J. Math. Soc. Japan
Vol. 39, No. 4, 1987

On Siegel domains of finite type

By Soji KANEYUKI

(Received March 27, 1986)

Introduction.

In this paper we introduce a new class of homogeneous Siegel domains, called Siegel domains of finite type. Let $D=D(V, F)\subset \boldsymbol{C}^N$ be a Siegel domain associated with a convex cone V and a V-hermitian form F. Let G_h (resp. G_a) be the identity component of the holomorphic (resp. affine) automorphism group of D. It is known (Nakajima [7]) that D is G_h-equivariantly and holomorphically imbedded, together with the ambient space $\boldsymbol{C}^N$, into a complex coset space M of the complexification of G_h. D is said to be *of finite type*, if there are only finitely many G_h-orbits in M. This concept is realization-free and is determined only by the holomorphic equivalence class of D. Let H be the identity component of the linear automorphism group of D. Then there exists a natural homomorphism ρ of H into the linear automorphism group of the cone V. The cone V is called of *$\rho(H)$-finite type,* if there exists only a finite number of $\rho(H)$-orbits in the ambient vector space in which V is imbedded as an open cone.

The first aim of this paper is to show that, if M has at most countably many G_h-orbits, then D is of finite type, and in this case each G_h-orbit is a semi-analytic set in M (Theorem 3.8). It follows that, if D is of finite type, then it is necessarily homogeneous (Proposition 3.11). As a consequence, if D is not homogeneous, then M has non-countably many G_h-orbits (Corollary 3.12). The main purpose of this paper is to prove the equivalence between finite type for D and $\rho(H)$-finite type for V (cf. Theorem 3.15). Thus D being of finite type or not is reduced to the problem on orbits under a group of linear transformations. We also show that every connected component of the intersection of a G_h-orbit with $\boldsymbol{C}^N$ is a G_a-orbit and conversely every G_a-orbit is obtained in this manner (Theorem 3.3). This yields a qualitative proof of a result of Kaup-Matsushima-Ochiai [5] which states that, if D is homogeneous, then it is affinely homogeneous (Corollary 3.5). Finally we remark that the class of Siegel domains of finite type properly contains the class of quasi-symmetric Siegel domains (Corollary 3.14 and Example 3.17). In this paper we make use of some

This research was partially supported by Grant-in-Aid for Scientific Research (No. 61540060), Ministry of Education, Science and Culture.

basic properties on semi-analytic sets and semi-algebraic sets (cf. Lojasiewicz [6], [14]). The complexification of a real vector space X will be denoted by X^C throughout this paper.

The author is grateful to H. Omori for helpful conversations.

§ 1. Basic facts on Siegel domains.

In this section we give a brief summary of basic facts on Siegel domains which are needed for later considerations (cf. [5], [12], [8], [7], [3]). Let V be an open convex cone in a real vector space R with vertex at the origin which contains no affine lines. We will call such a cone V simply a convex cone in R. Let W be a complex vector space and F be a V-hermitian form on W. Let us define a map Φ of the complex vector space $R^C \times W$ to R by putting

$$\Phi(z, u) = \operatorname{Im} z - F(u, u), \qquad z \in R^C, \quad u \in W. \tag{1.1}$$

Then the complete inverse image $\Phi^{-1}(V)$ is called a Siegel domain of the second kind or of the first kind, according as $W \neq (0)$ or $W = (0)$. Later on $\Phi^{-1}(V)$ will be usually denoted by $D(V, F)$ or briefly by D. If $\Phi^{-1}(V)$ is of the first kind, then it will usually be denoted by $D(V)$. Let G_h (resp. G_a) denote the identity component of the group of holomorphic (resp. affine) automorphisms of D. The Lie algebra $\mathfrak{g}_h$ of G_h has a structure of graded Lie algebra:

$$\mathfrak{g}_h = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2 . \tag{1.2}$$

The Lie algebra $\mathfrak{g}_a$ of G_a is the graded subalgebra of $\mathfrak{g}_h$:

$$\mathfrak{g}_a = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 . \tag{1.3}$$

