab+32b^2+32c^2-15)=0$,

(4.25) $16a^3b-80a^2b^2-272a^2c^2+15a^2+128ab^3+128abc^2-60ab-64b^4$
$-128b^2c^2+60b^2-64c^4+60c^2=0$.

In order to solve these equations, we consider the following cases.

Case 1: $a\neq0$, $b\neq0$, $c\neq0$, $d\neq0$. A contradiction follows if we compare (4.7) with (4.13).

Case 2: $a\neq0$, $b\neq0$, $c\neq0$, $d=0$. In this case, f_2 is identically zero. Therefore, e_2 is an eigenvector of A_{Je_1}. This implies that $c=0$. Therefore, this case cannot occur.

Case 3: $a\neq0$, $b\neq0$, $c=0$. Then the equation (4.7) becomes

(4.26) $$16ab-16b^2+15=0.$$

Combining with (4.23), we thus obtain that

(4.27) $$(16b^2-9)^2(16b^2-15)(16b^2-5)=0.$$

Since $ab\neq0$, we see from (4.26) that $16b^2-15\neq0$. Therefore, from (4.27), it follows that

$$b^2=\frac{9}{16} \quad \text{or} \quad b^2=\frac{5}{16}.$$

Using the fact that $a\geqq0$, we deduce from (4.26) that

$$b=-\frac{3}{4} \quad \text{and} \quad a=\frac{1}{2}$$

or

$$b=-\frac{\sqrt{5}}{4} \quad \text{and} \quad a=\frac{\sqrt{5}}{2}.$$

So we have to consider two subcases.

Subcase 3a: $a=\sqrt{5}/2$, $b=-\sqrt{5}/4$ and $c=0$. From (4.1) it then follows that either $d=0$ or $d=\sqrt{10}/4$. Then, after a straightforward calculation one sees that in both cases all the other equations are also satisfied.

Subcase 3b: $a=1/2$, $b=-3/4$ and $c=0$. From (4.1) we deduce in this case that $d=1/2$. Then putting $u=\sqrt{2}/2(e_2-e_1)$. We see that $\langle h(u, u), Ju\rangle=9\sqrt{2}/16>1/2$. This is in contradiction with the fact that e_1 is chosen as an absolute maximum of f_1.

Case 4: $a\neq 0$, $b=0$, $c=0$. Then, (4.25) immediately leads to a contradiction.

Case 5: $a\neq 0$, $b=0$, $c\neq 0$, $d=0$. Applying the same argument as in Case 2, we obtain a contradiction.

Case 6: $a\neq 0$, $b=0$, $c\neq 0$, $d\neq 0$. First, we deduce from (4.22) that $c^2=3/16$. Then, it follows from (4.21) that $a=1/2$. From (4.8), we then find that $d=1/2$. Now, putting $u=-1/\sqrt{5}\{e_1+e_2-\sqrt{3}e_3\}$ if $c=\sqrt{3}/4$, and $u=-1/\sqrt{5}\{e_1+e_2+\sqrt{3}e_3\}$ if $c=-\sqrt{3}/4$, we see that $\langle h(u, u), Ju\rangle=\sqrt{5}/2>1/2$. So, we obtain again a contradiction.

Case 7: $a=0$. This implies that f_1 is identically zero. By linearization, we then deduce that $b=c=d=0$. In this case, all the equations are trivially satisfied.

By combining this with Proposition 3.1, we immediately obtain the following lemma.

LEMMA 4.1. *If M is a* 3*-dimensional compact totally real submanifold of* $S^6(1)$ *and if all sectional curvatures K of M satisfy* $K\geqq 1/16$, *then, for each point p of M, there exists an orthonomal basis* $\{e_1, e_2, e_3\}$ *of T_pM such that either*

$$\text{(4.28)}\qquad \text{(i)}\quad h(e_1, e_1)=h(e_2, e_2)=h(e_3, e_3)=0\,,\quad h(e_1, e_2)=h(e_1, e_3)=h(e_2, e_3)=0\,,$$

or

$$\text{(4.29)}\qquad \text{(ii)}\quad h(e_1, e_1)=\frac{\sqrt{5}}{2}Je_1,\quad h(e_2, e_2)=-\frac{\sqrt{5}}{4}Je_1+\frac{\sqrt{10}}{4}Je_2\,,$$
$$h(e_3, e_3)=-\frac{\sqrt{5}}{4}Je_1-\frac{\sqrt{10}}{4}Je_2,\quad h(e_1, e_2)=-\frac{\sqrt{5}}{4}Je_2\,,$$
$$h(e_1, e_3)=-\frac{\sqrt{5}}{4}Je_3,\quad h(e_2, e_3)=-\frac{\sqrt{10}}{4}Je_3\,,$$

