J. Math. Soc. Japan
Vol. 32, No. 4, 1980

Dirichlet states

Dedicated to Professor K. Iseki on the occasion of his 60th birthday

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(Received Nov. 11, 1978)

The aim of this paper is the study of those states of a convex cone $F(X)$ of bounded functions on some set X which behave similar to the points of a Bauer simplex. These states are called Dirichlet states. The main results of the paper can be found in § 3, where those Dirichlet states which admit an integral representation on X are completely characterized by order properties. These results contain among others Choquet's theorem and the authors integral representation theorem [**4**].

The suitable Hahn-Banach methods and decomposition methods for linear functional which are necessary for treating Dirichlet states are gathered in § 1. In § 2 the Dirichlet states are completely characterized in terms of support properties and extension properties. § 3 and § 4 give individual integral representations for Dirichlet states and related states. And in § 5 problems—which seem to the author of some importance—are mentioned.

§ 1. Preliminaries.

Let X be a set and $F=F(X)$ a convex cone of bounded real-valued functions on X. Throughout this paper we assume that F *contains all constant functions*, or in other terms that $F\supset \boldsymbol{R}$. $\boldsymbol{R}^F$ is equipped with the pointwise order on F, this order relation is denoted by $\leqq$. $\rho\in \boldsymbol{R}^F$ is said to be *Y-monotone* if $\rho(f)\geqq\rho(g)$ whenever $f, g\in F$ such that $f(y)\geqq g(y)$ $\forall y\in Y$. If $Y\subset X$ then $\sup_Y$ denotes the sublinear functional on F given by $f\rightarrow\sup_{y\in Y}f(y)$.

A linear (additive and positive-homogeneous) $\nu: F\rightarrow \boldsymbol{R}$ is called *state* of F if ν is X-monotone and $\nu\leqq\sup_X$; this is equivalent to ν being X-monotone with $\nu(1)=1$.

LEMMA 1. *Assume that $\rho: F\rightarrow \boldsymbol{R}$ is linear and X-monotone and that $X=X_1\cup\cdots\cup X_n$. Then $\rho=\sum_{i=1}^{n}\rho_i$, where the ρ_i are linear and X_i-monotone.*

PROOF. We may assume that all $X_i\neq\emptyset$ because 0 is the only $\emptyset$-monotone linear functional. ρ can be extended to a linear X-monotone $\hat{\rho}$ on the vector space $E=F-F$. By [5, finite decomposition theorem] $\hat{\rho}$ is equal to $\sum_{i=1}^{n} \hat{\rho}_i$, where the $\hat{\rho}_i$ are linear and $\leqq \lambda_i \sup_{X_i}$ on E for some $\lambda_i \geqq 0$. This means that the $\hat{\rho}_i$ are X_i-monotone because E is a vector space. q. e. d.

The letters μ, ν will always stand for states and τ, ρ for linear monotone maps. Face(μ) is the *face generated by* μ in the state space, the means Face(μ) is the set of all those $\hat{\nu}$ such that there are ν and $1 \geqq \lambda > 0$ with $\mu = \lambda\hat{\nu} + (1-\lambda)\nu$.

By Cone(μ)$=\{\lambda\nu \mid \lambda \geqq 0,\ \nu \in$ Face(μ)$\}$ we denote the convex cone generated by Face(μ).

We shall say that μ is a *simplicial* state if for ρ_i, $\tau_i \in$ Cone(μ) ($i=1, 2$) with $\rho_1+\rho_2=\tau_1+\tau_2$ there are always $\rho_{ik} \in$ Cone(μ) (i, $k=1, 2$) such that $\sum_{k=1}^{2} \rho_{ik} = \tau_i$ and $\sum_{k=1}^{2} \rho_{ki} = \rho_i$ ($i=1, 2$). A simple inductive argument [1, p. 85] shows that this property goes over to converging sums:

LEMMA 2. *Let* μ *be simplicial and let* ρ_n, τ_n *be in* Cone(μ) *such that* $\sum_{n\in N} \rho_n = \sum_{n\in N} \tau_n$ *and* $\sum_{n\in N} \tau_n(1) < \infty$. *Then there are* $\rho_{n,m} \in$ Cone(μ) *with:*

$$\sum_{n\in N} \rho_{n,m} = \rho_m \quad \textit{and} \quad \sum_{n\in N} \rho_{m,n} = \tau_m \quad \forall m \in \boldsymbol{N}.$$

REMARK 1. [1, p. 84] If Cone(μ)$-$Cone(μ)$=V$ is a vector lattice with respect to the order relation given by $V_+=$Cone(μ) then μ is simplicial.

