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Strong topological transitivity and C^*-dynamical systems

By Ola BRATTELI, George A. ELLIOTT
and Derek W. ROBINSON

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0. Introduction.

Let T denote the action of a group H as homeomorphisms of a topological space X; then (X, H, T) is said to be topologically transitive if for each pair of non-empty open sets $A, B\subseteq X$ there exists an $h\in H$ such that $A\cap T_h(B)\neq\emptyset$ (see, for example, [**13**] Chapter 5). Following [**10**], we define the C^*-dynamical system $(\mathcal{A}, H, \tau)$ to be topologically transitive if for each pair of non-zero elements $x, y\in\mathcal{A}$ there exists an $h\in H$ such that $x\tau_h(y)\neq 0$.

The algebraic definition is particularly natural if $\mathcal{A}$ is abelian. In this case τ determines an action τ' of H as homeomorphisms of the spectrum X of $\mathcal{A}$ such that $(\tau_h x)(\omega)=x(\tau'_{h^{-1}}\omega)$, for $x\in\mathcal{A}$ and $\omega\in X$, and $(\mathcal{A}, H, \tau)$ is topologically transitive if and only if (X, H, τ') is topologically transitive.

In [**10**] the definition is given in a slightly different form. These authors require that the product $\mathcal{A}_1\mathcal{A}_2$ of each pair of non-zero τ-invariant hereditary C^*-subalgebras $\mathcal{A}_1, \mathcal{A}_2\subseteq\mathcal{A}$ is non-zero. This obviously follows from our definition but conversely if there exist non-zero $x, y\in\mathcal{A}$ such that $x\tau_h(y)=0$ for all $h\in H$ then the product $\mathcal{A}_1\mathcal{A}_2$ of the τ-invariant hereditary C^*-subalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ generated by $\{\tau_h(x)^*\mathcal{A}\tau_h(x); h\in H\}$ and $\{\tau_h(y)\mathcal{A}\tau_h(y)^*; h\in H\}$ must be zero.

Although the foregoing definition of transitivity is quite natural there is a seemingly stronger notion which appears to be more useful. The C^*-dynamical system $(\mathcal{A}, H, \tau)$ is defined to be strongly topologically transitive if for each finite sequence $\{(x_i, y_i); i=1, 2, \cdots, n\}$ of pairs of elements $x_i, y_i\in\mathcal{A}$ for which

$$\sum_{i=1}^{n} x_i\otimes y_i\neq 0,$$

in the algebraic tensor product $\mathcal{A}\otimes\mathcal{A}$, there exists an $h\in H$ such that

$$\sum_{i=1}^{n} x_i\tau_h(y_i)\neq 0$$

in $\mathcal{A}$.

Clearly strong topological transitivity implies topological transitivity; it suffices to apply the strong condition to a single pair (x, y). In Section 1 we show that the two properties are equivalent if $\mathcal{A}$ is abelian or finite-dimensional. We also show that strong topological transitivity follows from other ergodicity properties, but we do not know if strong topological transitivity is strictly stronger than topological transitivity.

In Section 2 we analyze the structure of the action α of a compact group G on a strongly transitive C^*-system $(\mathcal{A}, H, \tau)$ under the assumption that α and τ commute. More specifically, we show that $\{\alpha_g;\ g\in G\}$ consists of those automorphisms β of $\mathcal{A}$ which commute with τ and which reduce to the identity on the fixed point algebra $\mathcal{A}^\alpha$ of α. Then in Section 3 we examine the infinitesimal structure of (G, α). In particular we show that if δ is a closed symmetric derivation from the G-finite elements $\mathcal{A}_F$ into $\mathcal{A}$ then δ generates a one-parameter subgroup of α if, and only if, δ commutes with τ and δ is zero on the fixed points of α. Finally in Section 4 we make some remarks about the generation problem for dissipations. This analysis extends results recently obtained by Kishimoto and Robinson [**9**], Longo and Peligrad [**10**], and Robinson, Størmer and Takesaki [**11**]; see also [**1**], [**2**], [**3**], [**8**] and [**12**] for earlier results of a similar nature.

1. Topological transitivity.

In this section we analyze some basic properties of transitivity and strong transitivity as defined in the introduction. First we show that these properties are invariant under the adjunction of an identity.

Let $(\mathcal{A}, H, \tau)$ be a C^*-dynamical system. If $\mathcal{A}$ does not contain an identity one can adjoin such an element $\mathbf{1}$ by a canonical procedure and then extend (H, τ) to $\tilde{\mathcal{A}}=\mathcal{A}+\boldsymbol{C}\mathbf{1}$ by setting $\tilde{\tau}_h(x+\lambda\mathbf{1})=\tau_h(x)+\lambda\mathbf{1}$.

Lemma 1.1. *Let $(\mathcal{A}, H, \tau)$ be a C^*-dynamical system without identity and $(\tilde{\mathcal{A}}, H, \tilde{\tau})$ the system obtained by adjoining an identity. The following pairs of conditions are equivalent:*

1. (1s.) *$(\mathcal{A}, H, \tau)$ is (strongly) topologically transitive;*
2. (2s.) *$(\tilde{\mathcal{A}}, H, \tilde{\tau})$ is (strongly) topologically transitive.*

Proof. 1s$\Rightarrow$2s. Identify $\mathcal{A}$ with its universal representation. This gives a faithful representation of $\tilde{\mathcal{A}}$, and the tensor product Hilbert space gives a representation of $\tilde{\mathcal{A}}\otimes\tilde{\mathcal{A}}$. Assume

$$\sum_{i=1}^n \tilde{x}_i\otimes\tilde{y}_i\neq 0$$

for some $\tilde{x}_i$, $\tilde{y}_i\in\tilde{\mathcal{A}}$. Let (e_α) be an approximate identity of $\mathcal{A}$. Then $e_\alpha\tilde{x}\otimes\tilde{y}e_\alpha$ converges weakly to $\tilde{x}\otimes\tilde{y}$ for all $\tilde{x}$, $\tilde{y}\in\tilde{\mathcal{A}}$. Thus for some α sufficiently large,

$$\sum_{i=1}^n e_\alpha\tilde{x}_i\otimes\tilde{y}_i e_\alpha\neq 0$$

in $\mathcal{A}\otimes\mathcal{A}$. Hence there exists $h\in H$ such that

$$e_\alpha\Big(\sum_{i=1}^{n}\tilde{x}_i\tau_h(\tilde{y}_i)\Big)\tau_h(e_\alpha)=\sum_{i=1}^{n}e_\alpha\tilde{x}_i\tau_h(\tilde{y}_ie_\alpha)\neq 0$$

by Condition 1s. But this implies

$$\sum_{i=1}^{n}\tilde{x}_i\tau_h(\tilde{y}_i)\neq 0\,,$$

so Condition 2s is fulfilled. 2s$\Rightarrow$1s. This is evident from the embedding of $(\mathcal{A}, H, \tau)$ in $(\tilde{\mathcal{A}}, H, \tilde{\tau})$. 1$\Leftrightarrow$2. This follows from the above with $n=1$.