or

$$\text{(4.30)}\qquad \text{(iii)}\quad h(e_1, e_1)=\frac{\sqrt{5}}{2}Je_1,\quad h(e_2, e_2)=-\frac{\sqrt{5}}{4}Je_1\,,$$
$$h(e_3, e_3)=-\frac{\sqrt{5}}{4}Je_1,\quad h(e_1, e_2)=-\frac{\sqrt{5}}{4}Je_2\,,$$
$$h(e_1, e_3)=-\frac{\sqrt{5}}{4}Je_3,\quad h(e_2, e_3)=0\,.$$

Let M be as in Lemma 4.1. Then we have the following proposition.

PROPOSITION 4.1. *Let $p\in M$. Then we have that,*
(a) *if* (4.28) *holds, then* $K(p)\equiv 1$;
(b) *if* (4.29) *holds, then* $K(p)\equiv 1/16$;
(c) *if* (4.30) *holds, then* $1/16\leqq K(p)\leqq 21/16$,

where 1/16 *and* 21/16 *are actually obtained.*

PROOF. (a) In this case, $h=0$, so p is a geodesic point. From the Gauss equation, we obtain that $K(p)\equiv 1$.

(b) From [**E1**], we find that h_p has the same form as the second fundamental form of a totally real submanifold of constant curvature 1/16. So $K(p)=1/16$ by the Gauss equation.

(c) From the Gauss equation and (4.30), we obtain that

$$R(e_1, e_2)e_2 = R(e_1, e_3)e_3 = \frac{1}{16}e_1, \qquad R(e_2, e_3)e_3 = \frac{21}{16}e_2\,,$$

$$R(e_1, e_2)e_3 = R(e_2, e_3)e_1 = R(e_3, e_1)e_2 = 0\,.$$

Let σ be any plane section of T_pM. Then we can find an orthonormal basis $\{X, Y\}$ of σ such that $X=\cos\theta e_2+\sin\theta e_3$ and $Y=\sin\varphi e_1-\cos\varphi\sin\theta e_2+\cos\varphi\cos\theta e_3$, where $\theta, \varphi\in \boldsymbol{R}$. Then,

$$\begin{aligned}R(X, Y, Y, X) &= \cos^2\theta R(e_2, Y, Y, e_2)+2\cos\theta\sin\theta R(e_2, Y, Y, e_3)\\ &\quad +\sin^2\theta R(e_3, Y, Y, e_3)\\ &= \cos^2\theta\sin^2\varphi R(e_1, e_2, e_2, e_1)+\cos^2\varphi R(e_2, e_3, e_3, e_2)\\ &\quad +\sin^2\theta\sin^2\varphi R(e_1, e_3, e_3, e_1)\\ &= \frac{1}{16}+\frac{20}{16}\cos^2\varphi\,,\end{aligned}$$

and so we have

$$K(\sigma) = \frac{1}{16}+\frac{20}{16}\cos^2\varphi\,,$$

which gives us $1/16\leqq K\leqq 21/16$, where 1/16 is attained when $\cos\varphi=0$, i.e. when the plane σ passes through e_1, and 21/16 is attained only when $\cos\varphi=\pm 1$, i.e. by the plane spanned by e_2 and e_3. ■

The next statements follow easily from Proposition 4.1.

COROLLARY 4.1. *If M is a 3-dimensional compact totally real submanifold of $S^6(1)$ and if the sectional curvatures K of M satisfy either $1/16\leqq K\leqq 1$ or $1/16\leqq K<21/16$, then either $K\equiv 1$* (M is totally geodesic) *or $K\equiv 1/16$ on M.*

In the following proposition, we study more closely the case (c) of Proposition 4.1.

