J. Math. Soc. Japan
Vol. 41, No. 3, 1989

Two results in the affine hypersurface theory

By Takashi KUROSE

(Received April 4, 1988)
(Revised June 10, 1988)

Introduction.

In this paper, we shall show two results in the affine geometry.

The first theorem is on the structures of affine spheres in 3-dimensional real vector space $\boldsymbol{R}^3$. The complete affine spheres have been already classified by Calabi [2], Pogorelov [8], Cheng and Yau [4] and Sasaki [9]. On the other hand, we know little on the structures of affine spheres without the condition of completeness. We study non-complete affine spheres which satisfy a certain curvature condition for their affine metric.

In the second theorem, we shall study the hypersurfaces obtained as the graphs of the affine normal vector fields of affine hypersurfaces. It will be shown that the hypersurfaces thus obtained satisfy an integral formula in terms of the affine invariants. Conversely, it will be proved the integral formula is also the sufficient condition for affine hypersurfaces to be constructed in this way.

To explain our results more precisely, we review here some notations and facts in the affine geometry. (For more details, see [1], [2] and [5].) Let x: $M \to \boldsymbol{R}^{n+1}$ be a strictly convex hypersurface of $\boldsymbol{R}^{n+1}$. If we choose a vector field ξ of $\boldsymbol{R}^{n+1}$ along M such that $T(\boldsymbol{R}^{n+1})|M=T(M)+\boldsymbol{R}\cdot\xi$, we can define the induced affine connection ∇ and the second fundamental form h as follows: for arbitrary vector fields X and Y on M,

$$D_X Y = \nabla_X Y + h(X, Y)\cdot\xi,$$

where D is the standard affine connection of $\boldsymbol{R}^{n+1}$, and the vector field $\nabla_X Y$ is tangent to M.

Because of the strict convexity of M, h is definite and we can determine ξ uniquely by the following two conditions:

(i) For any vector field X of M, the vector field $A(X)=-\nabla_X\xi$ is tangent to M.

(ii) Let vol be the standard volume element of $\boldsymbol{R}^{n+1}$, and $(e_1, \cdots, e_n)$ the frame field of M. Then

$$\mathrm{vol}(e_1, \cdots, e_n, \xi) = \sqrt{\det((h(e_i, e_j))_{i,j=1,\dots,n})}.$$

The ξ satisfying (i) and (ii) is called the *affine normal vector field*. When we take the affine normal vector field as ξ, then h and A defined as above are called the *affine metric* and the *affine shape operator*, respectively. In this paper ξ always means the affine normal vector field.

Affine hyperspheres are by definition such strictly convex hypersurfaces that the affine shape operator is proportional to the identity operator. On affine hyperspheres, the *affine mean curvature* $L=(1/n)\cdot\mathrm{trace}(A)$ is constant.

Our first result concerns the local structure of affine spheres in $\boldsymbol{R}^3$ satisfying a curvature condition. We have:

THEOREM I. *Suppose that M is an affine sphere in $\boldsymbol{R}^3$ with constant Gaussian curvature K with respect to its affine metric. Then, M is a quadric surface or M is affinely equivalent to an open submanifold of the surface*

$$S(c) = \{(x^1, x^2, x^3)\in \boldsymbol{R}^3;\ \ x^1x^2x^3=c, \text{ and } x^1, x^2, x^3>0\},$$

where c is some positive real number.

In the recent paper, Nomizu and Pinkall classified the hypersurfaces whose induced connections are complete and flat for a suitable choice of the vector field ξ([7]). For a strictly convex hypersurface, the induced connection coincides with the Levi-Civita connection for the affine metric if and only if the hypersurface is a hyperquadric and ξ is proportional to the affine normal vector field. Hence, it remains a problem to classify complete affine hypersurfaces with the flat affine metric. We shall give a result in this direction as a corollary to Theorem I.