Lemma 1.1 is useful because it means that one can usually assume that $\mathcal{A}$ has an identity in the discussion of transitivity.

Next note that the use of two elements in the definition of topological transitivity is not particularly significant. In fact by iteration one readily sees that $(\mathcal{A}, H, \tau)$ is topologically transitive if, and only if, for each sequence $x_1, x_2, \cdots, x_k$ of non-zero elements of $\mathcal{A}$ there exist $h_1, h_2, \cdots, h_k\in H$ such that

$$\tau_{h_1}(x_1)\tau_{h_2}(x_2)\cdots\tau_{h_k}(x_k)\neq 0\,.$$

A similar conclusion is true for strong topological transitivity.

PROPOSITION 1.2. *The following conditions are equivalent:*

1. *The C^*-dynamical system $(\mathcal{A}, H, \tau)$ is strongly topologically transitive.*
2. *For each family of finite sequences $\{x_i^{(1)}, \cdots, x_i^{(k)}\,;\ i=1, \cdots, n\}$ of elements of $\mathcal{A}$ satisfying*

$$\sum_{i=1}^{n}x_i^{(1)}\otimes x_i^{(2)}\otimes\cdots\otimes x_i^{(k)}\neq 0\,,$$

in the k-fold algebraic tensor product $\otimes^k\mathcal{A}$, there exist $h_1, h_2, \cdots, h_k\in H$ such that

$$\sum_{i=1}^{n}\tau_{h_1}(x_i^{(1)})\tau_{h_2}(x_i^{(2)})\cdots\tau_{h_k}(x_i^{(k)})\neq 0\,.$$

PROOF. 1$\Rightarrow$2. We argue by induction. Assume Condition 2 is valid for $2\leqq k<N$. Now consider $x_i^{(j)}$ such that

$$\sum_{i=1}^{n}x_i^{(1)}\otimes x_i^{(2)}\otimes\cdots\otimes x_i^{(N)}\neq 0\,.$$

By making a linear rearrangement, if necessary, one can express the relation in the form

$$\sum_{i=1}^{p}y_i^{(1)}\otimes Y_i\neq 0\,,$$

where the $y_i^{(1)}$ are linearly independent and the Y_i are elements of $\otimes^{N-1}\mathcal{A}$ which can be written as

$$Y_i=\sum_{j=1}^{m}y_{ij}^{(2)}\otimes\cdots\otimes y_{ij}^{(N)}.$$

But we can also assume that $Y_1\neq 0$. Therefore by the induction hypothesis there exist $h_2, h_3, \cdots, h_N \in H$ such that

$$\sum_{j=1}^{m} \tau_{h_2}(y_{ij}^{(2)})\tau_{h_3}(y_{ij}^{(3)}) \cdots \tau_{h_N}(y_{ij}^{(N)}) \neq 0 .$$

Consequently

$$\sum_{i=1}^{n} \sum_{j=1}^{m} y_i^{(1)} \otimes \tau_{h_2}(y_{ij}^{(2)})\tau_{h_3}(y_{ij}^{(3)}) \cdots \tau_{h_N}(y_{ij}^{(N)}) \neq 0 ,$$

because the $y_i^{(1)}$ are linearly independent. Hence by Condition 1 there exists an $h_1 \in H$ such that

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \tau_{h_1}(y_i^{(1)})\tau_{h_2}(y_{ij}^{(2)}) \cdots \tau_{h_N}(y_{ij}^{(N)}) \neq 0$$

and by rearrangement this gives the conclusion

$$\sum_{i=1}^{n} \tau_{h_1}(x_i^{(1)})\tau_{h_2}(x_i^{(2)}) \cdots \tau_{h_N}(x_i^{(N)}) \neq 0 .$$

Thus the induction hypothesis is valid for $k=N$. $2 \Rightarrow 1$. This is evident.

Although strong topological transitivity appears to be a strictly stronger property than topological transitivity we do not know of any example in which this is established. In fact in many situations the two properties are equivalent.

THEOREM 1.3. *Let $(\mathcal{A}, H, \tau)$ be a C^*-dynamical system with $\mathcal{A}$ abelian. The following conditions are equivalent:*

1. *$(\mathcal{A}, H, \tau)$ is topologically transitive;*
2. *$(\mathcal{A}, H, \tau)$ is strongly topologically transitive;*
3. $\sup_{h \in H} \|x\tau_h(y)\| = \|x\| \cdot \|y\|, \qquad x, y \in \mathcal{A}$;
4. $\left\| \sum_{i=1}^{n} x_i^{(1)} \otimes \cdots \otimes x_i^{(k)} \right\|_k = \sup_{h_1, \cdots, h_k \in H} \left\| \sum_{i=1}^{n} \tau_{h_1}(x_i^{(1)}) \cdots \tau_{h_k}(x_i^{(k)}) \right\|$ *for all* $x_i^{(j)} \in \mathcal{A}$ *and all $k \geqq 2$, where $\|\cdot\|_k$ denotes the unique C^*-norm on the tensor product algebra $\otimes^k \mathcal{A}$.*

PROOF. Clearly $4 \Rightarrow 3 \Rightarrow 1$ and $4 \Rightarrow 2 \Rightarrow 1$. Hence it suffices to prove that $1 \Rightarrow 4$. But for this we can, by Lemma 1.1, assume that $\mathcal{A}$ has an identity.

The proof of $1 \Rightarrow 4$ is almost identical to the proof of Proposition 2.1 of [**9**]. First for $h=(h_1, h_2, \cdots, h_k) \in H^k$ one defines a linear map T_h from $\otimes^k \mathcal{A}$ into $\mathcal{A}$ by

$$T_h\left(\sum_{i=1}^{n} x_i^{(1)} \otimes \cdots \otimes x_i^{(k)}\right) = \sum_{i=1}^{n} \tau_{h_1}(x_i^{(1)}) \cdots \tau_{h_k}(x_i^{(k)}) .$$

It then follows that

$$a \in \otimes^k \mathcal{A} \longmapsto \|a\|_k = \sup_{h \in H^k} \|T_h a\|$$

defines a seminorm on $\otimes^k \mathcal{A}$. But since $T_h a^* = (T_h a)^*$, $T_h aa^* = (T_h a)(T_h a)^*$, and $T_h ab = (T_h a)(T_h b)$, one readily concludes that $\|\cdot\|_k$ is a C^*-seminorm. By the form of topological transitivity stated immediately before Proposition 1.2, for any elementary tensor $a=a_1\otimes\cdots\otimes a_k\in\otimes^k\mathcal{A}$ there exists $h\in H^k$ such that $T_h a\neq 0$, whence $\|a\|_k\neq 0$. Hence by Lemma 2.3 of [**9**] the $\|\cdot\|_k$ are in fact C^*-norms.

The next result establishes the equivalence of the two notions of transitivity for matrix algebras.