PROPOSITION 4.2. *Let M be a 3-dimensional compact totally real submanifold of $S^6(1)$ with K not constant and satisfying $K\geqq 1/16$. Then there exists globally a tangent vector field E_1 and locally tangent vector fields E_2 and E_3 such that*

(a) *$\{E_1, E_2, E_3\}$ is a local orthonormal frame such that $G(E_2, E_3)=JE_1$,*

(b) *for any* $p \in M$, f_1 *attains its maximum value at* $E_1(p)$,

(c) $h(E_1, E_1) = \frac{\sqrt{5}}{2} JE_1, \qquad h(E_2, E_2) = h(E_3, E_3) = -\frac{\sqrt{5}}{4} JE_1,$

$$h(E_1, E_2) = -\frac{\sqrt{5}}{4} JE_2, \qquad h(E_2, E_3) = 0, \qquad h(E_1, E_3) = -\frac{\sqrt{5}}{4} JE_3,$$

(d) $\nabla_{E_1} E_1 = \nabla_{E_2} E_2 = \nabla_{E_3} E_3 = 0, \qquad \nabla_{E_1} E_2 = -\frac{11}{4} E_3, \qquad \nabla_{E_2} E_1 = +\frac{1}{4} E_3,$

$$\nabla_{E_1} E_3 = \frac{11}{4} E_2, \qquad \nabla_{E_3} E_1 = -\frac{1}{4} E_2, \qquad \nabla_{E_2} E_3 = -\nabla_{E_3} E_2 = -\frac{1}{4} E_1.$$

PROOF. Since $K \geqq 1/16$ and K is not constant, it follows immediately from Proposition 4.1 and Lemma 4.1 that the vector field $E_1(p)$, where $E_1(p)$ is the maximum point of f_1 at each point p, is well defined on the whole of M and differentiable. Then we take E_2 and E_3 as locally defined orthonormal vector fields which are orthogonal to E_1. By changing, if necessary, the sign of E_3, it is then clear from Lemma 5.1 that (a), (b) and (c) are satisfied.

To prove (d), we first take a local orthonormal frame $\{E_1, E_2, E_3\}$ such that (a), (b) and (c) are satisfied. We write the connection in the following form:

$$\begin{aligned} \nabla_{E_1} E_1 &= a_{12} E_2 + a_{13} E_3, & \nabla_{E_2} E_2 &= a_{21} E_1 + a_{23} E_3, \\ \nabla_{E_3} E_3 &= a_{31} E_1 + a_{32} E_2, & \nabla_{E_1} E_2 &= -a_{12} E_1 + a_{11} E_3, \\ \nabla_{E_2} E_1 &= -a_{21} E_2 + a_{22} E_3, & \nabla_{E_3} E_1 &= -a_{31} E_3 + a_{33} E_2, \end{aligned}$$

where $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}$ are locally defined functions on M. Now, we can use the Codazzi equations: from $(\nabla h)(E_1, E_2, E_1) = (\nabla h)(E_2, E_1, E_1)$, we deduce that

$$a_{12} = a_{21} = 0 \quad \text{and} \quad a_{22} = \frac{1}{4}, \tag{4.31}$$

and from $(\nabla h)(E_1, E_3, E_1) = (\nabla h)(E_3, E_1, E_1)$ we obtain that

$$a_{13} = a_{31} = 0 \quad \text{and} \quad a_{33} = -\frac{1}{4}. \tag{4.32}$$

Using (4.31) and (4.32), the Gauss equation and the fact that $R(e_1, e_2, e_2, e_3) = R(e_1, e_3, e_3, e_2) = 0$ and $R(e_2, e_3, e_3, e_2) = 21/16$, we obtain that

$$E_1(a_{23}) - E_2(a_{11}) + a_{32}\left(a_{11} - \frac{1}{4}\right) = 0, \tag{4.33}$$

$$E_1(a_{32}) + E_3(a_{11}) + \left(\frac{1}{4} - a_{11}\right) a_{23} = 0, \tag{4.34}$$

$$E_2(a_{32}) + E_3(a_{23}) - \frac{1}{2} a_{11} - a_{23}^2 - a_{32}^2 = \frac{22}{16}. \tag{4.35}$$