COROLLARY. *Suppose that a strictly convex surface M of $\boldsymbol{R}^3$ is flat with respect to its affine metric. If M is affinely homogeneous, that is, M admits a transitive group of unimodular affine transformations, then M is a paraboloid or M is affinely equivalent to $S(c)$ for some positive real number c.*

REMARK. Affinely homogeneous surfaces of $\boldsymbol{R}^3$ were classified by Guggenheimer ([**6**], Theorem 12-4). We can also obtain the same result as in Corollary from this theorem with some calculations. In our proof, we do not need the classification theorem.

A strictly convex hypersurface $x: M\to\boldsymbol{R}^{n+1}$ is called *affinely strictly convex* if the affine shape operator has only positive eigenvalues. In this case, $\tilde{x}=-\xi$: $M\to\boldsymbol{R}^{n+1}$ is also a strictly convex hypersurface. It is a natural problem what properties $\tilde{x}$ has. In order to give an answer to this problem, we define two notations.

Let $\boldsymbol{R}_{n+1}$ be the dual space of $\boldsymbol{R}^{n+1}$. The *affine conormal vector field* χ of M is the $\boldsymbol{R}_{n+1}$-valued function on M uniquely determined by $\langle\chi, \xi\rangle\equiv 1$ on M and χ vanishes on $T(M)$. For a point p of M, the *affine distance* l at p is defined by the following equation:

$$l=-\mathrm{vol}(e_1, \cdots, e_n, p),$$

where $(e_1, \cdots, e_n)$ is the orthonormal frame for $T_p(M)$ with respect to the affine metric. Our second result is as follows:

THEOREM II. *Suppose that* $x: \boldsymbol{S}^n\to\boldsymbol{R}^{n+1}$ *is an affinely strictly convex hypersurface. Then*

$$\int_{S^n}\frac{\tilde{\chi}}{\tilde{l}^{n+2}}d\tilde{V}=0,$$

where $\tilde{l}$ *and* $\tilde{\chi}$ *are the affine distance and the affine conormal vector field of* $\tilde{x}=-\xi: \boldsymbol{S}^n\to\boldsymbol{R}^{n+1}$, *and* $d\tilde{V}$ *is the volume element with respect to the affine metric of* $\tilde{x}$.

Conversely, if a strictly convex hypersurface $\tilde{x}: \boldsymbol{S}^n\to\boldsymbol{R}^{n+1}$ *is given and satisfies the above integral formula, then there exists an affinely strictly convex hypersurface* $x: \boldsymbol{S}^n\to\boldsymbol{R}^{n+1}$ *such that* $\tilde{x}=-\xi$. *Moreover, the* x *is unique up to parallel transformations.*

In Section 1, we collect some more notations and facts in the affine hypersurface theory which are used in the proof of Theorem I. We shall give the proof of Theorem I and its corollary in Section 2 and the proof of Theorem II in Section 3.

ACKNOWLEDGEMENT. The author would like to thank Prof. T. Ochiai for his continuous encouragement and advice, and Prof. T. Sasaki whose lecture note invites the author to study the affine hypersurface theory.

§1. Preliminaries.

Let $x: M\to\boldsymbol{R}^{n+1}$ be a strictly convex hypersurface of $\boldsymbol{R}^{n+1}$, h the affine metric of M and A the affine shape operator of M. We define Φ, which is called the *Fubini-Pick form*, as follows: let X, Y and Z be arbitrary vector fields on M.

$$\Phi(X, Y, Z)=(\nabla h)(X, Y; Z). \tag{1.1}$$

It is easily verified that Φ is a symmetric (0, 3)-tensorfield on M, and that at any point p of M, Φ satisfies the apolarity condition:

$$\sum_{i=1}^{n}\Phi(e_i, e_i, e_j)=0 \qquad (j=1, \cdots, n), \tag{1.2}$$

where $(e_1, \cdots, e_n)$ is an orthonormal frame of $T_p(M)$ with respect to the affine metric h.