$\mathfrak{g}_h$ is naturally realized as a Lie algebra of polynomial vector fields on $R^C \times W$. The group G_h is center-free and so it can be identified with the adjoint group of $\mathfrak{g}_h$. The subspaces $\mathfrak{g}_{-2}$ and $\mathfrak{g}_{-1}$ may be naturally identified with R and W, respectively. $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ have the complex structures which are induced by the adjoint action of the vector field corresponding to the one-parameter subgroup of G_h sending a point $(z, u) \in R^C \times W$ to the point $(z, e^{it}u)$, $t \in \boldsymbol{R}$. Therefore the complexifications $\mathfrak{g}_{-1}^C$ and $\mathfrak{g}_1^C$ of $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ can be written in the form

$$\mathfrak{g}_{-1}^C = \mathfrak{g}_{-1}^+ + \mathfrak{g}_{-1}^- , \qquad \mathfrak{g}_1^C = \mathfrak{g}_1^+ + \mathfrak{g}_1^- , \tag{1.4}$$

where $\mathfrak{g}_1^+$ and $\mathfrak{g}_{-1}^+$ (resp. $\mathfrak{g}_1^-$ and $\mathfrak{g}_{-1}^-$) are the subspaces of $\mathfrak{g}_1^C$ and $\mathfrak{g}_{-1}^C$ consisting of holomorphic (resp. anti-holomorphic) vectors, respectively. Then the complexification $\mathfrak{g}_h^C$ of $\mathfrak{g}_h$ is written in the form of the graded Lie algebra:

$$\mathfrak{g}_h^C = \mathfrak{a}_{-1} + \mathfrak{a}_0 + \mathfrak{a}_1 , \tag{1.5}$$

where we put

$$(1.6)\qquad \mathfrak{a}_{-1}=\mathfrak{g}_{-2}^{C}+\mathfrak{g}_{-1}^{\pm}\,,\quad \mathfrak{a}_0=\mathfrak{g}_{-1}^{-}+\mathfrak{g}_0^{C}+\mathfrak{g}_1^{+}\,,\quad \mathfrak{a}_1=\mathfrak{g}_1^{-}+\mathfrak{g}_2^{C}\,.$$

Let G_h^C be the adjoint group of the Lie algebra $\mathfrak{g}_h^C$. G_h^C contains G_h as a subgroup. Let U be the normalizer of the complex subalgebra $\mathfrak{u}=\mathfrak{a}_0+\mathfrak{a}_1$ in G_h^C. We know that U is connected and $\mathrm{Lie}\,U=\mathfrak{u}$. Let us consider the complex homogeneous space $M=G_h^C/U$. Note that if the Siegel domain D is symmetric, then M is no other than its compact dual. Now let π be the natural projection of G_h^C onto M, and let us consider the composite map $\xi=\pi\cdot\exp$ of $\mathfrak{a}_{-1}$ into M. Then ξ is an open dense holomorphic imbedding of the vector space $\mathfrak{a}_{-1}$ into M. Since we are identifying $\mathfrak{g}_{-2}$ with R and $\mathfrak{g}_{-1}$ with W, the complex vector space $\mathfrak{a}_{-1}$ is identified with $R^C\times W$. Hence D is viewed as a domain in $\mathfrak{a}_{-1}$. Then it is known that the imbedding ξ is G_h-equivariant on D. The imbedding ξ of D is called the *Tanaka's imbedding*.

DEFINITION 1.1. With notations above, the Siegel domain D is said to be *of finite type*, if there are only finite number of G_h-orbits in M. D is said to be *of G_a-finite type*, if there are only finite number of G_a-orbits in $R^C\times W$.

We shall see later that every symmetric Siegel domain is of finite type. Let H denote the isotropy subgroup of G_a at the origin in $R^C\times W$. Then G_a can be expressed as

$$(1.7)\qquad G_a=H\cdot RW\,,\qquad \text{(semi-direct)}$$

where RW is the two-step nilpotent normal subgroup of G_a which is diffeomorphic to $R\times W$ and acts simply transitively on the Silov boundary of D. H consists of the pairs $(A, B)\in GL(R)\times GL(W)$ satisfying the two conditions:

$$(1.8)\qquad AF(u, v)=F(Bu, Bv)\qquad u, v\in W\,,$$

(1.9) A is in the identity component $G(V)$ of the (linear) automorphism group of the cone V.

We denote by ρ the homomorphism of H to $G(V)$ defined by $\rho(A, B)=A$.

