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On the structure of polarized manifolds with total deficiency one, II

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Introduction.

This is the second part of the study of polarized manifolds (M, L) with $\Delta(M, L)=1$. In this part we consider those with $d(M, L)=5$ and we prove the following

THEOREM. *Any polarized manifold (M, L) with $\Delta(M, L)=1$, $d(M, L)=5$ is isomorphic to a linear section of $Gr(5, 2)$ embedded by the Plücker coordinate. Here $Gr(5, 2)$ denotes the Grassmann variety parametrizing 2-dimensional linear subspaces of $\boldsymbol{C}^5$.*

Notation, convention and terminology.

We use the same notation as in the first part [**5**] except a few new symbols listed below. In particular, vector bundles are confused with locally free sheaves. Tensor products of line bundles are denoted by additive notation.

Example of symbols in the same use as in [**5**].

$\{Z\}$: The homology class defined by an analytic subset Z.

$|L|$: The complete linear system associated with a line bundle L.

L_T : The pull back of L to a space T by a given morphism.

However, when there is no danger of confusion, we OFTEN write L instead of L_T.

$[\Lambda]$: The line bundle defined by a linear system Λ.

$Bs\Lambda$: The intersection of all the members of Λ.

ρ_Λ : The rational mapping defined by Λ.

K^M : The canonical bundle of a manifold M.

$Q_C(M)$: The blowing up of M with center C.

E_C : The exceptional divisor on $Q_C(M)$ over C.

$E^\vee$: The dual bundle of a vector bundle E.

$\mathcal{F}[E]:=\mathcal{F}\otimes_{\mathcal{O}}\mathcal{E}$ for a coherent sheaf $\mathcal{F}$, where $\mathcal{E}$ is the locally free sheaf of sections of E.

$\boldsymbol{P}(E):=E^\vee-\{\text{0-section}\}/\boldsymbol{C}^\times$.

H^E: The relatively ample tautological line bundle on $\boldsymbol{P}(E)$.

H_α, H_β, $\cdots$: The line bundles defined by hyperplane sections on projective spaces $\boldsymbol{P}_\alpha$, $\boldsymbol{P}_\beta$, $\cdots$ indicated by the same suffixes.

$d(M, L) := (c_1(L))^n\{M\}$, where $n = \dim M$.

$\Delta(M, L) := n + d(M, L) - h^0(M, L)$.

Symbols introduced in this part.

Λ^{*Y}: The sub-linear system of Λ consisting of $D \in \Lambda$ such that $D \supset Y$, where Y is an analytic subset.

T^M: The tangent bundle of M.

Ω_M: The cotangent bundle $(T^M)^\vee$ of M.

$N^{C\backslash M}$: The normal bundle of a submanifold C in M.

$\Omega_{C\backslash M}$: The conormal bundle $(N^{C\backslash M})^\vee$.

$b_i(M)$: The i-th Betti number of M.

$s_j(E)$: The j-th Segre class of $E^\vee$.

Preliminaries.

Here, for the convenience of the reader, we review a couple of known results used (often without referring explicitly) in this paper.

(P. 1) Let E be a vector bundle. Then the total Segre class $s(E) = \sum_j s_j(E)$ is related to Chern classes by the formula $s(E)c(E^\vee) = 1$. One can take this to be a definition of $s(E)$. From this formula we see $s_1(E) = c_1(E)$, $s_2(E) = c_1(E)^2 - c_2(E)$, and so on. (For details, see [**10**]. But our notation is different from that of [**10**].)

(P. 2) If E is a direct sum of line bundles, then $s_n(E)$ is the sum of all the monomials of degree n of their Chern classes. For example, $s_2(A \oplus B \oplus C) = \alpha^2 + \beta^2 + \gamma^2 + \beta\gamma + \gamma\alpha + \alpha\beta$ and $s_3(A \oplus B) = \alpha^3 + \alpha^2\beta + \alpha\beta^2 + \beta^3$, where $\alpha = c_1(A)$, $\beta = c_1(B)$ and $\gamma = c_1(C)$.

(P. 3) The cohomology ring $H^*(\boldsymbol{P}(E); \boldsymbol{Z})$ of $\boldsymbol{P}(E)$ is generated by $c_1(H^E)$ as a $H^*(S; \boldsymbol{Z})$-module where S is the base space. The following formula is very useful for the calculation of intersection numbers.

(P. 4) $(H^E)^{a+r-1}\alpha\{\boldsymbol{P}(E)\} = s_a(E)\alpha\{S\}$ for any $\alpha \in H^{2(n-a)}(S; \boldsymbol{Z})$ where $n = \dim S$ and $r = \operatorname{rank} E$.

(P. 5) A vector bundle F on $\boldsymbol{P}(E)$ comes from a vector bundle on S if and only if the restriction of F to each fiber of $\boldsymbol{P}(E) \to S$ is trivial.

(P. 6) For any vector bundle V on S we have $H^p(S, V \otimes S^k E) \cong H^p(\boldsymbol{P}(E), V[kH^E])$, where k is a non-negative integer and $S^k E$ denotes the k-th symmetric product of E.

(P. 7) $H^p(\boldsymbol{P}(E), V[-kH^E]) = 0$ for $1 \leqq k \leqq r-1$.

(P. 8) Suppose that S is smooth and let $T^{\boldsymbol{P}(E)/S}$ be the relative tangent bundle $\mathrm{Ker}(T^{P(E)} \to T^{S}{}_{P(E)})$. Then there is a natural exact sequence $0 \to [0] \to E^{\vee}[H^{E}] \to T^{P(E)/S} \to 0$. From this follows the formula: $K^{P(E)} = -rH^{E} + [K^{S} + \det E]_{P(E)}$.

(P.9) Let C be a submanifold of a manifold M. Let $\tilde{M} = Q_C(M)$ be the blowing up of M with center C and let E_C be the exceptional divisor over C. Then $(E_C, [-E_C]) \cong (\boldsymbol{P}(\Omega), H^{\Omega})$ where $\Omega = \Omega_{C\setminus M}$ is the conormal bundle of C in M.

(P.10) The cohomology ring of $\tilde{M}$ is generated by $c_1(E_C)$ and $H^*(M; \boldsymbol{Z})$. Intersection numbers can be calculated by (P.9) and (P.4).

(P.11) A vector bundle F on $\tilde{M}$ comes from a vector bundle on M if and only if the restriction of F to each fiber of $E_C \to C$ is trivial.

(P.12) $K^{\tilde{M}} = K^{M} + (r-1)E_C$ where $r = \operatorname{codim} C$.

(P.13) On $\tilde{M}$ we have the following exact sequence: $0 \to \mathcal{O}_{\tilde{M}}(T^{\tilde{M}}) \to \mathcal{O}_{\tilde{M}}(T^{M}{}_{\tilde{M}}) \to \mathcal{O}_{E_C}(T^{E_C/C}[E_C]) \to 0$. So, combining (P.7), (P.8) and (P.9), we obtain an exact sequence $0 \to H^0(\tilde{M}, T^{\tilde{M}}) \to H^0(M, T^{M}) \to H^0(C, N^{C\setminus M}) \to H^1(\tilde{M}, T^{\tilde{M}}) \to H^1(M, T^{M}) \to H^1(C, N^{C\setminus M}) \to \cdots$.

(P.14) Let D be a smooth divisor on a manifold M. Suppose that D is a $\boldsymbol{P}^{r-1}$-bundle over a manifold B and that the restriction of $[D]$ to each fiber of $D \to B$ is $\mathcal{O}(-1)$. Then there is a manifold N containing B as a submanifold such that $Q_B(N) \cong M$, where the isomorphism induces $E_B \cong D$ preserving the $\boldsymbol{P}^{r-1}$-bundle structure over B.

§7. Examples of Del Pezzo manifolds of degree five and statement of main results.

(7.1) Let M be the Grassmann variety of two-dimensional linear subspaces of $\boldsymbol{C}^5$. Let L be the hyperplane section associated with the Plücker embedding. Then $\dim M = 6$, $d(M, L) = 5$ and $h^0(M, L) = 10$. So $\Delta(M, L) = 1$. Hence (M, L) is a Del Pezzo manifold (see (1.13) and (1.6)).

(7.2) When (M, L) is a Del Pezzo manifold, any smooth member of $|L|$ is also a Del Pezzo manifold. Hence we obtain Del Pezzo manifolds with $d=5$ by taking hyperplane sections of $Gr(5, 2)$ successively.

(7.3) Let S be a smooth hyperquadric in $P = \boldsymbol{P}^3_{\alpha}$. Then $S \cong \boldsymbol{P}^1_{\sigma} \times \boldsymbol{P}^1_{\tau}$ with $[H_{\alpha}]_S = H_{\sigma} + H_{\tau}$. Let C be a smooth member of $|H_{\sigma} + 3H_{\tau}|$ on S. Let $\tilde{P} = Q_C(P)$ be the blowing up of P with center C and let $\tilde{S}$ be the proper transform of S. Then $\tilde{S} \cong S \cong \boldsymbol{P}^1_{\sigma} \times \boldsymbol{P}^1_{\tau}$ and $N^{\tilde{S}\setminus\tilde{P}} = [S] - [E_C] = 2H_{\alpha} - (H_{\sigma} + 3H_{\tau}) = H_{\sigma} - H_{\tau}$. Considering $\tilde{S}$ as a $\boldsymbol{P}^1$-bundle over $\boldsymbol{P}^1_{\sigma}$, we apply (P.14) to obtain a manifold M which contains a submanifold $B \cong \boldsymbol{P}^1_{\sigma}$ such that $(Q_B(M), E_B) \cong (\tilde{P}, \tilde{S})$. By (P.11) we infer that $3H_{\alpha} - E_C = L_{\tilde{P}}$ for some $L \in Pic(M)$. We see easily that $2L - [\tilde{S}] = 4H_{\alpha} - E_C = L + H_{\alpha}$ is ample on $\tilde{P}$. Moreover, $LB = L_{\tilde{S}} H_{\tau}\{\tilde{S}\} = 2H_{\sigma}H_{\tau}\{\boldsymbol{P}^1_{\sigma} \times \boldsymbol{P}^1_{\tau}\} = 2$. So [**4**, (5.7)] applies to the effect that L is ample on M. $K^{M} + 2L = K^{\tilde{P}} - E_B + 2L = 0$

in $Pic(\check{P})$. This implies that (M, L) is a Del Pezzo manifold. Finally we infer $d(M, L)=L^3\{\tilde{P}\}=(3H_\alpha-E_C)^3=27H_\alpha^3-27H_\alpha^2E_C+9H_\alpha E_C^2-E_C^3=5$, since $H_\alpha^3=1$, $H_\alpha^2E_C=0$, $H_\alpha E_C^2=-H_\alpha(-E_C)\{E_C\}=-H_\alpha\{C\}=-(H_\sigma+H_\tau)(H_\sigma+3H_\tau)\{\boldsymbol{P}_\sigma^1\times\boldsymbol{P}_\tau^1\}=-4$ and $E_C^3=(-E_C)^2\{E_C\}=s_1(\Omega_{C\backslash P})=-(2H_\alpha+H_\sigma+3H_\tau)(H_\sigma+3H_\tau)\{\boldsymbol{P}_\sigma^1\times\boldsymbol{P}_\tau^1\}=-14$ (see (P. 4) and (P. 9)).