Now, we use the following transformation of the frame $\{E_1, E_2, E_3\}$:

$$U_1 = E_1, \quad U_2 = \cos\theta E_2 + \sin\theta E_3, \quad U_3 = -\sin\theta E_2 + \cos\theta E_3,$$

where θ is an arbitrary locally defined function on M. It is immediately clear that $\{U_1, U_2, U_3\}$ also satisfies (a), (b) and (c). Now, we look for a basis $\{U_1, U_2, U_3\}$ that also satisfies (d). Then the function θ must satisfy the following system of differential equations

$$\begin{cases} d\theta(E_1) + a_{11} + \dfrac{11}{4} = 0 \\ d\theta(E_2) + a_{23} = 0 \\ d\theta(E_3) - a_{32} = 0, \end{cases}$$

and conversely, if θ satisfies the system, then $\{U_1, U_2, U_3\}$ satisfies (d). Now the system has locally a solution if and only if the 1-form $\omega = (a_{11} + 11/4)\theta_1 + a_{23}\theta_2 - a_{32}\theta_3$, where $\{\theta_1, \theta_2, \theta_3\}$ is the dual basis of $\{E_1, E_2, E_3\}$, is closed. One can easily verify that $d\omega = 0$ is equivalent to (4.33), (4.34) and (4.35). ∎

5. The examples and the classification.

In this section, we give three examples of totally real 3-dimensional compact submanifolds of $S^6(1)$ satisfying $K \geqq 1/16$. Using the results of Section 4, we then prove that these examples are basically the only ones.

EXAMPLE 5.1. Consider the unit sphere $S^3 = \{(y_1, y_2, y_3, y_4) \in \boldsymbol{R}^4 \mid y_1^2 + y_2^2 + y_3^2 + y_4^2 = 1\}$ in $\boldsymbol{R}^4$. Let X_1, X_2, X_3 be the vector fields defined by $X_1(y_1, y_2, y_3, y_4) = (y_2, -y_1, y_4, -y_3)$, $X_2(y_1, y_2, y_3, y_4) = (y_3, -y_4, -y_1, y_2)$ and $X_3(y_1, y_2, y_3, y_4) = (y_4, y_3, -y_2, -y_1)$. Then X_1, X_2 and X_3 form a basis of tangent vector fields to S^3. Then we have that $[X_1, X_2] = 2X_3$, $[X_2, X_3] = 2X_1$ and $[X_3, X_1] = 2X_2$. We define a metric $\langle \cdot, \cdot \rangle$ on S^3 such that X_1, X_2 and X_3 are orthogonal and such that $\langle X_1, X_1 \rangle = 4/9$, $\langle X_2, X_2 \rangle = 8/3$ and $\langle X_3, X_3 \rangle = 8/3$. Then $E_1 = 3/2 X_1$, $E_2 = \sqrt{3}/2\sqrt{2} X_2$, $E_3 = -\sqrt{3}/2\sqrt{2} X_3$ form an orthonormal basis on S^3. We denote the Levi-Civita connection of $\langle \cdot, \cdot \rangle$ by ∇.

LEMMA 5.1.

$$\nabla_{E_1} E_1 = \nabla_{E_2} E_2 = \nabla_{E_3} E_3 = 0, \quad \nabla_{E_1} E_2 = -\frac{11}{4} E_3, \quad \nabla_{E_2} E_1 = \frac{1}{4} E_3,$$

$$\nabla_{E_1} E_3 = \frac{11}{4} E_2, \quad \nabla_{E_3} E_1 = -\frac{1}{4} E_2, \quad \nabla_{E_2} E_3 = -\nabla_{E_3} E_2 = -\frac{1}{4} E_1.$$

PROOF. From the definition of $\{E_1, E_2, E_3\}$ we obtain that $[E_1, E_2] = -3E_3$, $[E_2, E_3] = -1/2 E_1$ and $[E_3, E_1] = -3E_2$. Then ∇ is determined by the Koszul-formula

$$(5.1)\qquad \langle\nabla_X Y, Z\rangle=\frac{1}{2}\{X\langle Y, Z\rangle+Y\langle Z, X\rangle-Z\langle X, Y\rangle-\langle[Y, Z], X\rangle -\langle[X, Z], Y\rangle-\langle[Y, X], Z\rangle\}.$$

From (5.1) we can easily see that $\langle\nabla_{E_i}E_j, E_k\rangle=0$ for all i, j, k unless i, j and k are mutually different. Therefore $\nabla_{E_1}E_1=\nabla_{E_2}E_2=\nabla_{E_3}E_3=0$. We now compute $\nabla_{E_1}E_2$. We know that $\nabla_{E_1}E_2$ is in the direction of E_3. Therefore we obtain

$$\begin{aligned}\nabla_{E_1}E_2 &= -\frac{1}{2}\{\langle[E_2, E_3], E_1\rangle+\langle[E_1, E_3], E_2\rangle+\langle[E_2, E_1], E_3\rangle\}E_3\\ &= -\frac{1}{2}\left\{-\frac{1}{2}+3+3\right\}E_3=-\frac{11}{4}E_3,\end{aligned}$$

and $\nabla_{E_2}E_1=\nabla_{E_1}E_2-[E_1, E_2]=1/4E_3$.