DEFINITION 1.2. With notations above, the cone V is called *of finite type* (resp. *of $\rho(H)$-finite type*), if there exist only finitely many $G(V)$-orbits (resp. $\rho(H)$-orbits) in R.

Note that this definition of finite type is equivalent to the original definition of Pjaseckii [**9**].

§ 2. The relation between G_a-orbits and $\rho(H)$-orbits.

The following lemma and corollary are well-known [**10**].

LEMMA 2.1. (i) *Let* $g\in G_a$ *and put* $g=(h, a, c)$, *where* $h\in H$, $a\in R$, $c\in W$ (cf. (1.7)). *Then we have*

$$\Phi(g(z, u)) = \rho(h)\Phi(z, u)\,, \qquad (z, u)\in R^C\times W\,. \tag{2.1}$$

(ii) *Any point* $(z, u)\in R^C\times W$ *can be sent to the point* $(i\Phi(z, u), 0)$ *by a transformation in the group* RW.

COROLLARY 2.2. *Let* (z', u') *be the image of a point* $(z, u)\in R^C\times W$ *under a transformation* $(a, c)\in RW$. *Then we have*

$$\Phi(z', u') = \Phi(z, u)\,. \tag{2.2}$$

Conversely, if two points (z, u) *and* (z', u') *satisfy* (2.2), *then there exists a transformation in* RW *which sends* (z, u) *to* (z', u').

LEMMA 2.3. *Let* $\mathcal{V}$ *be a* $\rho(H)$*-orbit in* R. *Then* $\Phi^{-1}(\mathcal{V})$ *is a* G_a*-orbit in* $R^C\times W$.

PROOF. Take two points (z_1, u_1) and (z_2, u_2) in $\Phi^{-1}(\mathcal{V})$. Then there exists an element $h\in H$ such that

$$\Phi(z_2, u_2) = \rho(h)\Phi(z_1, u_1) = \Phi(h(z_1, u_1))\,.$$

On the other hand, by Lemma 2.1, the point (z_1, u_1) is sent to the point $(i\Phi(h(z_1, u_1)), 0)=(i\Phi(z_2, u_2), 0)$ by an element in G_a. Also (z_2, u_2) is sent to the same point by an element in RW. From these two it follows that $\Phi^{-1}(\mathcal{V})$ is contained in the orbit $G_a(z_1, u_1)$. Conversely take a point $g(z_1, u_1)\in G_a(z_1, u_1)$, where $g\in G_a$. If we write g in the form $(h, a, c)\in H\cdot RW$, then we have $\Phi(g(z_1, u_1))=\rho(h)\Phi(z_1, u_1)\in\rho(h)\mathcal{V}=\mathcal{V}$. This implies the inclusion $\Phi^{-1}(\mathcal{V})\supset G_a(z_1, u_1)$.

LEMMA 2.4. *The correspondence* $\Theta:\mathcal{V}\mapsto\Phi^{-1}(\mathcal{V})$ *is a bijection between the set of all* $\rho(H)$*-orbits in* R *and the set of all* G_a*-orbits in* $R^C\times W$.

PROOF. Let $\mathcal{D}$ be a G_a-orbit in $R^C\times W$, and put $\mathcal{V}=\Phi(\mathcal{D})$. Then $\mathcal{V}$ is a $\rho(H)$-orbit in R. Indeed, take two points $\Phi(z_i, u_i)$, where $(z_i, u_i)\in\mathcal{D}$, $i=1, 2$. Then there exists $g=(h, a, c)\in G_a$ such that $g(z_1, u_1)=(z_2, u_2)$. Therefore, by Lemma 2.1, we have $\Phi(z_2, u_2)=\Phi(g(z_1, u_1))=\rho(h)\Phi(z_1, u_1)$. Conversely, for an arbitrary element $h\in H$, we have $\rho(h)\Phi(z_1, u_1)=\Phi(h(z_1, u_1))\in\Phi(\mathcal{D})=\mathcal{V}$. Therefore we conclude $\mathcal{V}=\rho(H)\Phi(z_1, u_1)$. We want to show $\mathcal{D}=\Phi^{-1}(\mathcal{V})$. Take a point $(z, u)\in\Phi^{-1}(\mathcal{V})$. Then there exists a point $(z', u')\in\mathcal{D}$ such that $\Phi(z, u)=\Phi(z', u')$. By Corollary 2.2, we have $(z, u)\in RW(z', u')\subset G_a(z', u')=\mathcal{D}$, which implies $\Phi^{-1}(\mathcal{V})\subset\mathcal{D}$. Thus we get $\Phi^{-1}(\mathcal{V})=\mathcal{D}$. This means that the map Θ is surjective. That Θ is injective is trivial.