Thus (M, L) is a Del Pezzo 3-fold with $d=5$.

REMARK (not used in the following sections). Taking a smooth member of $|aH_\sigma+3H_\tau|$ with $a=0, 2, 3$ instead of C, we obtain a prepolarized manifold (M, L) by a similar procedure as above. L is ample on M if $a\leqq 2$ and $d(M, L)=6-a$, $\Delta(M, L)=1$. $M\cong\boldsymbol{P}^1\times\boldsymbol{P}^1\times\boldsymbol{P}^1$ if $a=0$. M is a complete intersection of type (2, 2) when $a=2$. When $a=3$, L is not ample because $LB=0$. But $B\cong\boldsymbol{P}^1$ can be contracted to an ordinary double point and we get a hypercubic. Of course L is the pull back of the hyperplane section bundle.

(7.4) Let Q be a smooth hyperquadric in $\boldsymbol{P}_\alpha^4$. Let S be a smooth hyperplane section. Then $S\cong\boldsymbol{P}_\sigma^1\times\boldsymbol{P}_\tau^1$ with $[H_\alpha]_S=H_\sigma+H_\tau$. Let C be a smooth member of $|H_\sigma+2H_\tau|$ on S. C is a Veronese curve of degree three in the hyperplane in $\boldsymbol{P}_\alpha^4$ containing S. Let $\tilde{Q}$ be the blowing up $Q_C(Q)$ and let $\tilde{S}$ be the proper transform of S. Then $\tilde{S}\cong S$ and $N^{\tilde{S}\backslash\tilde{Q}}=H_\alpha-E_C=-H_\tau$. Regarding S to be a $\boldsymbol{P}^1$-bundle over $\boldsymbol{P}_\sigma^1$, we blow down S to $l\cong\boldsymbol{P}^1$ and we get a manifold M containing l such that $(Q_l(M), E_l)\cong(\tilde{Q}, \tilde{S})$. As in (7.3), $2H_\alpha-E_C=L_Q$ for some $L\in Pic(M)$. $2L-E_l=3H_\alpha-E_C$ is ample on $\tilde{Q}$ and $L\{l\}=1$. Therefore L is ample on M. $K^M+2L=K^{\tilde{Q}}-E_l+2L=(-3H_\alpha+E_C)-(H_\alpha-E_C)+2(2H_\alpha-E_C)=0$ on $\tilde{Q}$. This implies that (M, L) is a Del Pezzo manifold. Finally we see $d(M, L)=(2H_\alpha-E_C)^3\{\tilde{Q}\}=8H_\alpha^3-12H_\alpha^2E_C+6H_\alpha E_C^2-E_C^3=5$, since $H_\alpha^3=2$, $H_\alpha^2E_C=0$, $H_\alpha E_C^2=-H_\alpha\{C\}=-(H_\sigma+H_\tau)(H_\sigma+2H_\tau)\{S\}=-3$ and $E_C^3=-s_1(N^{C\backslash Q})=-(H_\alpha+H_\sigma+2H_\tau)(H_\sigma+2H_\tau)\{S\}=-7$.

Thus we construct a Del Pezzo 3-fold with $d=5$.

(7.5) Let Q be a smooth hyperquadric in $\boldsymbol{P}_\alpha^4$. Let D be a hyperplane section with one ordinary double point p. Let Q_1 be the blowing up $Q_p(Q)$ and let D_1 be the proper transform of D. Then $[D_1]=H_\alpha-2E_p$ in $Pic(Q_1)$ and $D_1\cong\Sigma_2=\{((\zeta_0:\zeta_1:\zeta_2), (\xi_0:\xi_1))\in\boldsymbol{P}_\zeta^2\times\boldsymbol{P}_\xi^1 \mid \zeta_0:\zeta_1=\xi_0^2:\xi_1^2\}$. $\Delta=D_1\cap E_p$ is the subset $\{\zeta_0=\zeta_1=0\}$ of Σ_2, which defines a section of $\Sigma_2\to\boldsymbol{P}_\xi^1$ such that $\Delta^2=-2$. Let C be a smooth member of $|H_\zeta+H_\xi|$ on D_1 and let Q_2 be the blowing up $Q_C(Q_1)$. C intersects E_p transversally at a point q since $E_pC=(H_\zeta-2H_\xi)(H_\zeta+H_\xi)\{\Sigma_2\}=1$. So the proper transform $\tilde{E}$ of $E_p\cong\boldsymbol{P}_\beta^2$ on Q_2 is isomorphic to $Q_q(E_p)$. $|H_\beta-E_q|$ makes $\tilde{E}$ a $\boldsymbol{P}^1$-bundle over $\boldsymbol{P}^1$, and $\tilde{E}$ can be blown down with respect to this structure because $[\tilde{E}]_{\tilde{E}}=-H_\beta$. Thus we obtain a manifold $\tilde{Q}$. Let $\tilde{D}$ be the image of the proper transform D_2 of D_1 by the morphism $Q_2\to\tilde{Q}$. Then it is easy to see $\tilde{D}\xleftarrow{\sim}D_2\xrightarrow{\sim}D_1\cong\Sigma_2$.

A simpler way to construct $\tilde{Q}$ is this: Let C' be the image of C on Q.

Then C' is a Veronese curve of degree three in the hyperplane of $\boldsymbol{P}_\alpha^4$ containing D. It is easy to see that $\tilde{Q}$ is nothing other than $Q_{C'}(Q)$ and that $\tilde{D}$ is the proper transform of D on $\tilde{Q}$. (From this view point, however, it is not so easy to see that $\tilde{D}$ can be blown down.)

Any way, we see $\tilde{D}\cong\Sigma_2$ and $[\tilde{D}]_{\tilde{D}}=-H_\zeta+H_\xi$. So $\tilde{D}$ can be blown down to $l\cong\boldsymbol{P}^1$ and we obtain a manifold $M\supset l$ such that $(Q_l(M), E_l)\cong(\tilde{Q}, \tilde{D})$. $2H_\alpha-E_{C'}=L_Q$ for some $L\in Pic(M)$ and L is shown to be ample on M as in (7.4). Moreover, similarly as in (7.4), we see $K^M+2L=0$ and $d(M, L)=5$. Thus (M, L) is a Del Pezzo 3-fold with $d=5$.

(7.6) We have constructed Del Pezzo 3-folds with $d=5$ in several ways described in (7.2), (7.3), (7.4) and (7.5). However, they turn to be isomorphic to each other. Indeed, in § 9, we prove the following

THEOREM (Iskovskih [7]). *All the Del Pezzo 3-folds with $d=5$ are isomorphic to each other.*

(7.7) Now we explain a 4-dimensional version of the construction (7.5). Let Q be the smooth hyperquadric in $\boldsymbol{P}_\alpha^5=(\alpha_0:\cdots:\alpha_5)$ defined by $\alpha_0\alpha_5-\alpha_1\alpha_4+\alpha_2\alpha_3=0$. Let D be the hyperplane section $\alpha_5=0$, which has an ordinary double point $p=(1:0:\cdots:0)$. Let $Q_1=Q_p(Q)$ and let D_1 be the proper transform of D on Q_1. Then $D_1\cong Z$, here $Z=\{((\zeta_0:\cdots:\zeta_4), (\xi_1:\cdots:\xi_4))\in\boldsymbol{P}_\zeta^4\times\boldsymbol{P}_\xi^3 \mid \zeta_1:\zeta_2:\zeta_3:\zeta_4=\xi_1:\xi_2:\xi_3:\xi_4,\ \zeta_1\zeta_4=\zeta_2\zeta_3,\ \xi_1\xi_4=\xi_2\xi_3\}=\{((\zeta_0:\cdots:\zeta_4), (\sigma_0:\sigma_1), (\tau_0:\tau_1))\in\boldsymbol{P}_\zeta^4\times\boldsymbol{P}_\sigma^1\times\boldsymbol{P}_\tau^1 \mid \zeta_1:\zeta_2=\zeta_3:\zeta_4=\sigma_0:\sigma_1,\ \zeta_1:\zeta_3=\zeta_2:\zeta_4=\tau_0:\tau_1\}$. Note that $[H_\alpha]_Z=H_\zeta$ and $E_p\cap D_1=\{\zeta_1=\zeta_2=\zeta_3=\zeta_4=0\}\subset Z$. This defines a section of the natural $\boldsymbol{P}^1$-bundle structure $Z\to\boldsymbol{P}_\sigma^1\times\boldsymbol{P}_\tau^1$. Moreover we have $(Z, H_\zeta)\cong(\boldsymbol{P}(E), H^E)$, where E is the vector bundle $[0]\oplus[H_\sigma+H_\tau]$ on $\boldsymbol{P}_\sigma^1\times\boldsymbol{P}_\tau^1$.

Let C be smooth member of $|H_\zeta+H_\sigma|$ on D_1 and let $Q_2=Q_C(Q_1)$. Let E_2 and D_2 be proper transforms of E_p and D_1 on Q_2 respectively. Note that $C\cap E_p$ is a line l' on $E_p\cong\boldsymbol{P}_\xi^3$ since $[C]_{E_p\cap D_1}=H_\sigma$. We may assume that $l'=\{\xi_1=\xi_3=0\}\subset E_p$ by taking a linear change of coordinate if necessary. Then $E_2\cong Q_{l'}(E_p)\cong\{((\xi_1:\cdots:\xi_4), (\tau_0:\tau_1))\in\boldsymbol{P}_\xi^3\times\boldsymbol{P}_\tau^1 \mid \xi_1:\xi_3=\tau_0:\tau_1\}$. This is a $\boldsymbol{P}^2$-bundle over $\boldsymbol{P}^1$ and l' defines a section of it. We have $[E_2]_{E_2}=[E_p]_{E_2}=-H_\xi$. So (P. 14) applies to the effect that E_2 can be blown down to l'. Thus $(Q_2, E_2)\cong(Q_{l'}(\tilde{Q}), E_{l'})$ for a manifold $\tilde{Q}$ containing l' as a submanifold. Let $\tilde{D}$ be the image of D_2 on $\tilde{Q}$. Then we see $\tilde{D}\cong Z'=\{((\zeta_0:\cdots:\zeta_4), (\tau_0:\tau_1))\in\boldsymbol{P}_\zeta^4\times\boldsymbol{P}_\tau^1 \mid \zeta_1:\zeta_3=\zeta_2:\zeta_4=\tau_0:\tau_1\}$.