μ is said to have the *support property* if whenever $\nu \in$ Face(μ) is Y-monotone ($Y \subset X$) then all elements of Face(ν) (face generated by ν instead of μ!) are Y-monotone. Of course, all elements of Face(μ) have the support property when μ has it.

One may guess that the support property implies that a state is simplicial, but this is not so. In [2, example 1.9] is given an example of a compact convex set K such that for every $k \in K$ the measures representing k and living on $X=\overline{\mathrm{ex}(K)}$ (closure of the extreme points) do have equal support without K being a simplex. Hence every state of $F(X)=A(K)_{|X}$ (affine continuous functions restricted to X) has the support property. But there must be non-simplicial states, otherwise K would be a simplex (cf. next chapter).

We call μ *semidispersable*, if whenever

$$\mu \leqq \lambda \sup_{X_1} + (1-\lambda)\sup_{X_2}, \quad 0 \leqq \lambda \leqq 1, \quad X_1, X_2 \subset X, \tag{1}$$

then there are states $\mu_i \leqq \sup_{X_i}$ ($i=1, 2$) such that $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$. If in addition the μ_i can be assumed to be X_i-monotone ($i=1, 2$) then μ is said to be *monotone semidispersable*. Finally, μ is said to be (*monotone*) *dispersable* if all $\nu \in$ Face(μ) are (*monotone*) semidispersable.

REMARK 2. Every maximal μ (i. e. $\mu \leqq \nu \Rightarrow \nu = \mu$) is monotone dispersable.

This is a consequence of [**5**, sum theorem] and the facts that for a maximal μ all elements of Face(μ) are maximal, and that when $\mu \leqq \sup_Y$ $(Y \subset X)$ is maximal then μ has to be Y-monotone (sandwich theorem). Hence, μ is dispersable whenever $F(X)$ is a vector space.

With almost the same proof as in [**3**, theorem 3] one obtains a more general result:

LEMMA 3. *Inequality* (1) *implies the existence of states* $\mu_i \leqq \sup_{X_i}$ $(i=1, 2)$ *with* $\mu=\lambda\mu_1+(1-\lambda)\mu_2$ *if and only if*

$$\mu(f) \leqq \mu(g)+\lambda \sup_{X_1}(f_1)+(1-\lambda)\sup_{X_2}(f_2) \tag{2}$$

whenever $f, g, f_1, f_2 \in F$ *such that* $f(x) \leqq g(x)+f_i(x)$ $\forall x \in X$ $(i=1, 2)$. *The states* μ_i $(i=1, 2)$ *can be assumed to be* X_i*-monotone if and only if* (2) *holds whenever* $f, g, f_1, f_2 \in F$ *such that* $f(x) \leqq g(x)+f_i(x)$ $\forall x \in X_i$ $(i=1, 2)$.

§ 2. Dirichlet states.

DEFINITION 1. A simplicial state with support property is defined to be a *Dirichlet state.*

The reason for choosing this name becomes quite obvious from the following examples.

EXAMPLE 1. (i) Let K be a Choquet simplex then every state of $A(K)$ (affine continuous functions) is simplicial ([**1**, Proposition II. 3.3.]). (ii) Let K be a Bauer simplex and $X=\mathrm{ex}(K)$ (extreme points) then every state of $F(X)=A(K)_{|X}$ (restrictions to X) is a Dirichlet state. (This is a consequence of the following lemma and [**1**, Theorem II. 4.3]).