From Lemma 2.4 we have

PROPOSITION 2.5. *A Siegel domain $D(V, F)$ is of G_a-finite type if and only if the cone V is of $\rho(H)$-finite type.*

COROLLARY 2.6. *Every quasi-symmetric Siegel domain $D(V, F)$ is of G_a-finite type.*

PROOF. In this case the cone V is homogeneous self-dual, and consequently it is of finite type [9], [16]. Since $D(V, F)$ is quasi-symmetric, we have $\rho(H)=G(V)$ ([11]). The corollary now follows from Proposition 2.5.

COROLLARY 2.7. *A Siegel domain of the first kind $D(V)$ is of G_a-finite type, if and only if the cone V is of finite type.*

PROOF. Note that, for a Siegel domain of the first kind $D(V)$, we have $\rho(H)=G(V)$.

§3. The relation between G_h-orbits and G_a-orbits.

Under the identifications of R with $\mathfrak{g}_{-2}$ and of W with $\mathfrak{g}_{-1}$, Φ is viewed as a mapping of $\mathfrak{a}_{-1}$ to $\mathfrak{g}_{-2}$. In the sequel we shall identify $R^C\times W$ (viewed as $\mathfrak{a}_{-1}$) with its image under the imbedding ξ.

LEMMA 3.1 (Tanaka [12]). *Let $X\in\mathfrak{g}_h$. Then the Lie derivative $L_X\Phi$ of Φ with respect to X, viewed as a vector field on $\mathfrak{a}_{-1}$, is given by*

$$L_X\Phi=[B_X, \Phi], \tag{3.1}$$

where B_X is a $\mathfrak{g}_0$-valued function on $\mathfrak{a}_{-1}$ depending on X.

PROPOSITION 3.2. *Let Ω be a G_a-orbit in $R^C\times W$ and let $\mathcal{M}$ be a unique G_h-orbit in M containing Ω. Then we have* $\dim\Omega=\dim\mathcal{M}$.

PROOF. The proof is similar to that of Lemma 3.14 in Tanaka [13]: so we can omit the details. Here we use Lemmas 2.4, 3.1 and the fact that Φ is a submersion. Also note that the representation ρ of H is identified with its adjoint representation on $\mathfrak{g}_{-2}$.

The following theorem gives the relation between G_h-orbits and G_a-orbits.

THEOREM 3.3. *Let $M=\coprod_{\lambda\in\Lambda}\mathcal{M}_\lambda$ be the orbit decomposition of M under G_h, and let $\mathcal{D}_\lambda=\mathcal{M}_\lambda\cap(R^C\times W)$, $\lambda\in\Lambda$. Then, for each $\lambda\in\Lambda$, every connected component of $\mathcal{D}_\lambda$ is a G_a-orbit. Conversely every G_a-orbit in $R^C\times W$ is obtained in this manner.*

PROOF. By a result of Nakajima [7], $\mathcal{D}_\lambda$ is not empty for each $\lambda\in\Lambda$. Let

$\mathcal{D}_\lambda = \coprod_{\alpha\in A}\mathcal{D}_{\lambda\alpha}$ be the decomposition of $\mathcal{D}_\lambda$ into its connected components. Take a point $p\in\mathcal{D}_{\lambda\alpha}$. Then the orbit $G_a p$ is contained in $\mathcal{M}_\lambda\cap(R^C\times W)=\mathcal{D}_\lambda$. Therefore we have $G_a p\subset\mathcal{D}_{\lambda\alpha}$, which means that $\mathcal{D}_{\lambda\alpha}$ is G_a-stable. From Proposition 3.2 it follows that $G_a p$ is open in $\mathcal{D}_{\lambda\alpha}$. Suppose now that $G_a p\subsetneqq\mathcal{D}_{\lambda\alpha}$. Choose a point q in $\mathcal{D}_{\lambda\alpha}-G_a p$. Then the orbit $G_a q$ is contained in $\mathcal{D}_{\lambda\alpha}-G_a p$. Again by Proposition 3.2, $G_a q$ is open in $\mathcal{D}_{\lambda\alpha}$. Hence $\mathcal{D}_{\lambda\alpha}-G_a p$ is open in $\mathcal{D}_{\lambda\alpha}$. But this contradicts the fact that $\mathcal{D}_{\lambda\alpha}$ is connected. We have thus proved $G_a p =\mathcal{D}_{\lambda\alpha}$. The second assertion can be analogously proved.