A simpler way to obtain $\tilde{Q}$ is this: Let C' be the image of C on Q. $l'=C\cap E_p$ is an exceptional curve on the surface C and C' is the blowing down of it. Moreover we see that (C', H_α) is a polarized manifold with $d=3$, $\Delta=0$. Hence $C'\cong\Sigma_1$, the blowing up of $\boldsymbol{P}^2$ with center being a point. Similarly as in (7.5), $\tilde{Q}$ is nothing other than the blowing up $Q_{C'}(Q)$ and $\tilde{D}$ is the proper transform of D.

Now we see $[\tilde{D}]_{D_2}=[D_2+E_{l'}]_{D_2}=(H_\alpha-2E_p)-(H_\zeta+H_\sigma)+E_p=H_\tau-H_\zeta$ in

$Pic(Z)$, since $[E_p]_Z=H_\zeta-H_\xi$. So $N^{\tilde{D}\backslash\tilde{Q}}=H_\tau-H_\zeta$. Hence $\tilde{D}$ can be blown down to $l\cong \boldsymbol{P}^1_\tau$ with respect to the $\boldsymbol{P}^2$-bundle structure $D\cong Z'\to\boldsymbol{P}^1_\tau$. Thus we have a manifold $M\supset l$ such that $(Q_l(M), E_l)\cong(\tilde{Q}, \tilde{D})$. We see easily $[E_{C'}]_{\tilde{D}}=2H_\zeta-H_\tau$ and $2H_\alpha-E_{C'}\in Pic(\tilde{Q})$ comes from $L\in Pic(M)$. We have $L=[\tilde{D}]+H_\alpha$ on $\tilde{Q}$ and $L_{\tilde{D}}=H_\tau$. So it is easy to see that L is ample on M by [4, Appendix B]. Moreover we have $K^M+3L=K^{\tilde{Q}}-2E_l+3L=(-4H_\alpha+E_{C'})-2(H_\alpha-E_{C'})+3(2H_\alpha-E_{C'})=0$ on $\tilde{Q}$. This implies that (M, L) is a Del Pezzo manifold.

Finally we calculate $d=d(M, L)=(2H_\alpha-E_p-E_C)^4\{Q_2\}$. Putting $A=2H_\alpha-E_p$, we have $d=A^4-4A^3E_C+6A^2E_C^2-4AE_C^3+E_C^4$. Easily we have $A^4=(2H_\alpha)^4+E_p^4=31$ and $A^3E_C=0$. Recalling that $(Z, H_\zeta)\cong(\boldsymbol{P}(E), H^E)$ with $E=[0]\oplus[H_\sigma+H_\tau]$ on $\boldsymbol{P}^1_\sigma\times\boldsymbol{P}^1_\tau$ and that $A_Z=2H_\zeta-(H_\zeta-H_\xi)=H_\zeta+H_\xi$, we infer $A^2E_C^2=-A^2C=-(\zeta+\xi)^2(\zeta+\sigma)\{Z\}=-(\zeta^3+(3\sigma+2\tau)\zeta^2+4\sigma\tau\zeta)=-(s_2(E)+(3\sigma+2\tau)s_1(E)+4\sigma\tau)\{\boldsymbol{P}^1_\sigma\times\boldsymbol{P}^1_\tau\}=-(\sigma+\tau)^2-(3\sigma+2\tau)(\sigma+\tau)-4\sigma\tau=-11$, where ζ, ξ, σ and τ are the Chern classes of H_ζ, H_ξ, H_σ and H_τ. Using the exact sequence $0\to N^{C\backslash D_1}\to N^{C\backslash Q_1}\to N^{D_1\backslash Q_1}\to 0$, we get $c(N^{C\backslash Q_1})=(1+\zeta+\sigma)(1+2\xi-\zeta)$. So $s_1(\Omega_{C\backslash Q_1})=-2\xi-\sigma=-3\sigma-2\tau$ and $s_2(\Omega_{C\backslash Q_1})=(\zeta+\sigma)^2+(\zeta+\sigma)(2\xi-\zeta)+(2\xi-\zeta)^2=\zeta^2-(\sigma+2\tau)\zeta+10\sigma\tau$. Hence $AE_C^3=As_1(\Omega_{C\backslash Q_1})\{C\}=-(\zeta+\xi)(3\sigma+2\tau)(\zeta+\sigma)\{Z\}=-(3\sigma+2\tau)\zeta^2-7\sigma\tau\zeta=-((3\sigma+2\tau)(\sigma+\tau)+7\sigma\tau)\{\boldsymbol{P}^1_\sigma\times\boldsymbol{P}^1_\tau\}=-12$ and $E_C^4=-s_2(\Omega_{C\backslash Q_1})\{C\}=-(\zeta^2-(\sigma+2\tau)\zeta+10\sigma\tau)(\zeta+\sigma)\{Z\}=-\zeta^3+2\tau\zeta^2-8\sigma\tau\zeta=(-(\sigma+\tau)^2+2\tau(\sigma+\tau)-8\sigma\tau)\{\boldsymbol{P}^1_\sigma\times\boldsymbol{P}^1_\tau\}=-8$. Now, putting things together, we obtain $d=5$.

(7.8) We construct a Del Pezzo 4-fold with $d=5$ in another way. Let $P=\boldsymbol{P}^4_\alpha=(\alpha_0:\cdots:\alpha_4)$ and let D be the hyperplane $\{\alpha_4=0\}$. Let C be a Veronese curve of degree three in D, for example, $\{\alpha_0\alpha_2=\alpha_1^2,\ \alpha_0\alpha_3=\alpha_1\alpha_2,\ \alpha_1\alpha_3=\alpha_2^2\}\cong\boldsymbol{P}^1_\sigma$ with $[H_\alpha]_C=3H_\sigma$. Let $\tilde{P}$ be the blowing up of P with center C and let $\tilde{D}$ be the proper transform of D on P. Then, by definition of C, $\alpha_0\alpha_2-\alpha_1^2$, $\alpha_0\alpha_3-\alpha_1\alpha_2$, and $\alpha_1\alpha_3-\alpha_2^2$ define a linear system Λ on $\tilde{D}\cong Q_C(D)$ such that $[\Lambda]=2H_\alpha-E_C$ and $Bs\Lambda=\emptyset$. So we have a morphism $\rho_\Lambda:\tilde{D}\to\boldsymbol{P}^2_\xi$.

We claim that ρ_Λ makes $\tilde{D}$ a $\boldsymbol{P}^1$-bundle over $\boldsymbol{P}^2_\xi$. Indeed, every fiber Y of ρ_Λ is shown to be isomorphic to $\boldsymbol{P}^1$ as follows: There are two hyperquadrics Q_1, Q_2 in D containing C such that Y is the intersection of their proper transforms. Both Q_1 and Q_2 are irreducible since C cannot be contained in any hyperplane in D. Hence $Q_1\cap Q_2$ must be a curve of degree 4, and so $Q_1\cap Q_2=C\cup l$, where l is a line. It is easy to see that Y maps isomorphically onto l by $\tilde{D}\to D$.

We have $[2H_\alpha-E_C]_Y=0$ and $[\tilde{D}]_Y=[H_\alpha-E_C]_Y=-H_\alpha$. Therefore $\tilde{D}$ can be blown down with respect to ρ_Λ. Let M be the manifold containing $S\cong\boldsymbol{P}^2_\xi$ such that $(Q_S(M), E_S)\cong(\tilde{P}, \tilde{D})$. Then $2H_\alpha-E_C=L_{\tilde{P}}$ for some $L\in Pic(M)$. Using [4, Appendix B], we show L to be ample on M as before. $K^M+3L=K^{\tilde{P}}-\tilde{D}+3L=(-5H_\alpha+2E_C)-(H_\alpha-E_C)+3(2H_\alpha-E_C)=0$. So (M, L) is a Del Pezzo 4-fold.

Finally we have $d(M, L)=(2H_\alpha-E_C)^4\{\tilde{P}\}=16H_\alpha^4-32H_\alpha^3E_C+24H_\alpha^2E_C^2-8H_\alpha E_C^3$

$+E_C^4=5$, since $H_\alpha^4=1$, $H_\alpha^3E_C=H_\alpha^2E_C^2=0$, $H_\alpha E_C^3=H_\alpha C=3$ and $E_C^4=-(-E_C)^3\{E_C\}$ $=-s_1(\Omega_{C\backslash P})=-(K^PC-K^CC)=13$.

(7.9) The Del Pezzo 4-folds constructed in (7.7) and (7.8) are isomorphic to each other. Moreover, we prove in § 10 the following

THEOREM. *All the Del Pezzo 4-folds with $d=5$ are isomorphic to each other.*

(7.10) We describe a five dimensional version of (7.8). Let $P=\boldsymbol{P}_\alpha^5=\{(\alpha_0:\cdots:\alpha_5)\}$, let D be the hyperplane $\{\alpha_5=0\}$ and let $C\subset D$ be the subspace $\{\alpha_0\alpha_2=\alpha_1^2,\ \alpha_0\alpha_4=\alpha_1\alpha_3,\ \alpha_1\alpha_4=\alpha_2\alpha_3\}$. It is easy to see that $(C, H_\alpha)\cong(\boldsymbol{P}(E), H^E)$, where E is the vector bundle $[2H_\sigma]\oplus[H_\sigma]$ on $\boldsymbol{P}_\sigma^1$. Let $\tilde{P}$ be the blowing up $Q_C(P)$ and let $\tilde{D}$ be the proper transform of D. Similarly as in (7.8), we see that $\alpha_0\alpha_2-\alpha_1^2$, $\alpha_0\alpha_4-\alpha_1\alpha_3$, $\alpha_1\alpha_4-\alpha_2\alpha_3$ define a linear system Λ on $\tilde{D}\cong Q_C(D)$ such that $[\Lambda]=2H_\alpha-E_C$ and $Bs\Lambda=\emptyset$. ρ_Λ makes $\tilde{D}$ a $\boldsymbol{P}^2$-bundle over $\boldsymbol{P}_\xi^2$. Moreover, D is blown down with respect to ρ_Λ. So $(\tilde{P}, \tilde{D})\cong(Q_S(M), E_S)$ for a manifold M and $S\ (\subset M)\cong\boldsymbol{P}_\xi^2$. $2H_\alpha-E_C=L_{\tilde{P}}$ for some $L\in Pic(M)$. Similarly as in (7.8), we see that (M, L) is a Del Pezzo manifold. Finally, using $L^3{}_{\tilde{D}}=0$, we infer $d(M, L)=L^5=L^3(\tilde{D}+H_\alpha)^2=L^3H_\alpha^2=H_\alpha^2(2H_\alpha-E_C)^3=8H_\alpha^5-H_\alpha^2C=5$.