The Fubini-Pick form plays an important role in the affine geometry. It measures the difference between the hypersurface and hyperquadrics. In fact, we know:

FACT 1. *A strictly convex hypersurface is a hyperquadrics if and only if the Fubini-Pick form vanishes everywhere.*

We denote by $\tilde{\nabla}$ and R the Levi-Civita connection and the Riemannian curvature tensor of the Riemannian manifold (M, h). As in the Euclidean hypersurface theory, there are some relations among h, A, Φ and R: for arbitrary vector fields X, Y, Z and W on M,

(1.3) (the Gauss equation)

$$R(X, Y)Z = \frac{1}{2}\cdot\{h(Y, Z)\cdot A(X) - h(X, Z)\cdot A(Y) + h(A(Y), Z)\cdot X - h(A(X), Z)\cdot Y\} + \frac{1}{4}\cdot\{F(Y, F(X, Z)) - F(X, F(Y, Z))\},$$

(1.4) (the first Codazzi-Mainardi equation)

$$(\tilde{\nabla}\Phi)(X, Y, Z; W) - (\tilde{\nabla}\Phi)(X, Y, W; Z) = -h(Z, Y)\cdot h(A(W), X) + h(W, Y)\cdot h(A(Z), X) - h(Z, X)\cdot h(A(W), Y) + h(W, X)\cdot h(A(Z), Y),$$

(1.5) (the second Codazzi-Mainardi equation)

$$(\tilde{\nabla}A)(Y; X) - (\tilde{\nabla}A)(X; Y) = \frac{1}{2}\cdot\{F(X, A(Y)) - F(Y, A(X))\},$$

where F is a (1, 2)-tensor on M which is defined by $h(F(X, Y), Z) = \Phi(X, Y, Z)$. Conversely, the above equations are the sufficient condition of the existence of a hypersurface with given affine invariants. We have:

FACT 2. (The fundamental theorem of the affine geometry.) *Let (M, h) be a simply connected and connected Riemannian manifold of dimension n. Suppose that a* (1, 1)*-tensor A, which is symmetric with respect to h, and symmetric* (0, 3)*-tensor Φ are given on M and satisfy the equations* (1.2), (1.3), (1.4) *and* (1.5).

Then there exists a strictly convex immersion of M into $\boldsymbol{R}^{n+1}$ such that h, A and Φ are the affine metric, the affine shape operator and the Fubini-Pick form for this immersion, respectively. Moreover, such immersions are unique up to unimodular affine transformations of $\boldsymbol{R}^{n+1}$.

§2. Proof of Theorem I.

Suppose that an affine sphere $x: M \to \boldsymbol{R}^3$ has constant curvature K with respect to the affine metric h of M.

Let Φ be the Fubini-Pick form of M. We take a local orthonormal frame field (e_1, e_2) for the affine metric and write $\Phi(e_i, e_j, e_k)=h_{ijk}$ and $(\tilde{\nabla}\Phi)(e_i, e_j, e_k; e_l)=\tilde{\nabla}_l h_{ijk}$ $(i, j, k, l=1, 2)$.

Because of the symmetricity and the apolarity condition for Φ, any h_{ijk} is equal to either $\pm h_{111}$ or $\pm h_{222}$. Then the Gauss equation and the first Codazzi-Mainardi equation are:

$$K=L+\frac{1}{2}\cdot\{(h_{111})^2+(h_{222})^2\}, \tag{2.1}$$

$$\tilde{\nabla}_1 h_{111}=\tilde{\nabla}_2 h_{222}, \quad \text{and} \quad \tilde{\nabla}_2 h_{111}+\tilde{\nabla}_1 h_{222}=0, \tag{2.2}$$

where L is the affine mean curvature of M. The second Codazzi-Mainardi equation is trivial since M is an affine sphere.

Fact 2 in the section 1 says that affine spheres with constant curvature K are determined by h_{111} and h_{222} satisfying (2.1) and (2.2) for some constant L.