THEOREM 1.4. *Let $(\mathcal{A}, H, \tau)$ be a C^*-dynamical system with $\mathcal{A}$ finite-dimensional. The following conditions are equivalent:*

1. *τ is ergodic, i.e. $\mathcal{A}^\tau=\boldsymbol{C}\mathbf{1}$ where $\mathcal{A}^\tau$ denotes the fixed point algebra of τ;*
2. *$(\mathcal{A}, H, \tau)$ is topologically transitive;*
3. *$(\mathcal{A}, H, \tau)$ is strongly topologically transitive.*

PROOF. $2\Rightarrow 1$. If Condition 1 is false there must exist two non-zero orthogonal projections $x, y\in\mathcal{A}^\tau$. Hence $x\tau_h(y)=xy=0$ for all $h\in H$, and Condition 2 is false. $1\Rightarrow 3$. Assume $x_i, y_i\in\mathcal{A}$ are such that

$$\sum_{i=1}^n x_i\otimes y_i\neq 0$$

but

$$\sum_{i=1}^n x_i\tau_h(y_i)=0$$

for all $h\in H$. By linear rearrangement, if necessary, one may assume the x_i are linearly independent and $y_1\neq 0$. But it follows from Condition 1 that

$$\int dh\, \tau_h(y_i y_1^*)=\omega(y_i y_1^*)\mathbf{1}$$

where ω is the unique normalized τ-invariant trace on $\mathcal{A}$, and the integral is over the compact closure of H in Aut$\mathcal{A}$. Therefore

$$0=\int dh\Big(\sum_{i=1}^n x_i\tau_h(y_i y_1^*)\Big)=\sum_{i=1}^n x_i\omega(y_i y_1^*)\,.$$

Since $y_1\neq 0$ one has $\omega(y_1 y_1^*)>0$ and one concludes that the x_i are linearly dependent, which is a contradiction. Thus Condition 3 must be valid. $3\Rightarrow 2$. This is evident.

REMARK. For finite-dimensional $\mathcal{A}$ there are no analogues of Properties 3 and 4 of Theorem 1.3 for topological transitivity. In fact if $\mathcal{A}=M_2$ (the algebra of 2×2 matrices) and τ is an ergodic action of a finite group then there exist projections p, q such that

$$\sup_h \|p\tau_h(q)\|<1\,.$$

The above arguments establish two criteria for strong topological transitivity, both of which require the existence of a certain kind of ergodic state.

THEOREM 1.5. *Let $(\mathcal{A}, H, \tau)$ be a C^*-dynamical system for which there exists a τ-ergodic separating state ω, i.e. ω is τ-invariant and the cyclic covariant representation $(\mathcal{H}_\omega, \pi_\omega, U_\omega, \Omega_\omega)$ associated with ω satisfies*

1. *Ω_ω is the unique $U_\omega(H)$ invariant vector in $\mathcal{H}_\omega$,*
2. *π_ω is faithful and Ω_ω is separating for $\pi_\omega(\mathcal{A})''$.*

It follows that $(\mathcal{A}, H, \tau)$ is strongly topologically transitive.

PROOF. Again assume that

$$\sum_{i=1}^{n} x_i \otimes y_i \neq 0$$

but

$$\sum_{i=1}^{n} x_i \tau_h(y_i) = 0$$

for all $h \in H$, where the x_i are linearly independent and $y_1 \neq 0$. Therefore

$$\sum_{i=1}^{n} \pi_\omega(x_i)\pi_\omega(\tau_h(y_i y_1^*)) = 0$$

for all $h \in H$. Now it follows from the theory of τ-invariant states (see, for example, [5] Chapter IV, and in particular Theorem 4.3.23) that

$$\sum_{i=1}^{n} \pi_\omega(x_i)\omega(y_i y_1^*) = 0 .$$

But since Ω_ω is separating for $\pi_\omega(\mathcal{A})''$ and π_ω is faithful one concludes that

$$\sum_{i=1}^{n} x_i \omega(y_i y_1^*) = 0$$

and $\omega(y_1 y_1^*) > 0$. Thus the x_i are linearly dependent, which is a contradiction.

COROLLARY 1.6. *Let $(\mathcal{A}, H, \tau)$ be a C^*-dynamical system with H compact. The following conditions are equivalent:*

1. *τ is ergodic;*
2. *τ is topologically transitive;*
3. *τ is strongly topologically transitive.*

PROOF. $3 \Rightarrow 2 \Rightarrow 1$ is evident. $1 \Rightarrow 3$. Since τ is ergodic there is a unique τ-invariant state ω over $\mathcal{A}$ given by

$$\omega(x)\mathbf{1} = \int_H dh\, \tau_h(x) .$$

But ω is a trace by [7] and hence Ω_ω is separating for $\pi_\omega(\mathcal{A})''$. Thus Condition 3 follows from Theorem 1.5.

THEOREM 1.7. *Let $(\mathcal{A}, H, \tau)$ be a C^*-dynamical system. Assume there exists a τ-ergodic state ω such that the corresponding representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ is faithful. Moreover assume that $(\mathcal{A}, H, \tau)$ is asymptotically abelian in the sense*

$$\inf_{h \in H} \sum_{i=1}^{n} \|\pi_\omega([\tau_h(x_i), y_i])\psi_i\| = 0$$

for all finite sequences of elements $x_i, y_i \in \mathcal{A}$ and vectors $\psi_i \in \mathcal{H}_\omega$. It follows that $(\mathcal{A}, H, \tau)$ is strongly topologically transitive and furthermore

$$\sup_{h\in H}\|x\tau_h(y)\|=\|x\|\cdot\|y\|\,,\qquad x,\ y\in\mathcal{A}.$$

PROOF. Again assume

$$\sum_{i=1}^{n} x_i\otimes y_i\neq 0\,,$$

with the x_i linearly independent and $y_1\neq 0$, but

$$\sum_{i=1}^{n} x_i\tau_h(y_i)=0$$

for all $h\in H$. Then

$$\sum_{i=1}^{n}\pi_\omega(\tau_h(z)x_i\tau_h(y_iy_1^*z^*))=0$$

for all $h\in H$ and $z\in\mathcal{A}$. Therefore by taking a limit over a suitable net of convex combinations over h one concludes that

$$\sum_{i=1}^{n}\pi_\omega(x_i)\omega(zy_iy_1^*z^*)=0\,.$$

This again follows from the general theory of τ-invariant states as described in Section 4.3 of [5] together with our choice of the asymptotic abelianness condition. Now since π_ω is faithful, z can be chosen such that $\omega(zy_1y_1^*z^*)>0$. Hence the x_i must be linearly dependent, which is a contradiction. Consequently $(\mathcal{A},\ H,\ \tau)$ is strongly topologically transitive.

The last statement of the theorem follows by an argument given in [9]. One has

$$\omega((a\tau_h(b))^*(x\tau_h(y))^*(x\tau_h(y))(a\tau_h(b))\leqq\sup_h\|x\tau_h(y)\|^2\omega((a\tau_h(b))^*(a\tau_h(b)))\,.$$

It then follows from the conditions of asymptotic abelianness and ergodicity of ω that

$$\omega(a^*x^*xa)\omega(b^*y^*yb)\leqq\sup_h\|x\tau_h(y)\|^2\omega(a^*a)\omega(b^*b)\,.$$

Since π_ω is faithful it follows that

$$\|x\|^2\|y\|^2\leqq\sup_h\|x\tau_h(y)\|^2\leqq\|x\|^2\|y\|^2$$

which gives the desired conclusion.