The other cases can be computed similarly. ■

LEMMA 5.2. $R(E_1, E_2)E_3=R(E_2, E_3)E_1=R(E_3, E_1)E_2=0$,

$$R(E_1, E_2)E_2=\frac{1}{16}E_1=R(E_1, E_3)E_3,\quad R(E_2, E_3)E_3=\frac{21}{16}E_2.$$

PROOF. Straightforward from Lemma 5.1.

LEMMA 5.3. $\langle R(X, Y)W, Z\rangle=1/16(\langle X, Z\rangle\langle Y, W\rangle-\langle X, W\rangle\langle Y, Z\rangle)$

$$+\frac{20}{16}(\langle X^{\perp}, Z^{\perp}\rangle\langle Y^{\perp}, W^{\perp}\rangle-\langle X^{\perp}, W^{\perp}\rangle\langle Y^{\perp}, Z^{\perp}\rangle),$$

where $V^{\perp}$ denotes the orthogonal complement of a vector V with respect to E_1.

PROOF. We denote the expression on the right hand side by $Q(X, Y, W, Z)$. It is clear that Q is curvaturelike, in the sense of [**O'N**]. Therefore, to prove the lemma, we only have to prove that all sectional curvatures of R and Q are the same. Let σ be any plane in the tangent space of S^3. Then we can find an orthonormal basis $\{X, Y\}$ of σ such that $X=\cos\theta E_2+\sin\theta E_3$ and $Y=\sin\varphi E_1-\cos\varphi\sin\varphi E_2+\cos\varphi\cos\theta E_3$, where $\theta, \varphi\in\boldsymbol{R}$. The same calculation as in the proof of Proposition 4.1 shows that $R(X, Y, Y, X)=K(\sigma)=1/16+20/16\cos^2\varphi$. On the other hand, we obtain that also

$$Q(X, Y, Y, X)=\frac{1}{16}+\frac{20}{16}\cos^2\varphi. \qquad ■$$

From the proof of Lemma 5.3, we have that the sectional curvature of the plane σ is given by

$$K(\sigma)=\frac{1}{16}+\frac{20}{16}\cos^2\varphi.$$

It follows that $1/16\leqq K(\sigma)\leqq 21/16$, where $1/16$ is attained for every plane which contains E_1, and where $21/16$ is attained only for the plane spanned by

E_2 and E_3. Now, we define an immersion from $S^3(1)$, equipped with this metric $\langle , \rangle$ into $S^6(1)$ by

$$f:\ S^3(1) \longrightarrow S^6(1):\ (y_1, y_2, y_3, y_4) \longmapsto (x_1, x_2, x_3, x_4, x_5, x_6, x_7),$$

where

$$x_1 = \frac{1}{9}(5y_1^2+5y_2^2-5y_3^2-5y_4^2+4y_1), \quad x_2 = -\frac{2}{3}y_2, \tag{5.2}$$

$$x_3 = \frac{2\sqrt{5}}{9}(y_1^2+y_2^2-y_3^2-y_4^2-y_1), \quad x_4 = \frac{\sqrt{3}}{9\sqrt{2}}(-10y_3y_1-2y_3-10y_2y_4),$$

$$x_5 = \frac{\sqrt{3}\sqrt{5}}{9\sqrt{2}}(2y_1y_4-2y_4-2y_2y_3), \quad x_6 = \frac{\sqrt{3}\sqrt{5}}{9\sqrt{2}}(2y_1y_3-2y_3+2y_2y_4),$$

$$x_7 = -\frac{\sqrt{3}}{9\sqrt{2}}(10y_1y_4+2y_4-10y_2y_3),$$

and where $y_1^2+y_2^2+y_3^2+y_4^2=1$. The proof of the following theorem now follows straightforwardly from these formulas.

THEOREM 5.1. *f, as defined above, is an isometric, totally real embedding from* $(S^3, \langle , \rangle)$ *into* $S^6(1)$.

LEMMA 5.4. *This immersion satisfies the following equalities:*

$$h(E_1, E_1) = \frac{\sqrt{5}}{2} Jf_*(E_1), \qquad G(f_*E_1, f_*E_2) = Jf_*(E_3).$$

PROOF. This follows straightforwardly from the definitions and from (2.8).