(7.11) In § 11 we prove the following

THEOREM. *All the Del Pezzo 5-folds with $d=5$ are isomorphic to each other.*

(7.12) Now we outline a six dimensional version of (7.8). Let $P=\boldsymbol{P}_\alpha^6=\{(\alpha_0:\cdots:\alpha_6)\}$, let D be the hyperplane $\{\alpha_6=0\}$ and let $C\subset D$ be the subspace $\{\alpha_0\alpha_3=\alpha_1\alpha_2,\ \alpha_2\alpha_5=\alpha_3\alpha_4,\ \alpha_0\alpha_5=\alpha_1\alpha_4\}$. Then $(C, H_\alpha)\cong(\boldsymbol{P}_\sigma^1\times\boldsymbol{P}_\tau^2, H_\sigma+H_\tau)$. Let $\tilde{P}$ be the blowing up $Q_C(P)$ and let $\tilde{D}$ be the proper transform of D. Then $\alpha_0\alpha_3-\alpha_1\alpha_2$, $\alpha_2\alpha_5-\alpha_3\alpha_4$, $\alpha_0\alpha_5-\alpha_1\alpha_4$ define a linear system Λ on $\tilde{D}\cong Q_C(D)$ such that $[\Lambda]=2H_\alpha-E_C$ and $Bs\Lambda=\emptyset$. ρ_Λ makes $\tilde{D}$ a $\boldsymbol{P}^3$-bundle over $\boldsymbol{P}_\xi^2$. $\tilde{D}$ is blown down with respect to ρ_Λ and we have $(\tilde{P}, \tilde{D})\cong(Q_S(M), E_S)$ for $M\supset S\cong\boldsymbol{P}_\xi^2$. Moreover $2H_\alpha-E_C=L_P$ for some $L\in Pic(M)$ and (M, L) becomes a Del Pezzo 6-fold. Using $L^3{}_{\tilde{D}}=0$, we infer $d(M, L)=5$ as in (7.10).

(7.13) In § 12 we prove the following

THEOREM. *All the Del Pezzo 6-folds with $d=5$ are isomorphic to each other, hence isomorphic to $Gr(5, 2)$.*

(7.14) Recalling [**6**, (5.2)], we prove the following

COROLLARY. *There exists no Del Pezzo manifold (M, L) with* $\dim M\geqq7$ *and $d(M, L)=5$.*

(7.15) Combining (7.2), (7.6), (7.9), (7.11), (7.13) and (7.14) we get the Theorem stated in the introduction.

§ 8. Projective geometry on Del Pezzo manifolds.

Throughout this section M is an n-dimensional submanifold of $P \cong \boldsymbol{P}_{\xi}^{N}$, which is not contained in any hyperplane on P. Put $L=[H_{\xi}]_{M}$.

(8.1) PROPOSITION. *Let C be a linear submanifold of P lying on M. Then $Bs|L-E_C|=\emptyset$ on $M_1=Q_C(M)$.*

PROOF. Clearly $Bs|H_{\xi}-E_C|=\emptyset$ on $\tilde{P}=Q_C(P)$. Our assertion follows from this since M_1 is the proper transform of M on $\tilde{P}$.

(8.2) COROLLARY. *Let l be a line in P lying on M. Then $\deg F \geqq -1$ for any quotient line bundle F of the conormal bundle $\Omega_{l\backslash M}$ of l in M.*

PROOF. Let $\tilde{P}$ and M_1 be as above. Then $M_1 \cap E_l = \boldsymbol{P}(\Omega_{l\backslash M})$ and $[-E_l]_{E_l} = H^{\Omega_{l\backslash M}}$. $\Omega_{l\backslash M} \otimes [H_{\xi}]$ is semipositive since $Bs|L-E_l|=\emptyset$. Hence so is $F \otimes [H_{\xi}]$, which proves the assertion.

(8.3) PROPOSITION. *Suppose in addition that (M, L) is a Del Pezzo manifold with $d=d(M, L)$. Let W be the image of $\rho_{|L-E_l|}: M_1 \to \boldsymbol{P}_{\eta}^{\dim|L-E_l|}$. Then $d(W, H_{\eta})=d-3$ and $\Delta(W, H_{\eta})=0$ if $d \geqq 4$. In particular, W is a hyperquadric if $d=5$.*

PROOF. $(L-E_l)^n\{M_1\}=L^n+nL(-E_l)^{n-1}+(-E_l)^n=d-3$ since $L^n=d$, $L(-E_l)^{n-1}=-L\{l\}=-1$ and $(-E_l)^n=-(-E_l)^{n-1}\{E_l\}=-s_1(\Omega_{l\backslash M})=-K^M\cdot l+K^l\cdot l$ $=n-3$. Hence $\dim W=n$ if $d \geqq 4$. Put $w=\deg W$. Then $d-3=w\cdot\deg\rho$. On the other hand, $0 \leqq \Delta(W, H_{\eta}) \leqq n+w-(\dim|L-E_l|+1)=3+w-d$. Combining them we obtain $w=d-3$, $\deg\rho=1$ and $\Delta(W, H_{\eta})=0$.

(8.4) PROPOSITION. *Let T be a linear submanifold of P such that $\dim T+\dim(T\cap M)<\dim M$. Then any general member of $\Lambda=|H_{\xi}|_M^{*T}$ is smooth.*

PROOF. $\dim \Lambda=N-1-\dim T$ where $N=\dim P$. Note that $Bs\Lambda=T\cap M$ and that any general member of Λ is smooth in the outside of $T\cap M$. For any $x\in T\cap M$ let $\Lambda_x=\{D\in\Lambda \mid D \text{ is singular at } x\}$. Then $\dim\Lambda_x \leqq N-1-\dim M$. So $\dim\Lambda>\dim(\cup_{x\in T\cap M}\Lambda_x)$. Hence any general member of Λ is smooth at each $x\in T\cap M$, too.

(8.5) Let C be a smooth curve in P and let $x\in C\cap M$. We define the intersection multiplicity $\mu_x(C\cdot M)$ at x as follows.

Let P_1 be the blowing up $Q_x(P)$ and let M_1 and C_1 be the proper transforms of M and C. $C_1\cap E_x$ is a point x_2 on P_1. If $x_2 \notin M_1$, then we define $\mu_x(C\cdot M)=1$. If $x_2\in M_1$, then let $P_2=Q_{x_2}(P_1)$ and let M_2, C_2 be the proper transforms of M_1, C_1. $C_2\cap E_{x_2}$ is a point x_3. If $x_3\notin M_2$, then we define $\mu_x(C\cdot M)=2$. If $x_3\in M_2$, then we consider $P_3=Q_{x_3}(P_2)$ and make the similar procedure. Sooner or later we have $x_{k+1}\notin M_k$ unless $C\subset M$. Then we define $\mu_x(C\cdot M)=k$. If $C\subset M$, then $\mu_x(C\cdot M)=\infty$.

(8.6) THEOREM. *Let l be a line in P such that $l\not\subset M$. Put $\mu=\Sigma_{x\in l\cap M}\mu_x(l\cdot M)$. If $\mu\geqq 2\Delta(M, L)-d(M, L)+5$, then $g(M, L)\leqq\Delta(M, L)-\mu+2$ unless M is a hypersurface.*

PROOF. Over each $x \in l \cap M$ we blow up $\mu_x(l \cdot M)$-times successively in order to separate the proper transforms of l and M. Viewed on M, this is nothing other than the procedure of the elimination of base points of $|H_\xi|_M^{*l}$. Thus we obtain a manifold $\tilde{M}$ and a linear system Λ' on $\tilde{M}$ such that $\tilde{M}$ is a μ-times successive blowing up of M with center being a point at each step, namely, $\tilde{M}=Q_{p_\mu} \cdots Q_{p_1}(M)$, $Bs\Lambda'=\emptyset$, $\dim \Lambda'=\dim|H_\xi|^{*l}=N-2$ and $[\Lambda']=H_\xi-E_{p_1}-\cdots-E_{p_\mu}$. Then we have $d(\tilde{M}, \Lambda')=d(M, L)-\mu$, $g(\tilde{M}, \Lambda')=g(M, L)$ and $h^0(\tilde{M}, [\Lambda'])=h^0(M, L)-2$. Hence $\Delta(\tilde{M}, \Lambda')=\Delta(M, L)-\mu+2$. So $\mu \geqq 2\Delta(M, L)-d(M, L)+5$ implies that $d(\tilde{M}, \Lambda') \geqq 2\Delta(\tilde{M}, \Lambda')+1$. Therefore $g(\tilde{M}, \Lambda') \leqq \Delta(\tilde{M}, \Lambda')=\Delta(M, L)-\mu+2$ unless $d(\tilde{M}, \Lambda')=0$ (cf. [**3**, Theorem 4.1, c)]).

If $d(\tilde{M}, \Lambda')=0$, then $\dim W<n$ where $W=\rho_{\Lambda'}(M)$. On the other hand, $[\Lambda']^{n-1}\{E_{p_\mu}\}=1$. Therefore $\rho_{\Lambda'}(E_{p_\mu})$ is a linear subspace $V \cong \boldsymbol{P}^{n-1}$. Clearly $W \supset V$ and W is irreducible. Hence $W=V$. This implies $\dim \Lambda'=n-1$ and $\dim|L|=n+1$. So M is a hypersurface.

(8.7) COROLLARY. *Let l, M, μ be as in* (8.6). *Suppose in addition that M is a Del Pezzo manifold. Then $\mu \leqq 2$ unless M is a hypercubic.*

PROOF. $d(M, L) \geqq 4$ if M is not a hypercubic. So $\mu \geqq 3$ would imply $g(M, L) \leqq 1-\mu+2 \leqq 0$ by (8.6).

§9. Del Pezzo 3-folds with $d=5$.

Throughout this section let (M, L) be a Del Pezzo threefold with $d(M, L)=5$. By $\rho_{|L|}$ we regard M to be a submanifold in $P \cong \boldsymbol{P}_\xi^6$. Note that $[H_\xi]_M=L$.