The property of strong topological transitivity can also be expressed in terms of norms on tensor products, e.g., the system $(\mathcal{A},\ H,\ \tau)$ is strongly topologically transitive if, and only if,

$$\left\|\sum_{i=1}^{n}x_i^{(1)}\otimes x_i^{(2)}\otimes\cdots\otimes x_i^{(k)}\right\|_k=\sup_{h_1,\cdots,h_k\in H}\left\|\sum_{i=1}^{n}\tau_{h_1}(x_i^{(1)})\tau_{h_2}(x_i^{(2)})\cdots\tau_{h_k}(x_i^{(k)})\right\|$$

defines a norm on $\otimes^k\mathcal{A}$ for all $k\geqq 2$. This rephrasing follows directly from the original definition for $k=2$ and from Proposition 1.2 for higher k. Unfortunately these norms are not necessarily C^*-norms, although this is the case if $\mathcal{A}$ is abelian by Theorem 1.3. For example all C^*-norms on $\mathcal{A}\otimes\mathcal{A}$ satisfy the cross-norm property

$$\|x\otimes y\|=\|x\|\cdot\|y\|.$$

But the norm $\|\cdot\|_2$ has this property if, and only if,

$$(*)\qquad \sup_{h\in H}\|x\tau_h(y)\|=\|x\|\cdot\|y\|.$$

We have, however, already given an example where property $(*)$ fails (see the remark after Theorem 1.4). Thus $(*)$ is a necessary condition for the $\|\cdot\|_k$ to be C^*-norms. It is also sufficient to guarantee the more general cross-norm property

$$\|a\otimes b\|_{k+l}=\|a\|_k\|b\|_l$$

for $a\in\mathcal{A}_k$ and $b\in\mathcal{A}_l$. This follows by the argument used in the proof of Proposition 2.1 of [**9**]. It would be of interest to obtain necessary and sufficient conditions on $(\mathcal{A}, H, \tau)$ for the $\|\cdot\|_k$ to be C^*-norms. It would also be of interest to compare the $\|\cdot\|_k$ and the similar norms defined for norm asymptotically abelian systems in [**9**]. These latter norms are defined as above except the supremum is replaced by a limit supremum, and Proposition 2.1 of [**9**] establishes conditions under which the C^*-norm property is valid.

2. Topological transitivity and compact actions.

Next we consider a strongly topologically transitive C^*-system $(\mathcal{A}, H, \tau)$ and also an action α of a compact group G on $\mathcal{A}$. We assume α commutes with τ and our aim is to analyze the structure of (G, α).

THEOREM 2.1. *Let $(\mathcal{A}, H, \tau)$ be a strongly topologically transitive C^*-dynamical system and α a (faithful) continuous action of a compact group G as $*$-automorphisms of $\mathcal{A}$ such that $[\alpha, \tau]=0$. If β is a $*$-automorphism of $\mathcal{A}$ such that $[\beta, \tau]=0$ and $\beta(x)=x$ for all $x\in\mathcal{A}^\alpha$, the fixed point algebra of α, then $\beta=\alpha_g$ for some $g\in G$.*

REMARK. This theorem is a direct generalization of Theorem 1.1 of [**11**]. If $\mathcal{A}$ is a von Neumann algebra, the theorem remains true if strong topological transitivity is replaced by ergodicity ([**12**]; see also [**1**], [**2**], [**3**]).

If G is abelian, one may also replace strong topological transitivity by topological transitivity ([**10**], Theorem 3.1).

PROOF. Let $\mathcal{A}_F$ denote the set of G-finite elements in $\mathcal{A}$, i.e. the linear span of the spectral subspaces

$$\mathcal{A}^\alpha(U)=\left\{\int dg\,\mathrm{Tr}(U_g^{-1})\alpha_g(x)\,;\ x\in\mathcal{A}\right\}$$

corresponding to the irreducible representations U of G. Alternatively, $\mathcal{A}_F$ is characterized as the set of $x \in \mathcal{A}$ such that the linear span of $\{\alpha_g(x);\ g \in G\}$ is finite-dimensional. We note that $\mathcal{A}_F$ is a dense $*$-subalgebra of $\mathcal{A}$ and if $V \subseteq \mathcal{A}$ is a finite-dimensional α-invariant subspace then $V \subseteq \mathcal{A}_F$.

Next for each $h \in H$ let T_h denote the linear map from $\mathcal{A} \otimes \mathcal{A}$ into $\mathcal{A}$ defined by

$$T_h\Big(\sum_{i=1}^n x_i \otimes y_i\Big) = \sum_{i=1}^n x_i \tau_h(y_i)\,.$$

It follows immediately from $[\alpha, \tau]=0$ that

$$T_h(\alpha_g \otimes \alpha_g) = \alpha_g T_h$$

for all $g \in G$, and similarly since $[\beta, \tau]=0$,

$$T_h(\beta \otimes \beta) = \beta T_h\,.$$

Now let V be a finite-dimensional α-invariant subspace of $\mathcal{A}$ and introduce the finite-dimensional subspace $W=V+\beta(V)$. It follows from strong topological transitivity that there is a finite subset H_W of H such that the map

$$\bigoplus_{h \in H_W} T_h : W \otimes W \longrightarrow \bigoplus_{h \in H_W} \mathcal{A}$$

is injective.

OBSERVATION 1. *If $x, y \in \mathcal{A}_F$ and $\phi, \psi \in \mathcal{A}^*$ then*

$$\int_G dg\, \overline{\psi(\beta\alpha_g(x))}\, \phi(\beta\alpha_g(y)) = \int_G dg\, \overline{\psi(\alpha_g(x))}\, \phi(\alpha_g(y))\,.$$

PROOF. After replacing x by x^* and ψ by ψ^* it suffices to show

$$\int_G dg\, \psi(\beta\alpha_g(x))\phi(\beta\alpha_g(y)) = \int_G dg\, \psi(\alpha_g(x))\phi(\alpha_g(y))\,.$$

Since $x, y \in \mathcal{A}_F$, the linear α-invariant space V generated by $\{\alpha_g(x);\ g \in G\}$ and $\{\alpha_g(y);\ g \in G\}$ is finite-dimensional. Set $W=V+\beta(V)$. The injection

$$\bigoplus_{h \in H_W} T_h : W \otimes W \longrightarrow \bigoplus_{h \in H_W} \mathcal{A}$$

transports the linear functional $\psi \otimes \phi$ on $W \otimes W$ onto a linear functional ξ on the subspace

$$\Big(\bigoplus_{h \in H_W} T_h\Big)(W \otimes W) \subseteq \bigoplus_{h \in H_W} \mathcal{A}\,.$$