LEMMA 5.5. *The given orthonormal frame* $\{E_1, E_2, E_3\}$ *satisfies the conditions* (a), (b), (c) *and* (d) *of Proposition* 4.2.

PROOF. Since E_1 is always orthogonal to the only plane with sectional curvature 21/16, it follows that either E_1 or $-E_1$ satisfy (b). From Lemma 5.4 it follows that indeed E_1 satisfies (b). Also (a) follows from Lemma 5.4. (d) is Lemma 5.1 and (c) follows from the fact that M is totally real, and satisfies case (iii) of Lemma 4.1. ■

EXAMPLE 2. In [**E1**] N. Ejiri proved the existence of a totally real immersion $x: S^3(1/16) \to S^6(1)$. By [**D-W**], this immersion can be realized by using harmonic polynomials of degree 6. Here we give explicitly the immersion. Define a map x:

$$S^3\left(\frac{1}{16}\right) = \{(y_1, y_2, y_3, y_4) \in \boldsymbol{R}^4 \mid y_1^2+y_2^2+y_3^2+y_4^2=16\} \longrightarrow \mathcal{C}_+ = \boldsymbol{R}^7$$

by

$$x_1(y_1, y_2, y_3, y_4) = \sqrt{15}\, 2^{-10}(y_1y_3+y_2y_4)(y_1y_4-y_2y_3)(y_1^2+y_2^2-y_3^2-y_4^2)$$

$$x_2(y_1, y_2, y_3, y_4) = 2^{-12}[-\sum_j y_j^6+5\sum_{i<j} y_i^2y_j^2(y_i^2+y_j^2)-30\sum_{i<j<k} y_i^2y_j^2y_k^2]$$

$$x_3(y_1, y_2, y_3, y_4) = 2^{-10}[y_3y_4(y_3^2-y_4^2)(y_3^2+y_4^2-5y_1^2-5y_2^2) \\ +y_1y_2(y_1^2-y_2^2)(y_1^2+y_2^2-5y_3^2-5y_4^2)]$$

$$x_4(y_1, y_2, y_3, y_4) = 2^{-12}[y_2y_4(y_2^4+3y_3^4-y_4^4-3y_1^4)+y_1y_3(y_3^4+3y_2^4-y_1^4-3y_4^4) \\ +2(y_1y_3-y_2y_4)(y_1^2(y_2^2+4y_4^2)-y_3^2(y_4^2+4y_2^2))]$$

$$x_5(y_1, y_2, y_3, y_4) = x_4(y_2, -y_1, y_3, y_4)$$

$$x_6(y_1, y_2, y_3, y_4) = \sqrt{6}\; 2^{-12}[y_1y_3(y_2^4+5y_2^4-y_3^4-5y_4^4) \\ -y_2y_4(y_2^4+5y_1^4-y_4^4-5y_3^4)+10(y_1y_3-y_2y_4)(y_3^2y_4^2-y_1^2y_2^2)]$$

$$x_7(y_1, y_2, y_3, y_4) = x_6(y_2, -y_1, y_3, y_4).$$

THEOREM 5.2. *If we define x as above, then x is a totally real isometric immersion $x: S^3(1/16)\to S^6(1)$.*

PROOF. The theorem can be proved by a straightforward computation. ∎

REMARKS. 1. In [**M2**], K. Mashimo classifies the 3-dimensional compact totally real submanifold of S^6, which are obtained as orbits of closed subgroups of G_2. He proves that one of them has constant curvature 1/16. His description inspired us to find the above explicit expression. In fact $x(S^3(1/16))$ is nothing but such an orbit.

2. In the same paper, Mashimo proves that, if x_1 and x_2 are two isometric totally real immersions of $S^3(1/16)$ into $S^6(1)$, then $x_1=gx_2$ for some $g\in G_2$.