(9.1) LEMMA. *$Pic(M)$ is generated by L.*

PROOF. Let S be a general member of $|L|$ and let λ be the monodromy action on $\mathfrak{L}(S)$ induced by $|L|$ (cf. [**5**; (4.6), (4.23)]). Let Γ be the envelope of $\mathrm{Im}(\lambda)$. Then $F \in Pic(S)$ comes from $Pic(M)$ if and only if $x \cdot c_1(F)=0$ for any $x \in \Gamma$. So it suffices to show that $\mathfrak{R}(S)=\Gamma$. Note that $\mathfrak{L}(S) \cong \Lambda_4$ (cf. [**5**, (4.11)]) and let $\{h, e_1, \cdots, e_4\}$ be a normal base of $\mathfrak{L}(S)$ as in (4.12).

Claim a). *No exceptional cycle comes from $Pic(M)$.*

Indeed, if $c_1(F_S)$ is an exceptional cycle for $F \in Pic(M)$, then $(M, F) \cong (Q_p(N), E_p)$ for a manifold N and $p \in N$ by (5.5). Moreover $L+E_p=L'_M$ for $L' \in Pic(N)$ and (N, L') is a Del Pezzo three-fold with $d(N, L')=6$. But then there exists a line on N which passes p by (5.16). Then $L \cdot \tilde{l}=0$ for the proper transform $\tilde{l}$ of this line. Hence L cannot be ample. This contradiction proves our claim.

Claim b). *h does not come from $Pic(M)$.*

Assume that $h=c_1(H_S)$ for some $H \in Pic(M)$. $c_1(L-H)_S=2h-e_1-e_2-e_3-e_4$ and $\rho_{|L-H|}$ makes S a $\boldsymbol{P}^1$-ruled surface over $\boldsymbol{P}^1$. By [**4**, (2.8)] we obtain a holomorphic mapping $\rho_{|L-H|}: M \to \boldsymbol{P}^1$. On the other hand $|H|_S=|H_S|$ by [**4**, (2.3)]. Hence $Bs|H|\cap S=Bs|H_S|=\emptyset$. So $Bs|H|$ is a finite set and hence H is semipositive. However, $H^3\{M\}=H^3+(L-H)^3=L^3-3L^2H+3LH^2=(L^2-3LH+3H^2)\{S\}=5-9+3<0$. This contradiction proves our claim.

Claim c). *$h-e_i$ does not come from $Pic(M)$ for $i=1, \cdots, 4$.*

Assume that $h-e_4=c_1(F_S)$ for some $F\in Pic(M)$. Then $c_1(L-F)_S=2h-e_1-e_2-e_3$ and $\rho_{|L-F|}$ makes S a blowing up of $\boldsymbol{P}^2$ with center being four points. Therefore we can derive a contradiction quite similarly as in Claim b).

$\mathfrak{R}(S)\subset\Gamma$ follows from the above three claims. To see this, recall that $\mathfrak{R}(S)=\{\pm(h-e_i-e_j-e_k)\}_{i<j<k}\cup\{(e_i-e_j)\}_{i\neq j}$. By claim b), $xh\neq 0$ for some $x\in\Gamma\cap\mathfrak{R}(S)$. So we may assume $h-e_1-e_2-e_3\in\Gamma$ by changing the numbering if necessary. Note that $\alpha\in\Gamma$ if and only if $\alpha+(h-e_1-e_2-e_3)\in\Gamma$. By claim a), $ye_4\neq 0$ for some $y\in\Gamma\cap\mathfrak{R}(S)$. So $e_i-e_4\in\Gamma$ for some $i=1, 2, 3$. We may assume $i=3$. Again by claim a), $z(h-e_3-e_4)\neq 0$ for some $z\in\Gamma\cap\mathfrak{R}(S)$. So $e_j-e_k\in\Gamma$ for some $j=1, 2$ and $k=3, 4$. We may assume $e_2-e_3\in\Gamma$ without loss of generality. Hence $e_m-e_n\in\Gamma$ if $m\neq 1\neq n$. By claim c), $(h-e_1)u\neq 0$ for some $u\in\Gamma\cap\mathfrak{R}(S)$. So $e_1-e_n\in\Gamma$ for some $n=2, 3, 4$. Now it is easy to see $\mathfrak{R}(S)\subset\Gamma$.

Only the integral multiples of $c_1(L)$ are orthogonal to all the roots of $\mathfrak{L}(S)$. So $F\in Pic(S)$ comes from $Pic(M)$ only when $F=mL$ for some $m\in\boldsymbol{Z}$. This proves the Lemma.

(9.2) M contains many lines. Indeed, any general member of $|L|$ is a Del Pezzo surface and each exceptional curve on it is a line in P (cf. (5.3)).

(9.3) Let l be a line on M. Then $Bs|L-E_l|=\emptyset$ on $M_1=Q_l(M)$ and $\rho_\alpha=\rho_{|L-E_l|}$ is a birational morphism onto a hyperquadric Q in $\boldsymbol{P}^4_\alpha$ (cf. (8.3)).

(9.4) $(L-E_l)^2L=L^3-L\cdot l=4$ and $(L-E_l)^2E_l=2$. On the other hand, $h^0(E_l, L-E_l)=h^0(l, \Omega_{l\backslash M}\otimes[L])=4$. Hence $\rho_\alpha(E_l)$ is a hyperquadric and is a hyperplane section of Q.

(9.5) LEMMA. *$|aL-bE_l|\neq\emptyset$ only if $a\geqq 0$ and $2a\geqq b$.*

PROOF. Suppose $|aL-bE_l|\neq\emptyset$. Then $0\leqq L^2(aL-bE_l)=5a$. Similarly $0\leqq(L-E_l)^2(aL-bE_l)=4a-2b$ since $L-E_l$ is semipositive.

(9.6) Let H be the hyperplane on $\boldsymbol{P}^4_\alpha$ such that $H\cap Q=\rho_\alpha(E_l)$ (see (9.4)). Then $\rho_\alpha^*H=E_l+R$ for some $R\in|L-2E_l|$. By (9.1), $Pic(M_1)$ is generated by L and E_l. Using (9.5), we infer that R is irreducible and reduced.

(9.7) Put $C=\rho_\alpha(R)$. $(L-E_l)^2\{R\}=(L-E_l)^2(L-2E_l)=0$ implies $\dim C<2$. On the other hand, C is not contained in any hyperplane of $\boldsymbol{P}^4_\alpha$ other than H, because otherwise $2\leqq\dim\mathrm{Ker}(H^0(M_1, L-E_l)\to H^0(R, L-E_l))=h^0(M_1, E_l)=1$. So C is a curve and $h^0(C, H_\alpha)\geqq 4$.

Put $w=\deg C=H_\alpha C$. Let X be a general fiber of $R\to C$. Then $wLX=L(L-E_l)\{R\}=L(L-E_l)(L-2E_l)=3$. Clearly $w>1$ since $h^0(C, H_\alpha)\geqq 4$. So $w=3$ and $LX=1$. Hence C is a Veronese curve of degree three in $H\cong\boldsymbol{P}^3$.

For any fiber Y of $R\to C$, L_Y is ample since $L_Y=[E_l]_Y$ and $tL-E_l$ is ample on M_1 for $t \gg 0$. Hence $LY=LX=1$ implies that $Y \cong \boldsymbol{P}^1$. Thus R is a $\boldsymbol{P}^1$-bundle over C. Moreover, $[R]_Y=-LY=-1$ implies that R can be blown down with respect to this bundle structure.

(9.8) Let W be a manifold containing $C \cong \boldsymbol{P}^1$ such that $(M_1, R) \cong (Q_C(W), E_C)$. Then $L-E_l=A_{M_1}$ for some $A \in Pic(W)$. A is ample on C and $3A-E_C=L+(L-E_l)$ is ample on M_1. Hence A is ample on W by [**4**, (5.7)]. Moreover $K^W=K^{M_1}-E_C=-2L+E_l-(L-2E_l)=-3A$. Therefore (W, A) is a hyperquadric by [**2**, Theorem 2.2]. So W is naturally isomorphic to $Q \subset \boldsymbol{P}^4_\alpha$.

(9.9) Now we consider how is $\rho_\alpha(E_l)=H \cap Q$. We have $\det N^{l \backslash M}=0$ since $K^M=-2H_\xi$. So $N^{l \backslash M}=[H_\xi] \oplus [-H_\xi]$ or $[0] \oplus [0]$ by (8.2). l is said to be of special type (resp. non-special type) in the former (resp. latter) case.

(9.10) If l is of special type, then $E_l \cong \Sigma_2$ and we see easily that $\rho_\alpha(E_l)$ is a cone over a plane curve of degree two. Hence, in this case, the above procedure from M to $W=Q$ via M_1 is the inverse of the construction (7.5).

If l is of non-special type, then $E_l \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$ and $\rho_\alpha(E_l)$ is a smooth hyperquadric. In this case the procedure (9.7) and (9.8) is the inverse of (7.4).

(9.11) Now we have the following

PROPOSITION. *Let l be a line on a Del Pezzo* 3-*fold* (M, L) *with* $d(M, L)=5$. *Then* $Bs|L-E_l|=\emptyset$ *on* $M_1=Q_l(M)$ *and* $W=\rho_{|L-E_l|}(M_1)$ *is a smooth hyperquadric in* $\boldsymbol{P}^4$. $\rho_{|L-E_l|}(E_l)$ *is a hyperplane section of* W *and this is smooth if and only if* l *is of non-special type. In any case* M_1 *is the blowing up of* W *with center being a Veronese curve of degree three.*

(9.12) On any Del Pezzo 3-fold with $d=5$, there exists a line of non-special type (Iskovskih). Using this, he proved the following

THEOREM. *Every Del Pezzo threefold with* $d=5$ *is isomorphic to each other and is constructed as in* (7.4).

For a proof, see [**7**].

(9.13) COROLLARY. *Let* (M, L) *be a Del Pezzo threefold with* $d=5$. *Then* $H^1(M, T^M)=0$.

PROOF. $h^2(M, T^M)=h^1(M, \Omega^1_M \otimes [K^M])=0$ since $-K^M$ is ample (see [**0**] or [**11**]). So any infinitesimal deformation of M is not obstructed. On the other hand, by [**9**], one can easily see that any small deformation of a Del Pezzo manifold is also a Del Pezzo manifold. Combining them we obtain $h^1(M, T^M)=0$ from (9.12).

One can prove this by an elementary (=without deformation theory) method based on (P. 13), too. Details are left to the reader.

§ 10. Del Pezzo 4-folds with $d=5$.

Let (M, L) be a Del Pezzo 4-fold with $d(M, L)=5$. We regard M to be a submanifold in $\boldsymbol{P}_{\xi}^{7}$ by $\rho_{|L|}$ as before.