First, we shall show that Φ is parallel on M, that is, $\tilde{\nabla}_l h_{ijk}=0$ for any i, j, k and l. For K and L are constant on M, by derivating the both sides of (2.1), we get

$$\tilde{\nabla}_i h_{111}\cdot h_{111}+\tilde{\nabla}_i h_{222}\cdot h_{222}=0. \tag{2.3}$$

Substituting (2.2) into (2.3) with $i=1$, we have

$$\tilde{\nabla}_1 h_{222}\cdot h_{222}=-\tilde{\nabla}_2 h_{222}\cdot h_{111}. \tag{2.4}$$

We multiply h_{222} to the both sides of (2.4) and use (2.3) with $i=2$, we obtain

$$\tilde{\nabla}_1 h_{222}\cdot(h_{222})^2=\tilde{\nabla}_2 h_{111}\cdot(h_{111})^2. \tag{2.5}$$

(2.2) and (2.5) imply

$$\tilde{\nabla}_2 h_{111}\cdot\{(h_{111})^2+(h_{222})^2\}=0. \tag{2.6}$$

We may assume Φ is not a zero tensor field. Then $(h_{111})^2+(h_{222})^2\neq 0$, (2.6) and (2.2) imply

$$\tilde{\nabla}_2 h_{111}=-\tilde{\nabla}_1 h_{222}=0. \tag{2.7}$$

For either h_{111} or h_{222} is not zero, it follows from (2.7), (2.3) and (2.2) that

$$\tilde{\nabla}_1 h_{111}=\tilde{\nabla}_2 h_{222}=0. \tag{2.8}$$

(2.7) and (2.8) mean that Φ is parallel on M.

Now, we shall prove Theorem I. We divide the proof into two cases according to the value of K.

Suppose that $K\neq 0$. In this case, the local holonomy group of the Riemannian manifold (M, h) is isomorphic to $SO(2)$. Since Φ is parallel on M, Φ must be invariant under the action of this group. To see the action on Φ, we set:

(2.9) $$e_1' = \cos\theta \cdot e_1 + \sin\theta \cdot e_2, \quad e_2' = -\sin\theta \cdot e_1 + \cos\theta \cdot e_2.$$

Substituting (2.9) into $h'_{ijk} = \Phi(e'_i, e'_j, e'_k)$, we have:

(2.10) $$h'_{111} = \cos 3\theta \cdot h_{111} + \sin 3\theta \cdot h_{222}, \quad h'_{222} = -\sin 3\theta \cdot h_{111} + \cos 3\theta \cdot h_{222}.$$

(2.10) means that Φ is invariant if and only if Φ is zero. Therefore, M is an open submanifold of a quadric surface by Fact 1 of the section 1.

Next suppose that $K=0$. We may assume $\Phi \neq 0$. The parallel tensor field Φ is determined by the value of a point p of M. (2.10) implies that we can choose a suitable orthonormal frame (e_1, e_2) at p such that $h_{222}=0$ and $h_{111}>0$. By (2.1), h_{111} is equal to $\sqrt{-L}$. This means that Φ is unique up to rotations around a point of M for each L. On the other hand, the surface $S(c)$ has a flat affine metric and a non-zero Fubini-Pick form (cf. [2]). Hence, M must be affinely equivalent to $S(c)$ for some positive real number c. This completes the proof.

The corollary is directly deduced from Theorem I and the following lemma.

LEMMA. *Suppose that a strictly convex surface $x: M \to \boldsymbol{R}^3$ is flat with respect to its affine metric. Then M is a complete affine sphere if and only if M is affinely homogeneous.*

PROOF. First, we assume that M is a complete affine sphere. By Theorem I, M is a paraboloid or M is affinely equivalent to $S(c)$. It is easy to show that these two surfaces are affinely homogeneous.