It then follows from the Hahn-Banach theorem that ξ has a continuous extension to $\bigoplus_{h \in H_W} \mathcal{A}$ which we also denote by ξ. But ξ has a linear decomposition

$$\xi = \bigoplus_{h \in H_W} \xi_h\,.$$

Therefore

$$\begin{aligned}\int_G dg\, \psi(\beta\alpha_g(x))\phi(\beta\alpha_g(y)) &= \int_G dg(\psi\otimes\phi)(\beta\alpha_g(x)\otimes\beta\alpha_g(y)) \\ &= \int_G dg \sum_{h\in H_W} \xi_h(T_h(\beta\alpha_g(x)\otimes\beta\alpha_g(y))) \\ &= \int_G dg \sum_{h\in H_W} \xi_h(\beta\alpha_g(x\tau_h(y))) \\ &= \sum_{h\in H_W} \xi_h\Big(\beta\Big(\int_G dg\; \alpha_g(x\tau_h(y))\Big)\Big), \\ &= \sum_{h\in H_W} \xi_h\Big(\int_G dg\; \alpha_g(x\tau_h(y))\Big) \\ &= \int_G dg\, \psi(\alpha_g(x))\phi(\alpha_g(y))\end{aligned}$$

where the penultimate step uses the fact that β leaves the fixed points $\mathcal{A}^\alpha$ of α pointwise invariant, and the ultimate step follows from reversal of the previous steps.

Next let $C_F(G)\subseteqq C(G)$ denote the G-finite elements for the action of G as right (or, equivalently, left) translations on $C(G)$. Thus $C_F(G)$ is the set of continuous functions over G whose orbit under right translations spans a finite-dimensional subspace of $C(G)$. Again $C_F(G)$ is a dense $*$-subalgebra of $C(G)$.

OBSERVATION 2. *Every* $f\in C_F(G)$ *has the form*

$$f(g)=\sum_{i=1}^{n}\phi_i(\alpha_g(x_i))$$

where $x_i\in\mathcal{A}_F$ *and* $\phi_i\in\mathcal{A}^*$.

PROOF. Let $\mathcal{D}$ be the subspace of $C_F(G)$ of functions of the form

$$f(g)=\sum_{i=1}^{n}\phi_i(\alpha_g(x_i)).$$

In the proof of Observation 1 we established an identity of the form

$$\psi(\alpha_g(x))\phi(\alpha_g(y))=\sum_{h\in H_W}\xi_h(\alpha_g(x\tau_h(y)))$$

for all x, $y\in\mathcal{A}_F$ and ψ, $\phi\in\mathcal{A}^*$. The $\xi_h\in\mathcal{A}^*$ and H_W is a finite subset of H depending on the finite-dimensional subspace V spanned by the orbits $\alpha_g(x)$, and $\alpha_g(y)$, of x, and y. Since x, $y\in\mathcal{A}_F$ and τ commutes with α it follows that $x\tau_h(y)\in\mathcal{A}_F$. Therefore this identity establishes that $\mathcal{D}$ is an algebra. Also

$$\overline{\psi(\alpha_g(x))}=\psi^*(\alpha_g(x^*))$$

so $\mathcal{D}$ is a $*$-algebra. As α is a faithful representation of G it follows that the functions in $\mathcal{D}$ separate points of G, and hence $\mathcal{D}$ is dense in $C(G)$ by the Stone-Weierstrass theorem. Since $\mathcal{A}_F$ is closed under regularization by matrix elements of the irreducible representations of G it follows that $\mathcal{D}$ has the same property (with respect to the right regular representation). Then it easily follows from the orthogonality relations that $\mathcal{D}=C_F(G)=$the linear span of the matrix elements of the irreducible representations of G.

OBSERVATION 3. *There exists an isometric linear isomorphism* $B: C_F(G)\rightarrow C_F(G)$ *with the properties*

1. $B\left(\sum_{i=1}^{n}\phi_i(\alpha_g(x_i))\right)=\sum_{i=1}^{n}\phi_i(\beta\alpha_g(x_i))$, $\quad x_i\in\mathcal{A}_F,\ \phi_i\in\mathcal{A}^*$,
2. $B(f_1f_2)=B(f_1)B(f_2)$, $\quad f_1, f_2\in C_F(G)$,
3. $B(\bar{f})=\overline{B(f)}$, $\quad f\in C_F(G)$,
4. $B(r_g f)=r_g(B(f))$, $\quad f\in C_F(G)$,

where r denotes right translations.

PROOF. It follows from Observations 1 and 2 that there exists a unitary operator B on $L^2(G)$ with the action given by Property 1. Thus B is well defined as a linear operator from $C_F(G)$ into $C(G)$ by Observation 2. But since $\phi_i(\beta\alpha_g(x_i))=(\beta^*\phi_i)(\alpha_g(x_i))$, Observation 2 implies that B is in fact an operator from $C_F(G)$ into $C_F(G)$. Now if $m(f)$ denotes the operator of multiplication by $f\in C_F(G)$ on $L^2(G)$ then

$$(B^*F,\ m(f)B^*G)=(F,\ m(Bf)G)$$

for all F, $G\in L^2(G)$. Since B is unitary on $L^2(G)$ it then follows that

$$\|f\|_\infty=\|m(f)\|=\|m(Bf)\|=\|Bf\|_\infty,$$

i.e. B is an isometry from $C_F(G)$ into $C_F(G)$. By considering β^{-1} instead of β we see that B maps $C_F(G)$ onto $C_F(G)$.

The multiplicative property of B follows from the calculation

$$\begin{aligned} B(\phi(\alpha_g(x))\phi(\alpha_g(y)))&=B\Big(\sum_{h\in H_W}\xi_h(\alpha_g(x\tau_h(y)))\Big)\\ &=\sum_{h\in H_W}\xi_h(\beta\alpha_g(x\tau_h(y)))\\ &=\phi(\beta\alpha_g(x))\phi(\beta\alpha_g(y))\\ &=B(\phi(\alpha_g(x)))B(\phi(\alpha_g(y)))\end{aligned}$$

and B commutes with the involution because

$$\begin{aligned} B(\overline{\phi(\alpha_g(x))})&=B(\phi^*(\alpha_g(x^*)))\\ &=\phi^*(\beta\alpha_g(x^*))\\ &=\overline{\phi(\beta\alpha_g(x))}=\overline{B\phi(\alpha_g(x))}.\end{aligned}$$

Finally B commutes with right translations because

$$\begin{aligned}B(r_h(\psi(\alpha_g(x))))&=B(\psi(\alpha_g(\alpha_h(x))))\\&=\psi(\beta\alpha_g(\alpha_h(x)))\\&=r_h(\psi(\beta\alpha_g(x)))=r_h(B(\psi(\alpha_g(x))))\,.\end{aligned}$$

The proof of Theorem 2.1 is now straightforward.