The set $\mathcal{D}_\lambda$ $(\lambda\in\Lambda)$ is called a *truncated G_h-orbit* in $R^C\times W$. Note that the G_h-orbit through a point in D is thoroughly contained in D.

COROLLARY 3.4. *Let $\mathcal{M}_\lambda$ be a G_h-orbit contained in D. Then $\mathcal{M}_\lambda$ is also a G_a-orbit.*

PROOF. In this case we have $\mathcal{M}_\lambda=\mathcal{D}_\lambda$.

As a direct consequence we have the following corollary which was originally proved by Kaup-Matsushima-Ochiai [**5**].

COROLLARY 3.5. *If the group G_h is transitive on D, then so is G_a.*

Let $\tilde{\Omega}$ be the union of all k-dimensional G_h-orbits in M. Suppose that $\tilde{\Omega}$ is not empty. The intersection $\Omega=\tilde{\Omega}\cap(R^C\times W)$ is the union of all k-dimensional truncated G_h-orbits, and it is not empty (cf. the proof of Theorem 3.3).

LEMMA 3.6. *$\tilde{\Omega}$ is a semi-analytic set in the real analytic manifold M.*

PROOF. Let $\mathfrak{g}_h(p)$ denote the subspace of the tangent space $T_p(R^C\times W)$, $p\in R^C\times W$, which is spanned by the values at p of vector fields belonging to $\mathfrak{g}_h$. Let $\{X_1, \cdots, X_r\}$ be a basis of $\mathfrak{g}_h$, and choose a real linear coordinate system $(x_1, \cdots, x_n)$ of $R^C\times W$. Let us express X_i in the form $X_i=\sum_{j=1}^n\xi_{ji}\partial/\partial x_j$ $(1\leqq i\leqq r)$. Since X_i's are polynomial vector fields on $R^C\times W$ [**5**], the components ξ_{ji} are polynomials on $R^C\times W$. We then have

$$\Omega = \{p\in R^C\times W:\ \dim\mathfrak{g}_h(p) = \operatorname{rank}(\xi_{ji}(p)) = k\}. \tag{3.2}$$

This implies that Ω is a semi-algebraic set in $R^C\times W$, more precisely, Ω is defined in $R^C\times W$ by a finite number of polynomial equalities and polynomial inequalities. Let p be a point in M which does not belong to $R^C\times W$. Then there exists $a\in G_h$ such that $ap\in R^C\times W$ [**7**]. In the neighborhood $a^{-1}(R^C\times W)$ of p in M, the intersection $\tilde{\Omega}\cap a^{-1}(R^C\times W)$ is defined by a finite number of the equalities and the inequalities given by the real analytic functions which are the composites of the polynomial functions defining Ω and the transformation a. This implies that $\tilde{\Omega}$ is a semi-analytic set in M.

LEMMA 3.7. *$\tilde{\Omega}$ has only finitely many connected components.*

PROOF. Let $\tilde{\Omega}=\amalg_{\alpha\in A}\tilde{\Omega}_{\alpha}$ be the decomposition of $\tilde{\Omega}$ into its connected components. Then we have

$$\Omega = \coprod_{\alpha\in A} \tilde{\Omega}_{\alpha}\cap(R^{C}\times W)\,. \tag{3.3}$$

Note that each term of the right-hand side is never empty, since each G_h-orbit always meets $R^{C}\times W$ [7]. On the other hand each term of the right-hand side of (3.3) is open and closed in Ω. By decompoing each term of the right-hand side of (3.3) into the connected components, we have the decomposition of Ω into its connected components. But, since Ω is a semi-algebraic set in $R^{C}\times W$, Ω has only finitely many connected components [6]. Therefore we conclude that the index set A is a finite set.