Conversely, let M be an affinely homogeneous surface with flat affine metric h. Let G be the group of unimodular affine transformations acting on M transitively. Then G preserves the affine invariants of M, especially the affine metric h. This means the Riemannian manifold (M, h) admits a transitive group of isometries. Therefore, (M, h) is a complete flat space, so that it is isometric to the 2-dimensional Euclidean space $\boldsymbol{E}^2$. Since G acts on $\boldsymbol{E}^2$ transitively, G contains all parallel transformations of $\boldsymbol{E}^2$. Hence, the Fubini-Pick form, which is invariant under the action of G, must be parallel on M. It follows immediately from the first Codazzi-Mainardi equation (1.4) that the affine shape operator A is proportional to the identity operator. This implies M is an affine sphere. q. e. d.

§ 3. Proof of Theorem II.

We shall show Theorem II by reducing the problem to that of the Euclidean geometry. Suppose that $x: M \to \boldsymbol{R}^{n+1}$ is an affinely strictly convex hypersurface and $\tilde{x} = -\xi: M \to \boldsymbol{R}^{n+1}$. Note that $\tilde{x}_* = x_* \circ A$.

First, we represent the affine invariants of $\tilde{x}$ in terms of those of x. We denote by $\tilde{\xi}$ the affine normal vector field of $\tilde{x}$. We take a vector field W of M and a function ϕ on M such that

$$\tilde{\xi} = \phi \cdot \xi + \tilde{x}_* W. \tag{3.1}$$

Let $\tilde{\chi}$ be the conormal vector field and $\tilde{l}$ the affine distance of $\tilde{x}$. It follows from (3.1) and the definitions of $\tilde{\chi}$ and $\tilde{l}$ that:

$$\tilde{\chi} = \frac{1}{\phi} \cdot \chi, \tag{3.2}$$

$$\tilde{l} = \frac{1}{\phi}. \tag{3.3}$$

Substituting (3.1) into the definition of the affine metric $\tilde{h}$ of $\tilde{x}$, we have

$$h(X, A(Y)) = \phi \cdot \tilde{h}(X, Y)$$

for any vector fields X and Y on M. Hence,

$$\sqrt{\det(A)} \cdot dV = \phi^{n/2} \cdot d\tilde{V}, \tag{3.4}$$

where dV and $d\tilde{V}$ are the volume elements of (M, h) and $(M, \tilde{h})$, respectively.

On the other hand, the condition (ii) on the affine normal vector field implies that: let $(e_1, \cdots, e_n)$ be a frame field of M. Then

$$\mathrm{vol}(x_* e_1, \cdots, x_* e_n, \xi) = dV(e_1, \cdots, e_n),$$
$$\mathrm{vol}(\tilde{x}_* e_1, \cdots, \tilde{x}_* e_n, \tilde{\xi}) = d\tilde{V}(e_1, \cdots, e_n).$$

Using (3.4), we find

$$\det(A) = \phi^{-(n+2)}. \tag{3.5}$$

Combining (3.2), (3.3), (3.4) and (3.5), we obtain:

$$\int_M \frac{\tilde{\chi}}{\tilde{l}^{n+2}} d\tilde{V} = \int_M \chi dV. \tag{3.6}$$

Next, we shall represent the right-hand side of (3.6) in terms of the Euclidean invariants of $x: M \to \boldsymbol{R}^{n+1}$. Let g be the standard metric of $\boldsymbol{R}^{n+1}$ and ν the Euclidean normal vector field of x. We take a vector field Z of M and a function φ on M such that

$$\xi = \varphi \cdot \nu + x_* Z. \tag{3.7}$$

When we identify $\boldsymbol{R}^{n+1}$ and $\boldsymbol{R}_{n+1}$ by g, the χ can be written as follows:

$$\chi = \frac{1}{\varphi} \cdot \nu. \tag{3.8}$$

We denote by dV_ν the volume element of M with respect to the induced metric. For a frame field $(e_1, \cdots, e_n)$ of M,

$$dV_\nu(e_1, \cdots, e_n) = \mathrm{vol}(x_* e_1, \cdots, x_* e_n, \nu),$$
$$dV(e_1, \cdots, e_n) = \mathrm{vol}(x_* e_1, \cdots, x_* e_n, \xi).$$