(10.1) LEMMA. *M contains a plane.*

We divide the proof into several steps. Assuming to the contrary, we derive a contradiction in (10.8).

(10.2) Clearly M contains a line. Let l be a line on M. Then $Bs|L-E_l|=\emptyset$ on $M_1=Q_l(M)$ and $\rho_{\eta}=\rho_{|L-E_l|}$ is a birational morphism onto a hyperquadric W in $\boldsymbol{P}_{\eta}^{5}$. In view of $(L-E_l)^3E_l=2$ we infer that $\rho_{\eta}(E_l)$ is a hyperplane section of W as in (9.4).

(10.3) Let H be the hyperplane such that $H\cap W=\rho_{\eta}(E_l)$. Then $\rho_{\eta}^{*}H=E_l+R$ for some $R\in|L-2E_l|$. R is irreducible and reduced as in (9.6). Put $C=\rho_{\eta}(R)$.

(10.4) We claim that W is smooth. Indeed, let x be any point on $W\subset\boldsymbol{P}_{\eta}^{5}$. This defines a plane S in $\boldsymbol{P}_{\xi}^{7}$ containing l in a natural manner so that $|H_{\xi}|_{M}^{*S}$ corresponds $|H_{\eta}|^{*x}$. A general member T of $|H_{\xi}|_{M}^{*S}$ is smooth by (8.4) since we assume that M does not contain any plane. T is a Del Pezzo threefold and $l\subset T\subset M$. Let T_1 be the proper transform of T on M_1. Then $\rho_{\eta}(T_1)=Q$ is a hyperplane section of W such that $x\in Q$. In view of (9.11) we infer that Q is smooth. This implies that W is smooth at x.

(10.5) We claim that C is a smooth surface with $\Delta(C, H_{\eta})=0$, $d(H, H_{\eta})=3$.

Let x be any point on $C\subset\boldsymbol{P}_{\eta}^{5}$. Let S be the plane in $\boldsymbol{P}_{\xi}^{7}$ corresponding x and let T be a smooth member of $|H_{\xi}|_{M}^{*S}$. Then $\rho_{\eta}(T_1)\ni x$ where T_1 is the proper transform of T on M_1. In view of (9.11) we infer that $C\cap\rho_{\eta}(T_1)=\rho_{\eta}(R\cap T_1)$ is a Veronese curve of degree three. So C is smooth at x. Thus C is shown to be smooth. $d(C, H_{\eta})=3$ and $\Delta(C, H_{\eta})=0$ is now easy to prove.

(10.6) We claim that R is a $\boldsymbol{P}^1$-bundle over C. This follows from a similar argument as before since $R\cap T_1\to C\cap\rho_{\eta}(T_1)$ is a $\boldsymbol{P}^1$-bundle for any T_1 with $T\in|L|$ being smooth.

(10.7) M_1 is the blowing up of W with center C.

This is an easy consequence of the above observations and (P. 14). This procedure is the inverse of (7.7).

(10.8) Combining (10.5) and [2, Theorem 3.8], we infer that $(C, H_{\eta})\cong(\boldsymbol{P}(E), H^E)$ for the vector bundle $E=[H_{\sigma}]\oplus[2H_{\sigma}]$ on $\boldsymbol{P}_{\sigma}^{1}$. So C contains a smooth rational curve F such that $F^2=-1$ and $H_{\eta}F=1$. $0\to\Omega_{C\backslash W}\to\Omega_{F\backslash W}\to\Omega_{F\backslash C}\to 0$ is exact on F. Hence $c_1(\Omega_{C\backslash W})_F=c_1(\Omega_{F\backslash W})+c_1(N^{F\backslash C})=(K_F^W-K^F)+F^2=-3$. Therefore $L^2\{\rho_{\eta}^{-1}(F)\}=(2H_{\eta}-E_C)^2\{\rho_{\eta}^{-1}(F)\}=4H_{\eta}F+s_1(\Omega_{C\backslash W})_F=1$. This implies that the image of $\rho_{\eta}^{-1}(F)$ in M is a plane. Thus we have proved (10.1).

(10.9) Let S be a plane in $P=\boldsymbol{P}_{\xi}^{7}$ lying on M. Then $Bs|L-E_S|=\emptyset$ on $M_1=Q_S(M)$ by (8.1). So we have a morphism $\rho_{\alpha}:=\rho_{|L-E_S|}: M_1\to\boldsymbol{P}_{\alpha}^{4}$. We say that S is of vertex type (resp. non-vertex type) if ρ_α is not surjective (resp. surjective).

(10.10) For a while, till (10.17), suppose S to be a plane of non-vertex type. Namely $\rho_\alpha: M_1=Q_S(M)\to \boldsymbol{P}_\alpha^4$ is surjective.

$h^p(M_1, -2E_S)=h^{4-p}(M_1, K^{M_1}+2E_S)=h^{4-p}(M_1, -3L+3E_S)=0$ for any p by Ramanujam's vanishing theorem. Similarly we have $H^p(M_1, -E_S)=0$. Hence $H^p(E_S, -E_S)=0$ since $H^p(M_1, -E_S)\to H^p(E_S, -E_S)\to H^{p+1}(M_1, -2E_S)$ is exact. So $0=\chi(E_S, -E_S)=\chi(S, \Omega_{S\backslash M})=2\chi(S, \mathcal{O})+2^{-1}c_1(\Omega_{S\backslash M})c_1(S)+2^{-1}(c_1(\Omega_{S\backslash M})^2-2c_2(\Omega_{S\backslash M}))=2-c_2(\Omega_{S\backslash M})$ since $c_1(\Omega_{S\backslash M})=c_1(\Omega_M)-c_1(\Omega_S)=0$. Therefore $c_2(N^{S\backslash M})=c_2(\Omega_{S\backslash M})=2$.

(10.11) We have $(L-E_S)^4\{M_1\}=L^4-4L^3E_S+6L^2E_S^2-4LE_S^3+E_S^4=1$ since $L^4=5$, $L^3E_S=0$, $L^2E_S^2=-L^2S=-1$, $LE_S^3=L(-E_S)^2\{E_S\}=L\cdot s_1(\Omega_{S\backslash M})=0$ and $E_S^4=-(-E_S)^3\{E_S\}=-s_2(\Omega_{S\backslash M})=c_2-c_1^2=2$. Therefore ρ_α is a birational morphism.

(10.12) We have $(L-E_S)^3L=2$ and $(L-E_S)^3E_S=1$ as in (10.11). This implies that $H=\rho_\alpha(E_S)$ is a hyperplane on $\boldsymbol{P}_\alpha^4$. So we have $\rho_\alpha^*H=E_S+R$ for $R\in|L-2E_S|$.

(10.13) Using Lefschetz Theorem and (9.1), we infer that $Pic(M)$ is generated by L. So $Pic(M_1)$ is generated by L and E_S. On the other hand, $|aL-bE_S|=\emptyset$ unless $a\geqq 0$ and $2a\geqq b$ because $L-E_S$ is semipositive (compare (9.5)). Hence R is irreducible and reduced as in (9.6).

(10.14) Let $C=\rho_\alpha(R)$. C is not contained in any hyperplane of $\boldsymbol{P}_\alpha^4$ other than H. Indeed, if otherwise, $2\leqq\dim\operatorname{Ker}(H^0(M_1, H_\alpha)\to H^0(R, H_\alpha))=h^0(M_1, E_S)$. So $h^0(C, H_\alpha)\geqq 4$.

$H_\alpha^2L\{R\}=(L-E_S)^2L(L-2E_S)=0$ proves that $\dim C<2$. So C is an irreducible curve. Let X be a general fiber of $R\to C$ and put $w=\deg C$. Then $wL^2X=H_\alpha L^2\{R\}=(L-E_S)L^2(L-2E_S)=3$. So $w=3$ since C is not a line. Thus we infer that C is a Veronese curve of degree three in $H\cong\boldsymbol{P}^3$. Moreover $L^2X=1$.

(10.15) For any fiber Y of $R\to C$ we have $L^2Y=L^2X=1$. Moreover L_Y is ample since $L_Y=[E_S]_Y$ and $tL-E_S$ is ample on M_1 for $t\gg 0$. Therefore we infer $Y\cong\boldsymbol{P}^2$. Thus R is a $\boldsymbol{P}^2$-bundle over C.

(10.16) $[R]_Y=[L-2E_S]_Y=-L_Y$ implies that R can be blown down to C by (P. 14). It is easy to see that M_1 is the blowing up of $\boldsymbol{P}_\alpha^4$ with center C. The above procedure is the inverse of (7.8). Thus we have proved the following

(10.17) LEMMA. *If S is a plane of non-vertex type on M, then $Q_S(M)\cong Q_C(\boldsymbol{P}_\alpha^4)$ and M is of type* (7.8).

(10.18) In order to know whether a plane lying on M is of vertex type or not, we have the following

LEMMA. *Let S be a plane lying on M and let l be a line in S. Let $M'=Q_l(M)$ and $W=\rho_{|L-E_l|}(M')\subset\boldsymbol{P}_\eta^5$ (note that W is a hyperquadric by* (8.3)). *Let S' be the proper transform of S on M'. Then $X=\rho_{|L-E_l|}(S')$ is a point on W.*

Moreover, S is of vertex type if and only if x is a vertex of W.

PROOF. Obviously $\rho_{|L-E_l|}(S')$ is a point since $S'\cong S$ and $[L-E_l]_{S'}=0$. Let π be the projection $\boldsymbol{P}_\eta^5 \to$ {*lines passing through* x} $\cong \boldsymbol{P}_\alpha^4$. Then $\pi\circ\rho_{|L-E_l|}=\rho_{|L-E_S|}$ as rational mappings from M. So, $\rho_{|L-E_S|}$ is degenerate if and only if π_W is so, and π_W is degenerate if and only if x is a vertex of W.

(10.19) COROLLARY. *Let S_1 and S_2 be planes on M such that $S_1\cap S_2=l$ is a line. Then either S_1 or S_2 is of non-vertex type.*

PROOF. Let $M'=Q_l(M)$ and $W=\rho_{|L-E_l|}(M')$. By (8.4) there is a smooth member T of $|L|$ such that $l\subset T$. Let T' be the proper transform of T on M'. Then $\rho_{|L-E_l|}(T')$ is a hyperplane section of W and is a smooth hyperquadric by (9.11). Hence W is a hyperquadric with at most finite singular points. So W can have at most only one vertex. Therefore our assertion follows from (10.18).