THEOREM 3.8. *Suppose that M has at most countably many G_h-orbits. Then M has only finitely many G_h-orbits, and moreover each G_h-orbit is a semi-analytic set in M.*

PROOF. Let $\{\mathcal{M}_1, \mathcal{M}_2, \cdots\}$ be the totality of k-dimensional G_h-orbits in M. We have

$$\tilde{\Omega} = \coprod_{i=1}^{\infty} \mathcal{M}_i\,. \tag{3.4}$$

By Lemma 3.7 we can write

$$\tilde{\Omega} = \tilde{\Omega}_1 \amalg \tilde{\Omega}_2 \amalg \cdots \amalg \tilde{\Omega}_t\,, \tag{3.5}$$

where $\tilde{\Omega}_1, \cdots, \tilde{\Omega}_t$ are the totality of the connected components of $\tilde{\Omega}$. Since $\tilde{\Omega}$ is semi-analytic in M, so is each $\tilde{\Omega}_i$ [6]. $\tilde{\Omega}$ is obviously G_h-invariant. Therefore each $\tilde{\Omega}_i$ is also G_h-invariant, since G_h is connected. Thus each $\tilde{\Omega}_i$ is a disjoint union of G_h-orbits $\mathcal{M}_i$. From this it follows that $\dim\tilde{\Omega}_i \geqq k$ for each i. We have either one of the following two cases:

a) $\dim\tilde{\Omega}_i = k$ for $1\leqq i\leqq t$,

b) there exists an i_0 such that $\dim\tilde{\Omega}_{i_0}>k$.

Suppose that the case b) occurs. Let $k_0=\dim\tilde{\Omega}_{i_0}$, and let A be the set of k_0-dimensional regular points in $\tilde{\Omega}_{i_0}$. A is then a k_0-dimensional submanifold of M which is open and dense in $\tilde{\Omega}_{i_0}$ [6]. Let A_λ be a connected component of A. G_h leaves A stable. Therefore, G_h being connected, A_λ is G_h-stable, and hence A_λ can be written as a disjoint union of at most countably many k-dimensional G_h-orbits in M. Those G_h-orbits naturally define a k-dimensional involutive distribution on A_λ. Therefore, following the Chevalley's proof of the Frobenius theorem, one can find a cubic neighborhood Q of a point in A_λ in which each of those G_h-orbits can be expressed as k-dimensional slices, and furthermore the number of those slices are at most countable for each G_h-orbit. Consequently it follows finally that the k_0-dimensional cubic neighborhood Q is written as a disjoint union of at most a countable number of k-dimensional slices. But this is clearly a contradiction. Therefore we have to have the case a). In this case every G_h-orbit forming $\tilde{\Omega}_i$ is open in $\tilde{\Omega}_i$, and hence $\tilde{\Omega}_i$ must coincide with only one G_h-orbit, since $\tilde{\Omega}_i$ is connected. As a consequence the two decompositions (3.4) and (3.5) are identical. Thus we have proved the theorem.

COROLLARY 3.9. *Suppose that D is of finite type. Then every truncated G_h-orbit in $R^C \times W$ has only finitely many connected components.*

PROOF. As is seen in the proof of Theorem 3.8, (3.5) is the decomposition of $\tilde{\Omega}$ into k-dimensional G_h-orbits. We have

$$\Omega = \coprod_{i=1}^{t} \tilde{\Omega}_i \cap (R^C \times W) . \tag{3.6}$$

$\tilde{\Omega}_i \cap (R^C \times W)$ is a k-dimensional truncated G_h-orbit. Furthermore, $\tilde{\Omega}_i \cap (R^C \times W)$ is open and closed in Ω and hence it can be written as a disjoint union of connected components of Ω. Since Ω has only finitely many connected components, the number of connected components of Ω forming $\tilde{\Omega}_i \cap (R^C \times W)$ should be finite.