Hence, using (3.7), we have

$$dV = \varphi \cdot dV_\nu. \tag{3.9}$$

(3.8) and (3.9) yield

$$\int_M \chi dV = \int_M \nu dV_\nu. \tag{3.10}$$

We note that, by (3.7) and the definition of the second fundamental form h_ν of x, we have

$$h_\nu = \varphi \cdot h.$$

With (3.9), this equation yields

$$K = \varphi^{n+2}, \tag{3.11}$$

where K is the Gauss-Kronecker curvature of x. Finally, combining (3.6) and (3.10), we get:

$$\int_M \frac{\tilde{\chi}}{\tilde{l}^{n+2}} d\tilde{V} = \int_M \nu dV_\nu. \tag{3.12}$$

Now we are in position to prove Theorem II. When $x: \boldsymbol{S}^n \to \boldsymbol{R}^{n+1}$ is the strictly convex hypersurface, it is easy to show that the right-hand side of (3.12) is zero (cf. [**3**]). Then the integral formula in Theorem II is satisfied.

Conversely, suppose that a strictly convex hypersurface $\tilde{x}: \boldsymbol{S}^n \to \boldsymbol{R}^{n+1}$ is given and satisfies the integral formula. To show the existence of a hypersurface $x: \boldsymbol{S}^n \to \boldsymbol{R}^{n+1}$ such that $\tilde{x} = -\xi$, we use the following theorem due to Cheng and Yau.

THEOREM ([**3**]). *Let K be a positive function on $\boldsymbol{S}^n$. Suppose $\int_{S^n} x^\alpha \cdot K^{-1} = 0$ for all coordinate functions x^α. Then we can find a compact strictly convex hypersurface in $\boldsymbol{R}^{n+1}$ whose Gauss-Kronecker curvature is K. Moreover, any two such hypersurfaces must coincide after a translation.*

Let $\tilde{\nu}$ be the Euclidean normal vector field of $\tilde{x}$. We set

$$K = (g(\tilde{\chi}/\tilde{l}, \tilde{\nu}))^{-(n+2)}.$$

Then, the following equality holds:

$$\int_{S^n} \frac{\tilde{\nu}}{K} \nu^* dV_S = 0, \tag{3.13}$$

where dV_S is the standard volume element of the n-dimensional sphere. In fact, by the definition of the Gauss-Kronecker curvature $\tilde{K}$ of $\tilde{x}$,

$$\tilde{\nu}^* dV_S = \tilde{K} \cdot dV_{\tilde{\nu}}.$$

Using (3.7), (3.8) and (3.11) for $\tilde{x}$, we have

$$\tilde{\nu}^* dV_S = (g(\tilde{\chi}, \tilde{\nu}))^{-(n+2)} \cdot dV_{\tilde{\nu}}.$$

Then

$$\int_{S^n} \frac{\tilde{\nu}}{K} \nu^* dV_S = \int_{S^n} \frac{\tilde{\nu}}{\tilde{l}^{n+2}} dV_{\tilde{\nu}} = \int_{S^n} \frac{\tilde{\chi}}{\tilde{l}^{n+2}} d\tilde{V} = 0.$$

By the theorem of Cheng and Yau, (3.13) means that there exists a strictly convex hypersurface $x: S^n \to R^{n+1}$ such that the Gauss-Kronecker curvature of x is equal to K and the Euclidean normal vector field ν of x coincides with $\tilde{\nu}$.