The operator B as defined is an isometry from $C_F(G)$ onto $C_F(G)$. But as $C_F(G)$ is norm dense in $C(G)$ one can extend B by continuity to an isometry from $C(G)$ onto $C(G)$. The properties established in Observation 3 then extend by continuity. Hence B is a $*$-automorphism of $C(G)$ which commutes with right translations. Now let b be the homeomorphism of the spectrum G of $C(G)$ corresponding to B. If $b(e)=g$ then

$$b(h)=b(eh)=b(e)h=gh$$

for all $h\in G$, i.e. B is left translation by g^{-1}. Thus

$$\psi(\beta\alpha_h(x))=B(\psi(\alpha_h(x)))=\psi(\alpha_g\alpha_h(x))$$

for all $x\in\mathcal{A}_F$, $\psi\in\mathcal{A}^*$, and $h\in G$. Consequently

$$\beta(x)=\alpha_g(x)$$

for all $x\in\mathcal{A}_F$ which implies $\beta=\alpha_g$.

The above procedure of constructing $C_F(G)$ from elements of $\mathcal{A}_F$ gives information about the spectral subspaces of $(\mathcal{A}, G, \alpha)$.

Corollary 2.2. *Let $(\mathcal{A}, H, \tau)$ be a strongly topologically transitive C^*-dynamical system and α a (faithful) continuous action of a compact group G as $*$-automorphisms of $\mathcal{A}$ such that $[\alpha, \tau]=0$. Further let $\mathcal{U}(G)$ denote the set of irreducible representations of G and for each $U\in\mathcal{U}(G)$ define the spectral subspace $\mathcal{A}^\alpha(U)$ by*

$$\mathcal{A}^\alpha(U)=\left\{\int dg\ \mathrm{Tr}(U_g^{-1})\alpha_g(x)\,;\ x\in\mathcal{A}\right\}.$$

It follows that

1. $\mathcal{A}^\alpha(U)\neq\{0\}$, $U\in\mathcal{U}(G)$,
2. *If $U_1, U_2\in\mathcal{U}(G)$ and $V\in\mathcal{U}(G)$ occurs in the decomposition of $U_1\otimes U_2$ then*

$$(\mathcal{A}^\alpha(U_1)\mathcal{A}^\alpha(U_2))\cap\mathcal{A}^\alpha(V)\neq\{0\}\,.$$

We conclude this section with two examples which demonstrate the difficulty in characterizing the automorphisms α_g. In both examples one has a $*$-automorphism β which leaves invariant each finite-dimensional α-invariant subspace of $\mathcal{A}$ but $\beta\notin\alpha_G$.

Example 2.3 (Longo and Peligrad [**10**]). Let S_3 be the permutation group on 3 elements; S_3 has order 6 and is generated by two elements r, s with the relations $r^3=e$, $s^2=e$, $rs=sr^2$. The dual $\hat{S}_3$ consists of 3 representations γ_1, γ_2, γ_3 with $\dim\gamma_1=\dim\gamma_2=1$, and $\dim\gamma_3=2$, where

$$\gamma_1(r)=1\,,\qquad \gamma_1(s)=1$$

$$\gamma_2(r)=1\,,\qquad \gamma_2(s)=-1$$

$$\gamma_3(r)=\begin{pmatrix}\cos 2\pi/3 & \sin 2\pi/3\\ -\sin 2\pi/3 & \cos 2\pi/3\end{pmatrix},\qquad \gamma_3(s)=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}.$$

Let $\mathcal{A}=M_2$ be the algebra of 2×2 matrices and define $\alpha_g=\mathrm{Ad}(\gamma_3(g))$. Then α is ergodic, and hence strongly topologically transitive by Theorem 1.4. The representation α of G has the decomposition

$$\alpha=\gamma_1\oplus\gamma_2\oplus\gamma_3$$

into irreducibles and there is a unitary operator $V\in\mathcal{A}$ such that $\alpha_g(V)=\gamma_2(g)V$. The operator V is determined up to a phase factor by the requirement that V is in the γ_2-subspace of $\mathcal{A}$. Let $\beta=\mathrm{Ad}(V)$. It follows that $[\beta,\alpha]=0$. Moreover as each of the representations γ_1, γ_2, γ_3 occurs with multiplicity one it follows that the only α-invariant subspaces of $\mathcal{A}$ are the subspaces corresponding to these three representations, and all linear combinations of these subspaces. Therefore β leaves all these α-invariant subspaces invariant, but nevertheless $\beta\notin\alpha_G$.

EXAMPLE 2.4. Let $\mathcal{A}=C(S_2)$ denote the continuous functions over the two sphere S_2 and α the canonical action of the group $G=SO(3)$ of rotations on $\mathcal{A}$. The system $(\mathcal{A}, G, \alpha)$ is topologically transitive, and hence strongly topologically transitive by Theorem 1.3. Let β be the $*$-automorphism of $C(S_2)$ corresponding to reflection about the origin. Then $\beta\notin\alpha_G$. Nevertheless $[\beta,\alpha]=0$ and a calculation with spherical harmonics shows that all the irreducible representations of $SO(3)$ occur in the decomposition of α with multiplicity one and hence β leaves all the finite-dimensional α-invariant subspaces invariant.

3. The infinitesimal structure of (G, α).

The next result gives an infinitesimal characterization of the one-parameter subgroups of the group action (G, α) considered in Theorem 2.1. It is similar to Theorem 2.1 of [11]. It is possible to give a proof roughly following the lines of the proof of Theorem 2.1 of [11], but we give a shorter proof based upon the relation between $\mathcal{A}_F$ and $C_F(G)$ established in the previous section.

THEOREM 3.1. *Let $(\mathcal{A}, H, \tau)$ be a strongly topologically transitive C^*-dynamical system and α a (faithful) continuous action of a compact group G as $*$-automorphisms of $\mathcal{A}$ such that $[\alpha, \tau]=0$. Further let δ be a symmetric derivation of*

$\mathcal{A}$ with domain $D(\delta)=\mathcal{A}_F$, the G-finite elements of $\mathcal{A}$. The following conditions are equivalent:

1. *δ is closable and its closure $\bar{\delta}$ generates a one-parameter subgroup of α_G.*
2. a. $\delta(x)=0, \quad x\in\mathcal{A}^\alpha$,
   b. $\delta\tau_h(x)=\tau_h\delta(x), \quad x\in\mathcal{A}_F,\ h\in H.$

REMARKS. 1. The drawback of this result as opposed to the comparable results of [**9**] and [**11**] is that in Condition 2 we must explicitly assume that δ is zero on the fixed point algebra $\mathcal{A}^\alpha$. In [**9**] and [**11**] this was a consequence of $D(\delta)=\mathcal{A}_F$, simplicity of $\mathcal{A}$, and asymptotic abelianness of $(\mathcal{A}, H, \tau)$. Also this holds if $\mathcal{A}$ is abelian.

2. If G is abelian, the techniques of [**10**] show that the assumption of strong topological transitivity may be replaced by topological transitivity.