LEMMA 4. *If* $F(X)$ *is a vector lattice with respect to pointwise order then every state of* $F(X)$ *is a Dirichlet state.*

PROOF. By the Stone-Kakutani theorem [**1**, p. 76] there is a lattice isomorphism from $F(X)$ onto a dense subspace of $C(\bar{X})$, where $\bar{X}$ is compact and can be identified with the set of lattice-preserving states of $F(X)$. So, the vector space generated by the states of $F(X)$ is order isomorphic to the dual of $C(\bar{X})$ and therefore a vector lattice. Hence, remark 1 implies that every state of $F(X)$ is simplicial. According to the Riesz-representation theorem every state μ has a unique representing measure m_μ on $\bar{X}$. Furthermore the uniqueness of m_μ implies that $\mu \rightarrow m_\mu$ is affine. Again by the Riesz-representation theorem ν is Y-monotone if and only if m_ν has its support in the closure of $\beta(Y)$, where $\beta: X \rightarrow \bar{X}$ is the canonical embedding. Now, if $\nu=\lambda\nu_1+(1-\lambda)\nu_2$ then $m_\nu=\lambda m_{\nu_1}+(1-\lambda)m_{\nu_2}$ and the support of m_{ν_1} is contained in the support of m_ν. Hence, ν_1 is Y-monotone if ν is. q.e.d.

By VF we denote the max-stable cone generated by F, that is the smallest convex cone $\supset F$ such that $f\vee g\in VF$ whenever $f, g\in VF$. $VF-VF$ is then the vector lattice generated by F.

THEOREM 1. *μ is a Dirichlet state of F if and only if μ has a unique extension to a state of VF.*

PROOF. Let μ have a unique extension. Then μ has a unique extension to a state $\hat{\mu}$ of $E=VF-VF$; and every $\rho\in\mathrm{Cone}(\mu)$ has to have a *unique* extension to an element $\hat{\rho}\in\mathrm{Cone}(\hat{\mu})$ and the map $\rho\rightarrow\hat{\rho}$ must be a bijective linear map from $\mathrm{Cone}(\mu)$ to $\mathrm{Cone}(\hat{\mu})$. Using lemma 4 we may conclude that μ is simplicial. Now, consider $\nu_1\in\mathrm{Face}(\nu)$ and let $\nu\in\mathrm{Face}(\mu)$ be Y-monotone. Then by [4, lemma 2] $\hat{\nu}$ must be Y-monotone on E and again from lemma 4 we obtain that $\hat{\nu}_1\in\mathrm{Face}(\hat{\nu})$ is Y-monotone. Hence ν_1 (restriction of $\hat{\nu}_1$ to F) is Y-monotone and μ has the support property.

If μ is a Dirichlet state then we consider arbitrary extensions $\hat{\mu}_1$, $\hat{\mu}_2$ to states of VF. For $\varphi\in VF$ we show $\hat{\mu}_1(\varphi)\geqq\hat{\mu}_2(\varphi)$. This clearly proves the theorem. φ can be written as $f_1\vee f_2\vee\cdots\vee f_m$, where $f_1, \cdots, f_m\in F$. Let $X_i=\{x\,|\,f_i(x)\geqq\varphi(x)\}$ $(i=1, \cdots, m)$, then according to lemma 1 we may decompose

$$\hat{\mu}_1=\sum_{i=1}^{m}\hat{\rho}_{1i} \quad\text{and}\quad \hat{\mu}_2=\sum_{i=1}^{m}\hat{\rho}_{2i}\,,$$

where $\hat{\rho}_{1,i}$ and $\hat{\rho}_{2,i}$ are X_i-monotone. Then $\rho_{ki}\in\mathrm{Cone}(\mu)$ (restriction of $\hat{\rho}_{ki}$ to F) is X_i-monotone and we can find $\tau_{ij}\in\mathrm{Cone}(\mu)$ such that

$$\sum_{j=1}^{m}\tau_{ji}=\rho_{1i} \quad\text{and}\quad \sum_{j=1}^{m}\tau_{ij}=\rho_{2i}$$