3. Since x has degree 6, we can define a totally real immersion of $\boldsymbol{RP}^3(1/16)$ in S^6, but $x(S^3(1/16))$ is neither an embedded sphere, nor an embedded projective space. This is already proved by Mashimo in [**M1**], where he shows that the immersion is at least 6-fold. Using our description we can prove that x is 24-fold. Indeed, let p be the point $(4, 0, 0, 0)$ in S^3. Then $x(p)=(0, -1, 0, \cdots, 0)$, and we show that there are exactly 23 other points in S^3 which are mapped onto the same point. Using the fact that $y_1^2+y_2^2+y_3^2+y_4^2=16$, we easily see that

$$x_2(y_1, y_2, y_3, y_4) = -1+2^{-9}\cdot N(y_1, y_2, y_3, y_4),$$

where N is given by

$$N(y_1, y_2, y_3, y_4) = \sum_{i<j} y_i^2y_j^2(y_i^2+y_j^2)-3\sum_{i<j<k} y_i^2y_j^2y_k^2.$$

It is sufficient to prove that there are exactly 24 solutions (y_1, y_2, y_3, y_4) such that $N(y_1, y_2, y_3, y_4)=0$ and $y_1^2+y_2^2+y_3^2+y_4^2=16$. Put $\lambda_i=y_i^2$ and suppose that $\lambda_1\geqq\lambda_2\geqq\lambda_3\geqq\lambda_4\geqq 0$. Then $N=0$ means

$$(\lambda_1-\lambda_2)(\lambda_2(\lambda_1-\lambda_4)+\lambda_3(\lambda_1-\lambda_4)+\lambda_4(\lambda_1-\lambda_2))+(\lambda_2-\lambda_3)(\lambda_2(\lambda_1-\lambda_4)$$
$$+\lambda_3(\lambda_2-\lambda_4)+\lambda_1\lambda_2-\lambda_3\lambda_4)+(\lambda_3-\lambda_4)(\lambda_1(\lambda_3-\lambda_4)+\lambda_2(\lambda_3-\lambda_4)+\lambda_3(\lambda_2-\lambda_4))=0\,.$$

This equation has non-zero solutions $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)=(\lambda, \lambda, \lambda, \lambda)$, $\lambda>0$, or $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)=(\lambda, 0, 0, 0)$, $\lambda>0$.

Since N is invariant under permutation of $\{y_1, y_2, y_3, y_4\}$ and under the change of sign of one or more y_k, we obtain as the set of solutions of $N=0$ and $\sum_k y_k^2=16$:

$$S=\{(4, 0, 0, 0), (-4, 0, 0, 0), (0, 4, 0, 0), \cdots, (0, 0, 0, -4), (2, 2, 2, 2),$$
$$(-2, 2, 2, 2), \cdots, (-2, -2, -2, -2)\}.$$

It is clear that $\#S=24$.

EXAMPLE 5.3. If i denotes the inclusion map of $M=\{x\in S^6(1)\mid x=x_1e_1+x_3e_3+x_5e_5+x_7e_7\}$, then $i: M\to S^6(1)$ is a totally real totally geodesic immersion.

Before proving the classification theorem, we first need one more lemma.

LEMMA 5.6. *Let M^n and $\tilde{M}^n$ be Riemannian manifolds with Levi-Civita connections ∇ and $\tilde{\nabla}$. Suppose that there exist c_{ij}^k, $i, j, k\in\{1, \cdots, n\}$ such that for all $p\in M$ and $\tilde{p}\in\tilde{M}$ there exist orthonormal frame fields $\{E_i\}$ around $\tilde{p}$ and $\{\tilde{E}_i\}$ around $\tilde{p}$, such that $\nabla_{E_i}E_j=\sum_k c_{ij}^k E_k$ and $\tilde{\nabla}_{\tilde{E}_i}\tilde{E}_j=\sum_k c_{ij}^k\tilde{E}_k$ for all i, j. Then for every $p\in M$ and $\tilde{p}\in\tilde{M}$ there exists a local isometry f which maps a neighbourhood of p onto a neighbourhood of $\tilde{p}$, and E_i on $\tilde{E}_i$.*

PROOF. The lemma can be proved similarly as the local version of the Cartan-Ambrose-Hicks theorem, cfr. (the proof of) Theorem 1.7.18 of [**Wo**, p. 30]. ■

If M satisfies the condition of the lemma, then we obtain, applying the lemma for $M=\tilde{M}$, that M is locally homogeneous. This could also be proved using the fact that M is strongly curvature homogeneous [**Si**]. If in addition, M is complete and simply connected, then M is homogeneous.

In the following, $x_1: M_1\to S^6(1)$, $x_2: M_2\to S^6(1)$ and $x_3: M_3\to S^6(1)$ denote respectively the first, second and third examples of this section.