COROLLARY 3.10. *Suppose that D is of finite type. Then every G_a-orbit and every truncated G_h-orbit are both semi-algebraic sets in $R^C \times W$.*

PROOF. Let $\mathcal{D}_i$ be a G_a-orbit of dimension k. Then $\mathcal{D}_i$ is a connected component of a truncated G_h-orbit $\tilde{\Omega}_i \cap (R^C \times W)$ (Theorem 3.3). That connected component is semi-algebraic, since it is a connected component of the semi-algebraic set Ω [**6**]. By Corollary 3.9, every truncated G_h-orbit has only finitely many connected components which are all semi-algebraic, as was shown above. Therefore the truncated G_h-orbit is also semi-algebraic [**6**].

REMARK. Suppose that D is of finite type. Then, by Theorem 3.8 and Corollary 3.10, every G_h-orbit (resp. every truncated G_h-orbit) is locally closed in M (resp. $R^C \times W$) and so it is a regular submanifold of M (resp. $R^C \times W$) [**1**].

PROPOSITION 3.11. *Suppose that D is of finite type. Then D is homogeneous, that is, the group G_h acts transitively on D.*

PROOF. D can be written as the disjoint union of G_h-orbits $\mathcal{M}_i$ which are contained in D:

$$D = \coprod_{i=1}^{s} \mathcal{M}_i . \tag{3.7}$$

By Corollary 3.10, each $\mathcal{M}_i$ is semi-algebraic in $R^C \times W$. Therefore D itself is a semi-algebraic set in $R^C \times W$; we have $\dim D = \max_{1 \leq i \leq s} \dim \mathcal{M}_i$. Consequently there exists a G_h-orbit $\mathcal{M}_{i_0} \subset D$ which contains an open set in $R^C \times W$. D has the Bergman metric, with respect to which G_h is a group of isometries of D. Hence it follows that $D = \mathcal{M}_{i_0}$.

COROLLARY 3.12. *Suppose that D is not homogeneous. Then there exist non-countably infinite number of G_h-orbits in M.*

PROOF. Otherwise D has to be of finite type (Theorem 3.8). Therefore, by Proposition 3.11 D is homogeneous, which is a contradiction.

PROPOSITION 3.13. *D is of finite type if and only if it is of G_a-finite type.*

PROOF. Suppose that D is of G_a-finite type. Let $M = \amalg_{\lambda \in \Lambda} \mathcal{M}_\lambda$ be the G_h-orbit decomposition of M. As was remarked before (cf. the proof of Theorem 3.3), $\mathcal{D}_\lambda = \mathcal{M}_\lambda \cap (R^C \times W)$ is not empty for every $\lambda \in \Lambda$. By Theorem 3.3, all connected components of all $\mathcal{D}_\lambda$ $(\lambda \in \Lambda)$ exhaust all G_a-orbits in $R^C \times W$. That G_a-orbits are finite in number implies that Λ is a finite set. Conversely suppose that M has only finitely many G_h-orbits $\mathcal{M}_1, \cdots, \mathcal{M}_s$. Set $\mathcal{D}_i = \mathcal{M}_i \cap (R^C \times W)$, $1 \leqq i \leqq s$. By Corollary 3.9, every $\mathcal{D}_i$ has only finitely many connected components each of which is a G_a-orbit (cf. Theorem 3.3). Therefore D is of G_a-finite type.

From the above proposition and Corollary 2.6, we have

COROLLARY 3.14. *Every quasi-symmetric Siegel domain is of finite type.*

Combining Proposition 3.13 with Proposition 2.5, we finally obtain

THEOREM 3.15. *Let $D = D(V, F)$ be a Siegel domain. Then D is of finite type if and only if the cone V is of $\rho(H)$-finite type. In particular, a Siegel domain of the first kind $D(V)$ is of finite type if and only if the cone V is of finite type.*

Let $H(n, \boldsymbol{R})$ be the vector space of real symmetric $n \times n$ matrices and $V = H^+(n, \boldsymbol{R})$ be the convex cone in $H(n, \boldsymbol{R})$ consisting of all positive definite elements. Then we have

COROLLARY 3.16. *If a Siegel domain $D = D(H^+(n, \boldsymbol{R}), F)$ is homogeneous, then it is of finite type.*

PROOF. By Corollary 3.5, the group G_a is transitive on D. Hence $\rho(H)$ contains a maximal (connected) $\boldsymbol{R}$-triangular subgroup T_1 of $G(V)$ ([**2**]). It is known [**15**] that there are only finitely many T_1-orbits in $H(n, \boldsymbol{R})$. Consequently $H^+(n, \boldsymbol{R})$ is of $\rho(H)$-finite type, which implies that D is of finite type.