We shall show that $-\tilde{x}$ is the affine normal vector field of x. By (3.8) and (3.11), the conormal vector field χ of x satisfies:

$$\chi = K^{-1/(n+2)} \cdot \nu = K^{-1/(n+2)} \cdot \tilde{\nu}. \tag{3.14}$$

Therefore, for any vector field X on S^n,

$$g(\chi, D_X \tilde{x}) = g(\chi, \tilde{x}_* X) = 0. \tag{3.15}$$

Here, we claim that

$$g(\chi, \tilde{x}) = -1. \tag{3.16}$$

If (3.16) holds, it follows from (3.15) that:

$$g(D_X \chi, \tilde{x}) = 0. \tag{3.17}$$

On the other hand, by the definition of χ, $-\xi$ also satisfies (3.16) and (3.17). Since (3.14) implies that χ_* is non-degenerate, $\tilde{x}$ must coincide with $-\xi$.

Let us prove (3.16). By (3.14), we have

$$g(\chi, \tilde{x}) = K^{-1/(n+2)} \cdot g(\nu, \tilde{x}) = g(\tilde{\chi}/\tilde{l}, \tilde{\nu}) \cdot g(\tilde{\nu}, \tilde{x}). \tag{3.17}$$

By (3.8), we get

$$g(\tilde{\chi}, \tilde{\nu}) = \tilde{\varphi}^{-1}. \tag{3.18}$$

To represent $\tilde{l}$ in terms of the Euclidean invariants, we take an orthonormal frame field $(e_1, \cdots, e_n)$ with respect to $\tilde{h}$. Then, using (3.9), we obtain

$$\begin{aligned} \tilde{l} &= -\det(e_1, \cdots, e_n, \tilde{x}) = -g(\tilde{\nu}, \tilde{x}) \cdot \det(e_1, \cdots, e_n, \tilde{\nu}) \\ &= -g(\tilde{\nu}, \tilde{x}) \cdot dV_{\tilde{\nu}}(e_1, \cdots, e_n) = -\tilde{\varphi}^{-1} \cdot g(\tilde{\nu}, \tilde{x}), \end{aligned} \tag{3.19}$$

(3.16) follows from (3.17), (3.18) and (3.19).

Finally, we show the uniqueness part of Theorem II. Assume that two affinely strictly convex hypersurfaces x and $y: S^n \to R^{n+1}$ have the same affine normal vector field ξ. The Euclidean normal vector field at a point p of S^n is perpendicular to $\xi_*(T_p(S^n))$. Since ξ_* is non-degenerate, the Euclidean normal vector field is determined by ξ. Moreover, ξ determines the Gauss-Kronecker determined by ξ. Moreover, (3.11) implies that the Gauss-Kronecker curvature is equal to $(g(\xi, \nu))^{n+2}$. Hence, the two hypersurfaces x and y have the same Euclidean normal vector field and the same Gauss-Kronecker curvature. Cheng and Yau's theorem says the y is obtained from the x by a parallel translation of $\boldsymbol{R}^{n+1}$. This completes the proof.

References

[1] W. Blaschke, Vorlesungen über Differentialgeometrie II, Springer, 1923.
[2] E. Calabi, Complete affine hypersphere I, Sympos. Math., **10** (1972), 19-38.
[3] S.Y Cheng and S.T. Yau, On the regularity of the solution of the n-dimensional Minkowski problem, Comm Pure Appl. Math., **29** (1976), 495-516.
[4] ———, Complete affine hypersurfaces, Part I The completeness of affine metrics, Comm. Pure Appl. Math., **39** (1986), 839-866.
[5] H. Flanders, Local theory of affine hypersurfaces, J. Analyse Math., **15** (1965), 353-387.
[6] H. Guggenheimer, Differential Geometry, McGraw-Hill, 1963.
[7] K. Nomizu and U. Pinkall, On the geometry of affine immersions, Math. Z., **195** (1987), 165-178.
[8] A.V. Pogorelov, The Minkowski Multidimensional Problem, V.H. Winston & Sons, 1978.
[9] T. Sasaki, Hyperbolic affine hyperspheres, Nagoya Math J., **77** (1980), 107-123.

Takashi KUROSE
Department of Applied Mathematics
Fukuoka University
Nanakuma, Jonan-ku,
Fukuoka 814-01
Japan