PROOF. $1\Rightarrow2$ is evident. $2\Rightarrow1$. If $x, y\in\mathcal{A}_F$ and $\phi, \psi\in\mathcal{A}^*$ then using the notation of Section 2 with $W=V+\delta(V)$ where V is the α-invariant span of x and y, one calculates that

$$\begin{aligned}&\int dg\,\{\psi(\alpha_g(x))\phi(\delta(\alpha_g(y)))+\psi(\delta(\alpha_g(x)))\phi(\alpha_g(y))\}\\&=\int dg \sum_{h\in H_W}\xi_h(\alpha_g(x)\tau_h(\delta\alpha_g(y))+\delta(\alpha_g(x))\tau_h(\alpha_g(y)))\\&=\sum_{h\in H_W}\xi_h\Big(\int dg\ \delta(\alpha_g(x\tau_h(y)))\Big)\\&=0.\end{aligned}$$

The last step follows because δ is zero on $\mathcal{A}^\alpha$. This establishes that one can define a linear operator $D: C_F(G)\to C(G)$ by

$$D\Big(\sum_{i=1}^n \phi_i(\alpha_g(x_i))\Big)=i\sum_{i=1}^n \phi_i(\delta(\alpha_g(x_i)))$$

and D is symmetric on $L^2(G)$. (Use Observation 2 of Section 3.) But further calculations analogous to those in the proof of Observation 3 of Section 2 then establish that

$$\begin{aligned}&D(f_1f_2)=D(f_1)f_2+f_1D(f_2), && f_1, f_2\in C_F(G),\\&D(\bar{f})=\overline{D(f)}, && f\in C_F(G),\\&D(r_gf)=r_g(D(f)), && f\in C_F(G).\end{aligned}$$

Since D commutes with right translations it leaves the corresponding finite-dimensional spectral subspaces of $C(G)$ invariant. Hence D is essentially self-adjoint since it is the direct sum of bounded symmetric operators. Consequently the closure $\bar{D}$ of D generates a strongly continuous one-parameter group β of $*$-automorphisms of $C(G)$ (see, for example, the discussion in Example 3.2.67 of

[5]). Moreover β must commute with right translations. Therefore β is a one-parameter subgroup of left translations by the argument used in the proof of Theorem 2.1.

Let l denote left translations on $C(G)$; then there is a one-parameter subgroup $t\mapsto h_t$ of G such that

$$\beta_t\phi(\alpha_g(x))=l_{h_t}\phi(\alpha_g(x))=\phi(\alpha_{h_t}^{-1}\alpha_g(x))$$

for $x\in\mathcal{A}_F$ and $\phi\in\mathcal{A}^*$. But if $\hat{\delta}$ denotes the generator of the one-parameter group of $*$-automorphisms $t\mapsto\alpha_{h_t}^{-1}$ one finds

$$\phi(\delta(\alpha_g(x)))=D\phi(\alpha_g(x))=\phi(\hat{\delta}(\alpha_g(x)))$$

by differentiation. Thus $\delta=\hat{\delta}$ on $\mathcal{A}_F$. Finally since $\mathcal{A}_F$ is invariant under α_{h_t} for all t it follows that $\mathcal{A}_F$ is a core for $\hat{\delta}$; see Chapter 3 of [5]. Hence δ is closable and $\bar{\delta}=\hat{\delta}$. (Compare [6], where the group G is only assumed to be locally compact, but the derivation D is assumed to be closed.)

Remark. The generation property in Theorem 2.1 of [11] was established by first proving that δ leaves invariant each finite-dimensional α-invariant subspace. This can be deduced directly from strong topological transitivity as follows.

Let $x_1, x_2, \cdots, x_n$ be a basis of linearly independent elements of the α-invariant subspace $\mathcal{M}$. Then the action α is given on $\mathcal{M}$ by a matrix,

$$\alpha_g(x_i)=\sum_{j=1}^n U_{ji}(g)x_j\,,$$

and by linear rearrangement, using the orthogonality relations in the group, one can suppose that (U_{ji}) is in fact unitary. Then since α commutes with τ one calculates that

$$\alpha_g\Big(\sum_{i=1}^n x_i^*\tau_h(x_i)\Big)=\sum_{i=1}^n x_i^*\tau_h(x_i)\,.$$

Thus

$$0=\delta\Big(\sum_{i=1}^n x_i^*\tau_h(x_i)\Big)=\sum_{i=1}^n \delta(x_i)^*\tau_h(x_i)+x_i^*\tau_h(\delta(x_i))$$

for all $h\in H$. But strong topological transitivity then implies that

$$\sum_{i=1}^n \delta(x_i^*)\otimes x_i+x_i^*\otimes\delta(x_i)=0\,.$$

Consequently

$$\sum_{i=1}^n \delta(x_i)\omega_j(x_i^*)=-\sum_{i=1}^n x_i\omega_j(\delta(x_i^*))\in\mathcal{M}$$

for any state ω_j over $\mathcal{A}$ or, by linear algebra,

$$\delta(x_i)\,\mathrm{Det}\,\overline{\omega_j(x_i)}\in\mathcal{M}\,.$$

But since the x_i are linearly independent the ω_j can be chosen such that the determinant is non-zero and hence $\delta(x_i)\in\mathcal{M}$.

4. Dissipations.

We conclude with some remarks on the generation problem for dissipations.

Consider the assumptions of Theorem 3.1 but with δ a symmetric dissipation, i.e.

$$\delta(x^*x)\leqq x^*\delta(x)+\delta(x)^*x$$

for all $x\in\mathcal{A}_F$. It is then natural to ask whether the conditions $\delta(\mathcal{A}^\alpha)=\{0\}$ and $[\delta, \tau]=0$ imply that δ is closable and its closure $\bar{\delta}$ generates a strongly positive semigroup β.

If $\mathcal{A}$ is simple with identity, τ is norm asymptotically abelian, and G is abelian, then this question is answered in the affirmative by Theorem 3.1 of [**9**] and in fact β is completely positive. On the other hand the example in Section 3 of [**4**] with $\mathcal{A}=M_2$, $G=H=\boldsymbol{Z}_2\times\boldsymbol{Z}_2$, $\tau=\alpha$, and $\delta=H$ shows that even if δ generates a positive semigroup strong topological transitivity of $(\mathcal{A}, H, \tau)$ does not necessarily imply that the semigroup is completely positive.

If one tries to tackle this problem with the techniques of the present paper then it is not clear that the dissipation δ lifts to an operator D on $C_F(G)$, as in the proof of Theorem 3.1, i.e. by the definition

$$D(\phi(\alpha_g(x)))=\phi(\delta(\alpha_g(x)))\,, \qquad x\in\mathcal{A}_F,\ \phi\in\mathcal{A}^*.$$

If, however, G is abelian and τ is only assumed to be topologically transitive then it follows from the techniques used in the proof of Theorem 3.1 in [**10**], combined with techniques of [**3**], that there exists a function $\phi:\hat{G}\mapsto\boldsymbol{C}$ such that

$$\delta(x)=\phi(\gamma)x$$

for all $x\in\mathcal{A}^\alpha(\gamma)$, the α-spectral subspace of $\mathcal{A}$ corresponding to the character $\gamma\in\hat{G}$. Using this, it is easy to see that D is well defined, and in fact is given by

$$D(\gamma(g))=\phi(\gamma)\gamma(g)$$

for $\gamma\in\hat{G}$. But D is not generally a dissipation, or, equivalently, ϕ is not generally negative definite. This is clear from the example on $\mathcal{A}=M_2$ mentioned above. In this example D is a dissipation if δ is a complete dissipation, and then δ is generally a generator of a completely positive semigroup (see, for example, [**3**], [**4**]).