EXAMPLE 3.17. Let D be a Siegel domain over the cone $H^{+}(2, \boldsymbol{R})$ in $\boldsymbol{C}^4$:

$$D=\{(z_1, z_2, z_3, u)\in \boldsymbol{C}^4 : (y_1-|u|^2)y_2-y_3^2>0,\ y_2>0\}, \tag{3.8}$$

where $z_k=x_k+iy_k$ $(k=1, 2, 3)$. As is well-known, D is the lowest dimensional non-symmetric homogeneous Siegel domain. D serves an example of Siegel domains of finite type which are not quasi-symmetric.

EXAMPLE 3.18. Let $D=D(V, F)$ be a homogeneous Siegel domain in $\boldsymbol{C}^8$ formed by all points $(z_1, \cdots, z_7, u)$ satisfying

$$\begin{cases} ((y_1-|u|^2)y_3-y_6^2-y_7^2)(y_2y_3-y_5^2)-(y_3y_4-y_5y_6)^2>0\,, \\ y_2y_3-y_5^2>0\,, \quad y_3>0\,, \end{cases} \tag{3.9}$$

where $z_k=x_k+iy_k$ ([4]). Then the underlying cone V is a homogeneous cone of rank 3 which are not self-dual [4]. One can verify by using a result of [9] that V is not of finite type, and so it is not of $\rho(H)$-finite type. D provides an example of Siegel domains which are homogeneous but not of finite type.

Bibliography

[1] N. Bourbaki, Groupes et Algèbres de Lie, Chap. 2-3, Hermann, Paris, 1972.

[2] S. Kaneyuki, On the automorphism groups of homogeneous bounded domains, J. Fac. Sci. Univ. Tokyo, **14** (1967), 89-130.

[3] S. Kaneyuki, Homogeneous Bounded Domains and Siegel Domains, Lecture Notes in Math., **241**, Springer, 1971.

[4] S. Kaneyuki and T. Tsuji, Classification of homogeneous bounded domains of lower dimension, Nagoya Math. J., **53** (1974), 1-46.

[5] W. Kaup, Y. Matsushima and T. Ochiai, On the automorphisms of generalized Siegel domains, Amer. J. Math., **92** (1970), 475-497.

[6] S. Lojasiewicz, Ensembles semi-analytiques, Lecture Notes, IHES, Bures-sur-Yvette, 1965.

[7] K. Nakajima, On Tanaka's imbeddings of Siegel domains, J. Math. Kyoto Univ., **14** (1974), 533-548.

[8] K. Nakajima, On equivariant holomorphic imbeddings of Siegel domains to compact complex homogeneous spaces, J. Math. Kyoto Univ., **19** (1979), 471-480.

[9] V.S. Pjaseckii, Classification of convex cones of finite type of rank three, Trudy Sem. Vektor. Tenzor. Anal., **19** (1979), 202-217.

[10] I.I. Pyatetskii-Shapiro, Automorphic Functions and the Geometry of Classical Domains, Gordon and Breach, New York, 1969.

[11] I. Satake, Algebraic Structure of Symmetric Domains, Iwanami Shoten Publishers, Tokyo and Princeton Univ. Press, Princeton, 1980.

[12] N. Tanaka, On infinitesimal automorphisms of Siegel domains, J. Math. Soc. Japan, **22** (1970), 180-212.

[13] N. Tanaka, On affine symmetric spaces and the automorphism groups of product manifolds, **14** (1985), 277-351.

[14] S. Lojasiewicz, Triangulation of semi-analytic sets, Ann. Scuola Norm. Sup. Pisa Cl. Sci., **18** (1964), 449-474.

[15] V. S. Pjaseckii, Classification of convex cones of T-finite type, Trudy Moscow Math. Soc., **41** (1980), 37–82.
[16] V. S. Pjaseckii, Finiteness problem in the theory of homogeneous convex cones, Trudy Moscow Math. Soc., **47** (1984), 219–244.

Soji KANEYUKI
Department of Mathematics
Sophia University
Kioicho, Tokyo 102
Japan