(10.20) For a while, till (10.25), let S be a plane on M of vertex type. Then $0=(L-E_S)^4\{M_1\}$ for $M_1=Q_S(M)$. $L^4=5$, $L^3E_S=0$, $L^2E_S^2=-1$ and $LE_S^3=0$ as in (10.11). So we have $1=E_S^4=c_2(\Omega_{S\backslash M})$. Combining this with (10.10) we obtain the following

LEMMA. *S is of vertex type if and only if $c_2(\Omega_{S\backslash M})=1$, and S is of non-vertex type if and only if $c_2(\Omega_{S\backslash M})=2$.*

(10.21) *Again S is a plane of vertex type and let $V=\rho_{|L-E_S|}(M_1)\subset \boldsymbol{P}_\alpha^4$. Then V is a smooth hyperquadric.*

PROOF. Let T be a general member of $|L|$ on M and let T_1 be the proper transform of T on M_1. Then $T_1\cong Q_l(T)$ where l is the line $S\cap T$. $W=\rho_{|L-E_S|}(T_1)$ is a smooth hyperquadric in $\boldsymbol{P}_\alpha^4$ by (9.11). Clearly $V\supset W$, $\dim V<4$ and V is irreducible. So $V=W$.

(10.22) *There exists a fiber Y of $\rho_{|L-E_S|}: M_1\to V$ such that* $\dim Y\geqq 2$.

PROOF. If otherwise, $\dim Y=1$ for every fiber Y. $2LY=LH_\alpha^3\{M_1\}=L(L-E_S)^3=2$ implies $LY=1$. L_Y is ample since $tL-E_S$ is ample on M_1 for $t\gg 0$. Therefore we infer that $Y\cong \boldsymbol{P}^1$. So M_1 is a $\boldsymbol{P}^1$-bundle over V. Hence $c_4(M_1)=c_1(\boldsymbol{P}^1)c_3(V)=8$. So $c_4(M)=c_4(M_1)-c_3(E_S)+c_2(S)=5$. This contradicts the following

(10.23) LEMMA. *$b_j(M)=1$ for $j=0, 2, 6, 8$ and $b_4(M)\geqq 2$ and $b_j(M)=0$ for other j.*

PROOF. Let T be a smooth member of $|L|$. T is a Del Pezzo 3-fold with $d=5$. So by §9 we have $b_i(T)=1$ for $i=0, 2, 4, 6$ and $b_i(T)=0$ for other i. By Lefschetz Theorem $H_p(T)\to H_p(M)$ is surjective for $p\leqq 3$. So $b_0(M)=b_2(M)=1$, $b_1(M)=b_3(M)=0$. By duality we obtain $b_6(M)=b_8(M)=1$ and $b_5(M)=b_7(M)=0$. Assume that $b_4(M)=1$. Then the free part of $H^4(M;\boldsymbol{Z})$ is isomorphic to $\boldsymbol{Z}$. Let γ be the integral base of it such that $(c_1(L))^2=k\gamma$ for a positive integer k. By the Poincare duality we infer that $\gamma^2=\pm 1$ since $H^4(M;\boldsymbol{Z})$ is self-dual by the intersection pairing. But then $5=L^4=(k\gamma)^2=\pm k^2$. This is absurd. Thus we infer $b_4(M)\geqq 2$.

(10.24) Let Y be a fiber of $\rho_{|L-E_S|}: M_1\to V$ over $y\in V$ such that $\dim Y\geqq 2$. y corresponds to a three-dimensional linear subspace B in $\boldsymbol{P}^7_\xi$ such that $S\subset B$. Note that $B\not\subset M$ since $Pic(M)$ is generated by L. Let B_1 be the proper transform of B in $Q_S(\boldsymbol{P}^7_\xi)$. Then $B_1\cong B\cong \boldsymbol{P}^3$ and $Y=B_1\cap M_1$. By (8.7) we have $\mu(l\cdot M)\leqq 2$ for any line l in B with $l\not\subset M$. From this we infer that Y is a plane in B_1, because $\mu(l\cdot M)\geqq l_1\cdot Y+1$ where l_1 is the proper transform of l and $l_1\cdot Y$ is the intersection number in $B_1\cong \boldsymbol{P}^3$. So Y maps onto a plane on M.

(10.25) Assume that Y maps onto S. Then $Y=E_S\cap B_1$ and this defines a section of $E_S\to S$. This corresponds a sub-bundle of $N^{S\backslash M}$ of rank one. This gives a splitting of $N^{S\backslash M}$ into line bundles since $S\cong \boldsymbol{P}^2$. But this is impossible since $c_1(\Omega_{S\backslash M})=0$ and $c_2(\Omega_{S\backslash M})=1$ by (10.20). Thus we conclude that $Y\not\subset E_S$ and $Y\cap E_S$ is a line in B_1. Putting things together we get the following

LEMMA. *Y maps onto a plane Y' on M such that $Y'\cap S$ is a line.*

(10.26) Now we can prove the following

THEOREM. *Any Del Pezzo 4-fold (M, L) with $d(M, L)=5$ can be obtained as in (7.8).*

PROOF. By (10.1) M contains a plane S. If S is of non-vertex type, then (10.17) applies. If S is of vertex type, then by (10.25), there is a plane Y' such that $Y'\cap S$ is a line. By (10.19), Y' is of non-vertex type. Hence (10.17) applies.

(10.27) COROLLARY. *All the Del Pezzo 4-folds with $d(M, L)=5$ are isomorphic to each other.*

PROOF. All the pairs (C, D) each of which consists of a hyperplane D in $\boldsymbol{P}^4$ and of a Veronese curve C of degree three in D are projectively equivalent to each other. Hence all the Del Pezzo 4-folds constructed as in (7.8) are isomorphic to each other.

(10.28) COROLLARY. *$H^1(M, T^M)=0$ for any Del Pezzo 4-fold (M, L) with $d(M, L)=5$.*

Proof is similar to that of (9.13).

(10.29) Let S be a plane of non-vertex type. Then $Q_S(M)=M_1\cong Q_C(\boldsymbol{P}^4)$ where C is a Veronese curve of degree three. Each fiber of $E_C\to C$ maps onto a plane on M. They are easily seen to be of vertex type.

On the other hand, one can see that any plane on M other than S is of the above type. In particular, S is the unique plane of non-vertex type on M. We can also show that $H^0(N^{S\backslash M})=0$ and that S has no infinitesimal non-trivial displacement.

Proofs of the above facts are omitted since we don't use them in the following sections.

§ 11. Del Pezzo 5-folds with $d=5$.

Let (M, L) be a Del Pezzo 5-fold with $d(M, L)=5$. We regard M to be a submanifold of $\boldsymbol{P}^8_\xi$ by $\rho_{|L|}$ as before.

(11.1) LEMMA. *Let Γ be a smooth member of $|L|$ and let S be a plane on Γ. Then S is of non-vertex type on Γ if and only if $c_2(N^{S\setminus M})=2$.*

This follows from Lemma (10.20).

(11.2) Suppose, for a while, S to be a plane of the above type. Let $M_1=Q_S(M)$. Then $\rho_\alpha=\rho_{|L-E_S|}: M_1\to\boldsymbol{P}^5_\alpha$ is a birational morphism.

Indeed, we have $(L-E_S)^5\{M_1\}=(L-E_S)^4\{\Gamma_1\}=1$ for the proper transform Γ_1 of Γ, where Γ is a smooth member of $|L|^{*S}$.

(11.3) We can easily see $L(L-E_S)^4\{M_1\}=2$. So $(L-E_S)^4E_S=1$. This implies that $H=\rho_\alpha(E_S)$ is a hyperplane on $\boldsymbol{P}^5_\alpha$. Hence $\rho_\alpha^*H=E_S+R$ for some $R\in|L-2E_S|$. R is shown to be irreducible and reduced similarly as in (10.13).

(11.4) Let $W=\rho_\alpha(R)$. $LH^3_\alpha\{R\}=L(L-E_S)^3(L-2E_S)=L^5-5L^4E_S+9L^3E_S^2-7L^2E_S^3+2LE_S^4=0$ since $L^5=5$, $L^4E_S=L^3E_S=0$, $L^2E_S^3=L^2S=1$ and $LE_S^4=-L(-E_S)^3\{E_S\}=-L\cdot s_1(\Omega_{S\setminus M})=1$. This implies $\dim W<3$. On the other hand, $L^2H^2_\alpha\{R\}=L^2(L-E_S)^2(L-2E_S)=3$. So we infer that $\dim W=2$.

(11.5) W cannot be contained in any hyperplane in $\boldsymbol{P}^5_\alpha$ other than H, because otherwise $2\leqq\dim\operatorname{Ker}(\Gamma(M_1, H_\alpha)\to\Gamma(R, H_\alpha))=h^0(M_1, E_S)$. Therefore $h^0(W, H_\alpha)\geqq5$.

(11.6) Put $w=\deg W$ and let X be a general fiber of $R\to W$. Then $3=L^2H^2_\alpha\{R\}=wL^2X$. $w>1$ by (11.5). So $w=3$ and $L^2X=1$. From this we infer $\Delta(W, H_\alpha)=0$.

(11.7) Now we come to the crucial step of showing W to be smooth. First we prove the following

LEMMA. *There exists no linear threefold T such that $S\subset T\subset M$.*

Assume that such a threefold T exists. Let M' be the blowing up of M with center T and let $\rho_\eta=\rho_{|L-E_T|}: M'\to\boldsymbol{P}^4_\eta$ be the morphism as in (8.1). We claim that this is surjective.

To see this, take a smooth member Γ of $|L|$ such that $S=T\cap\Gamma$. Let Γ' be the proper transform of Γ on M'. Then $\Gamma'\cong Q_S(\Gamma)$ since $T\cap\Gamma=S$. S is of non-vertex type on Γ by (11.1). Hence $\rho_\eta(\Gamma')=\boldsymbol{P}^4_\eta$. So of course $\rho_\eta(M')=\boldsymbol{P}^4_\eta$. Thus ρ_η is surjective.

Let X be a general fiber of ρ_η. Then $LX=L(L-E_T)^4\{M'\}=(L-E_S)^4\{\Gamma'\}=1$. So $X\cong\boldsymbol{P}^1$ since $Bs|L|=\emptyset$. On the other hand, $\deg K^X=K^{M'}\cdot X=(-4L+E_T)X=-3LX=-3$. This absurdity proves the non-existence of T.

(11.8) In order to show that W is smooth, it suffices to show that W is not a cone, since $\Delta(W, H_\alpha)=0$ (cf. [2, Corollary 4.7]). So assume to the contrary and let x be a vertex of W. Let T be the linear threefold containing S corresponding

x. In particular, $\pi_x \circ \rho_{|L-E_S|} = \rho_\Lambda$ as rational mappings on M, where $\Lambda = |H_\xi|^{*T}$ and π_x is the projection $\boldsymbol{P}^5_\alpha \to \{\text{lines passing through } x\} \cong \boldsymbol{P}^4$.