MAIN THEOREM. *Let $x: M^3\to S^6(1)$ be a totally real isometric immersion of a 3-dimensional complete Riemannian manifold M into the nearly Kaehler $S^6(1)$. If the sectional curvatures K of M satisfy $K\geqq 1/16$, then either M is simply connected and x is congruent to*

$$\text{(i)}\quad x_1: M_1\longrightarrow S^6(1),\qquad i.e.\ \frac{1}{16}\leqq K\leqq\frac{21}{16}$$

or

(ii) $x_3: M_3 \longrightarrow S^6(1)$, *i.e.* $K \equiv 1$,

or $\tilde{x}$, *the composition of the universal covering map of* M *with* x, *is congruent to*

$$x_2: M_2 \longrightarrow S^6(1), \qquad i.e.\ K \equiv \frac{1}{16}.$$

PROOF. Let $\tilde{x} = x \circ \pi$, where π is the universal covering map $\pi: \tilde{M} \to M$. By the Bonnet-Myers-theorem, we know that $\tilde{M}$ (as well as M) is compact.

By Proposition 4.1, we obtain that either $\tilde{M}$ is totally geodesic, such that $\tilde{x}$ is congruent to x_3, or $\tilde{M}$ has constant curvature 1/16 such that $\tilde{x}$ is congruent to x_2, or the sectional curvatures K of M vary between 1/16 and 21/16.

In the last case, from Proposition 4.2(d), Lemma 5.1 and Lemma 5.6, we obtain that $\tilde{M}$ is homogeneous and locally isometric to M_1. M_1 being analytic, we obtain that there is an isometry between M_1 and $\tilde{M}$. Therefore there exists an orthonormal basis $\{E_1, E_2, E_3\}$ of M_1 and $\{F_1, F_2, F_3\}$ of $\tilde{M}$, both defined globally and satisfying Proposition 4.2, and an isometry $\varphi: M_1 \to \tilde{M}$ such that $\varphi_* E_i = F_i$, $i=1, 2, 3$.

Let ϕ be the map between the normal bundles of M_1 and $\tilde{M}$ defined by $\phi(JE_i) = JF_i$. Then ϕ preserves the bundle metric, the second fundamental form and the normal connection. By the rigidity theorem of submanifolds, $\tilde{x}$ and x_1 are congruent. Since x_1 is an embedding, it follows that $\tilde{x}$ is an embedding in case (i), and therefore that π is an isometry. ∎

FINAL REMARKS. 1. It's good to remark that the nearly Kaehler structure J used by Mashimo and the almost complex structure $\tilde{J}$ used by Ejiri are different, namely $\tilde{J} = AJA$, where A is the isometry defined by $Ae_k = e_k$, $k=1, \cdots, 6$, $Ae_7 = -e_7$. In this paper we have always used $\tilde{J}$.

2. It's easy to prove that the isometry σ in the proof of the main theorem belongs to G_2. Indeed, since $\sigma(E_i) = F_i$ and $\sigma(JE_i) = JF_i$, $i=1, 2, 3$, we obtain that $\{u_0 = p,\ u_1 = E_1(p),\ u_2 = E_2(p),\ u_3 = E_3(p),\ u_4 = JE_1(p) = u_0 \times u_1 = u_2 \times u_3,\ u_5 = JE_2(p) = u_0 \times u_2 = u_3 \times u_1,\ u_6 = JE_3(p) = u_0 \times u_3 = u_1 \times u_2\}$ is mapped by σ into $\{v_0 = \sigma(p) = p',\ v_1 = F_1(p'),\ v_2 = F_2(p'),\ v_3 = F_3(p'),\ v_4 = JF_1(p'),\ v_5 = JF_2(p'),\ v_6 = JF_3(p')\}$. Using the definition of J and (2.8), we see that $\sigma(u_i \times u_j) = v_i \times v_j = \sigma(u_i) \times \sigma(u_j)$ for $i, j = 0, \cdots, 6$. This means that $\sigma \in G_2$. In the same way one can prove that two totally geodesic totally real 3-dimensional submanifolds are congruent by an element of G_2. Therefore we can replace the word "congruent" in the main theorem by the words "congruent by an automorphism of $\mathcal{C}$", or shorter "G_2-congruent".

3. By the same arguments as used for proving the rigidity, one can prove that M_1 and M_2 as well as M_3 are orbits under some subgroup of G_2. In particular M_1 is congruent to the orbit M_1 of Mashimo's paper [**M2**].

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