There is one important special case where D is a dissipation, the case that both $\mathcal{A}$ and G are abelian.

PROPOSITION 4.1. *Let $(\mathcal{A}, H, \tau)$ be a topologically transitive C^*-dynamical system where $\mathcal{A}$ is abelian, and let α be a faithful continuous action of a compact abelian group G as $*$-automorphisms of $\mathcal{A}$ such that $[\alpha, \tau]=0$. Further, let δ be a symmetric operator on $\mathcal{A}$ with domain $D(\delta)=\mathcal{A}_F$, the G-finite elements of $\mathcal{A}$. Assume that*

i. *δ is a dissipation, i.e.*

$$\delta(x^*x)\leqq\delta(x)^*x+x^*\delta(x)$$

for all $x\in\mathcal{A}_F$,

ii. *$\delta(x)=0$ for all $x\in\mathcal{A}^\alpha$,*

iii. *$\delta\tau_h(x)=\tau_h\delta(x)$ for all $x\in\mathcal{A}_F$, $h\in H$.*

It follows that δ is closable and its closure $\bar{\delta}$ generates a one-parameter semigroup $t\geqq 0\mapsto\exp\{-t\bar{\delta}\}$ of completely positive contractions. Furthermore, there exists a convolution semigroup $t\geqq 0\mapsto\mu_t$ of probability measures on G such that

$$e^{-t\bar{\delta}}(x)=\int_G d\mu_t(g)\alpha_g(x)$$

for all $x\in\mathcal{A}$.

PROOF. The proof combines the tensor product characterization of topological transitivity on abelian C^*-algebras in Theorem 1.3 with ideas from the proof of Theorem 3.1 in [**9**], but the present case is simpler.

First note that by Lemma 1.1 we may assume $\mathcal{A}$ has an identity. Next we have already remarked that topological transitivity of τ implies the existence of a function $\phi: G\to \boldsymbol{C}$ such that

$$\delta(x)=\phi(\gamma)x$$

for all $x\in\mathcal{A}^\alpha(\gamma)$, $\gamma\in\hat{G}$. Next for $x_i\in\mathcal{A}^\alpha(\gamma_i)$ and $h_i\in H$, $i=1, 2, \cdots, k$, set $x=\sum\tau_{h_i}(x_i)$; then

$$\delta(x^*)x+x^*\delta(x)-\delta(x^*x)=\sum_{i,j=1}^{k} M_{ij}\tau_{h_i}(x_i)^*\tau_{h_j}(x_j)\geqq 0$$

where

$$M_{ij}=\overline{\phi(\gamma_i)}+\phi(\gamma_j)-\phi(\gamma_j-\gamma_i)\,.$$

Now, if $y_i^{(j)}$ are elements in $\mathcal{A}$ such that

$$\begin{aligned}&T_h\Big(\sum_{i=1}^{n} y_i^{(1)}\otimes y_i^{(2)}\otimes\cdots\otimes y_i^{(k)}\Big)\\&=\sum_{i=1}^{n}\tau_{h_1}(y_i^{(1)})\tau_{h_2}(y_i^{(2)})\cdots\tau_{h_k}(y_i^{(k)})\\&\geqq 0\end{aligned}$$

for all $h\in H^k$, then it follows from the isometric nature of the morphism

$$\bigoplus_{h\in H^k} T_h:\otimes^k\mathcal{A}\longrightarrow\bigoplus_{h\in H^k}\mathcal{A}$$

(Theorem 1.3) that

$$\sum_{i=1}^{n} y_i^{(1)} \otimes y_i^{(2)} \otimes \cdots \otimes y_i^{(k)} \geqq 0$$

in $(\otimes^k \mathcal{A})^{\wedge}$, the C^*-algebra completion of the algebraic tensor product $\otimes^k \mathcal{A}$. Applying this to the inequality for M_{ij} above, we deduce that

$$(*) \qquad \sum_{i,j=1}^{k} M_{ij} X_i^* X_j \geqq 0$$

in $(\otimes^k \mathcal{A})^{\wedge}$, where $X_i = \mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes x_i \otimes \cdots \otimes \mathbf{1}$ with the x_i occurring in the i'th position.

As $\mathcal{A}$ is abelian there exist pure states ω_i on $\mathcal{A}$ with $|\omega_i(x_i)| = \|x_i\|$. Hence, taking $x_i \neq 0$, replacing x_i by $\lambda_i x_i / \omega_i(x_i)$ in $(*)$, where $\lambda_i \in \boldsymbol{C}$, and applying the product state $\omega_1 \otimes \cdots \otimes \omega_k$ we find

$$\sum_{i,j=1}^{k} M_{ij} \bar{\lambda}_i \lambda_j \geqq 0 .$$

Thus the function ϕ is negative definite on $\hat{G}$. The rest of the proof is exactly as in the last part of the proof of Theorem 5.1 in [3].

Finally we note that if $(\mathcal{A}, H, \tau)$ satisfies the strong condition

$$\sup_{h \in H} \|x \tau_h(y)\| = \|x\| \cdot \|y\| , \qquad x, y \in \mathcal{A}$$

of transitivity, and if (G, α) is the action of a compact abelian group which commutes with τ, and if δ is a symmetric dissipation satisfying $\delta(\mathcal{A}^\alpha) = \{0\}$ and $[\delta, \tau] = 0$, then one can deduce that the associated function $\phi : \hat{G} \to \boldsymbol{C}$ satisfies

$$(*) \qquad \sqrt{|\phi(\gamma+\mu)|} \leqq \sqrt{|\phi(\gamma)|} + \sqrt{|\phi(\mu)|} .$$

This follows by examining the inequalities

$$\omega(x^* \delta(x) + \delta(x)^* x - \delta(x^* x)) \geqq 0$$

for states ω and elements $x = \lambda x_\gamma + \tau_h(x_\mu)$ where $\lambda \in \boldsymbol{C}$, $x_\gamma \in \mathcal{A}^\alpha(\gamma)$, and $x_\mu \in \mathcal{A}^\alpha(\mu)$.

It is an interesting question whether $(*)$ is sufficient to ensure that $\exp\{-t\delta\}$ is contractive as a map from $\mathcal{A}_F$ to $\mathcal{A}_F$ since this would ensure that it extends to a positive semigroup.

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Ola BRATTELI
Institute of Mathematics
University of Trondheim
N-7034 Trondheim-NTH
Norway

George A. ELLIOTT
Mathematics Institute
University of Copenhagen
Universitetsparken 5
DK-2100, Copenhagen Ø
Denmark

Derek W. ROBINSON
Department of Matnematics
Institute of Advanced Studies
Australian National University
Canberra, A. C. T.
Australia