(11.9) $T \not\subset M$ by (11.7). Let T_1 be the proper transform of T in $P_1 = Q_S(\boldsymbol{P}^8_\xi)$. Then we have $T_1 \cap M_1 = \rho_\alpha^{-1}(x)$, which we denote by Y.

(11.10) Y is a plane on $T_1 \cong T \cong \boldsymbol{P}^3$. Indeed, for any line l on T with $l \not\subset M$, we have $2 \geqq \mu(l \cdot M) \geqq 1 + l_1 \cdot Y$ by (8.7), where l_1 is the proper transform of l in T_1. This implies $\deg Y = 1$. $\dim Y = 2$ is clear since Y is a fiber of $R \to W$.

(11.11) Let $P_2 = Q_Y(P_1)$ be the blowing up of P_1 with center Y and let M_2, T_2 be the proper transforms of M_1, T_1. Then $T_2 \cap M_2 = \emptyset$. Indeed, if $T_2 \cap M_2 \ni u$, we can find a line l on T such that $l \not\subset M$ and $l_2 \ni u$ for the proper transform l_2 of l in P_2. Then $\mu(l \cdot M) \geqq 2 + \mu(l_2 \cdot M_2) > 2$. This contradicts (8.7).

(11.12) $Bs|L - E_S - E_Y| = \emptyset$ *on* M_2.

Indeed, $\Lambda = |H_\xi|^{*T}$ corresponds $|L - E_S - E_Y|$ on P_2 and $Bs|L - E_S - E_Y| = T_2$. So this assertion follows from $T_2 \cap M_2 = \emptyset$.

(11.13) Let $\rho_\beta = \rho_{|L-E_S-E_Y|} : M_2 \to \boldsymbol{P}^4_\beta$ be the morphism. In view of $(L - E_S)_Y = 0$, we infer $0 = H^5_\beta = (L - E_S - E_Y)^5 = (L - E_S)^5 - E^5_Y$. So $E^5_Y = (L - E_S)^5 = 1$.

(11.14) Let R_2 be the proper transform of R on M_2. Then $[R_2] = L - 2E_S - \delta E_Y$ with $\delta > 0$ since $Y \subset R$.

(11.15) $s_1(\Omega_{Y \setminus M_1}) = K^{M_1}_Y - K^Y = (-4L + 2E_S) - (-3L) = L_Y$. Hence $LE^4_Y = -L(-E_Y)^3\{E_Y\} = -1$. Using $(L - E_S)_Y = 0$, we obtain $(L - 2E_S)(L - E_S - E_Y)^4 = (L - 2E_S)(L - E_S)^4 - LE^4_Y = 1$. We have also $(L - E_S - E_Y)^4 E_Y = E^5_Y = 1$ by (11.13). Therefore $H^4_\beta\{R_2\} = (L - E_S - E_Y)^4(L - 2E_S - \delta E_Y) = 1 - \delta$. So $\delta = 1$ since H_β is semipositive.

(11.16) REMARK. We have not yet used the assumption that x is a vertex of W.

(11.17) $\pi_x \circ \rho_{|L-E_S|} = \rho_\Lambda = \rho_{|L-E_S-E_Y|}$ as rational mappings on M (cf. (11.8)). Since x is a vertex of W and $\pi_x(W)$ is a curve, we infer that $\rho_\beta(R_2)$ is a curve in $\boldsymbol{P}^4_\beta$. However we have $L^2 H^2_\beta\{R_2\} = L^2(L - E_S - E_Y)^2(L - 2E_S - E_Y) = L^2(L - E_S)^2 (L - 2E_S) + L^2(-E_Y)^3 = 2 > 0$. This contradiction proves the smoothness of W.

(11.18) By (11.16) and (11.10) we infer that R is a $\boldsymbol{P}^2$-bundle over W. $[R]_Y = [L - 2E_S]_Y = -L_Y$ for any fiber Y of $R \to W$. So R can be blown down with respect to this structure. Now it is easy to see that ρ_α gives an isomorphism $M_1 \cong Q_W(\boldsymbol{P}^5_\alpha)$.

(11.19) Clearly the above procedure from M to $\boldsymbol{P}^5_\alpha$ via M_1 is the inverse of (7.10). Thus we prove the following

THEOREM. *Let* (M, L) *be any Del Pezzo* 5-*fold with* $d(M, L) = 5$. *Then* $M, L)$ *can be constructed as in* (7.10).

(11.20) COROLLARY. *All the Del Pezzo* 5-*folds with* $d = 5$ *are isomorphic to each other.*

Proof is similar to that of (10.27). Note that any polarized smooth surface

(W, H_α) with $\varDelta(W, H_\alpha)=0$, $d(W, H_\alpha)=3$ is isomorphic to $(\boldsymbol{P}(E), H^E)$, where E is the vector bundle $[H_\sigma]\oplus[2H_\sigma]$ on $\boldsymbol{P}_\sigma^1$ (cf. [2, Theorem 3.8]).

(11.21) COROLLARY. $H^1(M, T^M)=0$.

Proof is similar to that of (9.13). Using (P. 13), one can prove this without deformation theory, too.

§ 12. Del Pezzo 6-folds with $d=5$.

Let (M, L) be a Del Pezzo 6-fold with $d(M, L)=5$. We regard M as a submanifold in $\boldsymbol{P}_\xi^9$ by $\rho_{|L|}$ as before. The argument in this section is very similar to that in § 11. So we just sketch the outline of it.

(12.1) LEMMA. *Let D be a smooth member of $|L|$ and let $\varGamma$ be a smooth member of $|L|_D$. Let S be a plane on $\varGamma$. Then S is of non-vertex type on $\varGamma$ if and only if $c_2(N^{S\backslash M})=3$.*

Proof is easy.

(12.2) LEMMA. *Let S be a plane on M such that $c_2(N^{S\backslash M})=3$. Then there exists no linear threefold T such that $S\subset T\subset M$.*

PROOF. Assume that such a threefold T exists. We see that $\rho_\eta:=\rho_{|L-E_T|}: M'=Q_T(M)\to\boldsymbol{P}_\eta^5$ is surjective as in (11.7). For a general fiber X of ρ_η we have $LX=1$ and $\deg K^X=-3$ as in (11.7). This is absurd.

(12.3) From now on let S be a plane on M as in (12.2). Let $M_1=Q_S(M)$. Then $\rho_\alpha=\rho_{|L-E_S|}: M_1\to\boldsymbol{P}_\alpha^6$ is a birational morphism.

(12.4) We have $(L-E_S)^6=1$, $L(L-E_S)^5=2$ and $(L-E_S)^5E_S=1$. So $H=\rho_\alpha(E_S)$ is a hyperplane on $\boldsymbol{P}_\alpha^6$. Hence $\rho_\alpha^*H=E_S+R$ for $R\in|L-2E_S|$. R is irreducible and reduced as in (10.13).

(12.5) Let $W=\rho_\alpha(R)$. Then $LH^4\{R\}=0$ implies $\dim W<4$. On the other hand $L^2H_\alpha^3\{R\}=3$. So $\dim W=3$.

(12.6) W cannot be contained in any hyperplane other than H. Therefore $h^0(W, H_\alpha)\geqq 6$. Hence $w=\deg W>1$. Moreover, as in (11.6), we have $w=3$, $\varDelta(W, H_\alpha)=0$ and $L^2X=1$ for a general fiber X of $R\to W$.

(12.7) Let x be a point on W and let T be the corresponding linear threefold containing S as in (11.8). Then $T\not\subset M$ by (12.2).

(12.8) Let T_1 be the proper transform of T in $P_1=Q_S(\boldsymbol{P}_\xi^9)$. Then $T_1\cap M_1=\rho_\alpha^{-1}(x)$, which we denote by Y. Y is a plane on $T_1\cong\boldsymbol{P}^3$ similarly as in (11.10).

(12.9) *$Bs|L-E_S-E_Y|=\emptyset$ on $M_2=Q_Y(M_1)$.*

Proof is similar to that of (11.12).

(12.10) Let $\rho_\beta=\rho_{|L-E_S-E_Y|}: M_2\to\boldsymbol{P}_\beta^5$ be the morphism. Then we have $H_\beta^6=0$ and $E_Y^6=-1$ as in (11.13).

(12.11) Let R_2 be the proper transform of R on M_2. Then we have $[R_2]=L-2E_S-E_Y$, $H_\beta^5(L-2E_S)=1$ and $H_\beta^5E_Y=1$ as in (11.15).

(12.12) We have $L^2H_\beta^3\{R_2\}=2>0$. This implies $\dim\rho_\beta(R_2)=\dim\pi(W)=3$, where π is the rational mapping $\boldsymbol{P}_\alpha^6\to\{\textit{lines passing through } x\}\cong\boldsymbol{P}^5$. Hence x is not a vertex of W.

(12.13) The above argument (12.7)~(12.12) proves that W is not a cone. Hence W is smooth because $\Delta(W, H_\alpha)=0$. Moreover we see $(W, H_\alpha)\cong(\boldsymbol{P}_\sigma^1\times\boldsymbol{P}_\tau^2, H_\sigma+H_\tau)$ by [**2**, Corollary 3.9].

(12.14) R is a $\boldsymbol{P}^2$-bundle over W as in (11.18). Moreover ρ_α gives an isomorphism $M_1\cong Q_W(\boldsymbol{P}_\alpha^6)$.

(12.15) The above procedure is the inverse of (7.12). Thus we show that any Del Pezzo 6-fold with $d=5$ can be constructed as in (7.12). Moreover they are all isomorphic to each other as in (11.20). Now, in view of (7.1), we obtain the following

(12.16) THEOREM. *Any Del Pezzo* 6-*fold with* $d=5$ *is isomorphic to* $Gr(5, 2)$.

(12.17) COROLLARY. *There exists no Del Pezzo manifold* (M, L) *with* $d(M, L)=5$ *and* $\dim M\geqq 7$.

PROOF. Suppose to the contrary and let $\{D_i\}$ be a smooth ladder of (M, L) with $\dim D_i=i$. Then D_6 is an ample divisor on D_7 and $D_6\cong Gr(5, 2)$ by (12.16). This contradicts [**6**, (5.2)].

(12.18) THEOREM. *Any Del Pezzo manifold with* $d=5$ *is a linear section of* $Gr(5, 2)$.

For a proof, combine (7.1), (7.2), (9.12), (10.27), (11.20), (12.16) and (12.17).

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