because μ is simplicial. Furthermore the support property of μ implies that τ_{ji} is X_i-monotone. This clearly implies $\tau_{ji}(f_i)\geqq\tau_{ji}(f_j)$ and from this we obtain:

$$\hat{\mu}_1(\varphi)=\sum_{i=1}^{m}\hat{\rho}_{1i}(f_i)=\sum_{i=1}^{m}\rho_{1i}(f_i)=\sum_{j,i=1}^{m}\tau_{ji}(f_i)\geqq\sum_{j,i=1}^{m}\tau_{ji}(f_j)$$

$$=\sum_{j=1}^{m}\rho_{2j}(f_j)=\sum_{j=1}^{m}\hat{\rho}_{2j}(f_j)=\hat{\mu}_2(\varphi)\,. \qquad \text{q.e.d.}$$

COROLLARY 1. *If F is max-stable then every state is Dirichlet.*

COROLLARY 2. *Let $Y\subset X$ be compact such that F separates the points of Y and every $f\in F$ is continuous on Y. Then the following are equivalent:*

(i) *Every state $\leqq\sup_Y$ is Dirichlet.*

(ii) *$F(X)-F(X)$ restricted to Y is a dense subspace of $C(Y)$ (continuous real-valued functions on Y).*

PROOF. Theorem 1 together with Hahn-Banach and Stone-Weierstrass.

q.e.d.

§ 3. Representation by integrals.

We fix $Z\subset X$ and we say that μ is *partially decomposable* on Z if for $Z_n\subset Z$ with $\bigcup_{n=1}^{\infty} Z_n=Z$ there are always $\lambda_0, \lambda_n\geqq 0$ with $0\leqq\lambda_0<1$ and $1=\lambda_0+\sum_{n=1}^{\infty}\lambda_n$ such that

$$\mu\leqq\lambda_0 \sup_Z+\sum_{n=1}^{\infty}\lambda_n \sup_{Z_n}$$

μ is said to be *decomposable on* Z if we can always have $\lambda_0=0$. Here, of course, we define $0\cdot\sup_\emptyset=0$. It is useful to consider *decompositions* (with respect to $(Z_n)_{n\in N}$) of the form:

$$\mu=\sum_{n=1}^{\infty}\lambda_n\mu_n+\lambda_0\mu_0$$

where $\mu_0\leqq\sup_X$ and $\mu_n\leqq\sup_{Z_n}$ are states and the $\lambda_0, \lambda_n\geqq 0$ are such that $\sum_{k=0}^{\infty}\lambda_k=1$. If, in addition, the μ_n are Z_n-monotone ($n=1, 2, \cdots$) then this decomposition is called *monotone* (with respect to $(Z_n)_{n\in N}$, of course). Such a decomposition is said to be *maximal* if we have for all N that, whenever $\beta_N=\left(1-\sum_{n=1}^{N}\lambda_n\right)>0$, then λ_{N+1} is β_N times the maximum of those $0\leqq\lambda\leqq 1$ such that $\nu_{N+1}\leqq\lambda\sup_{Z_{N+1}}+(1-\lambda)\sup_X$, where ν_{N+1} is the state

$$\nu_{N+1}=\frac{1}{\beta_N}\left(\mu-\sum_{n=1}^{N}\lambda_n\mu_n\right)=\frac{1}{\beta_N}\left(\sum_{N+1}^{\infty}\lambda_n\mu_n+\lambda_0\mu_0\right).$$

A trivial inductive argument shows that for a (monotone) dispersable μ there is always such a (monotone) maximal decomposition.

LEMMA 5. *Let μ be dispersable. Then the following are equivalent:*

(i) *Every $\nu\in\mathrm{Face}(\mu)$ is decomposable on X.*

(ii) *Every $\nu\in\mathrm{Face}(\mu)$ is partially decomposable on X.*

(iii) *For all $\nu\in\mathrm{Face}(\mu)$ we have $\sum_{n=1}^{\infty}\nu(f_n)=-\infty$ whenever $0\geqq f_n\in F$ such that $\sum_{n=1}^{\infty}f_n(x)=-\infty$ $\forall x\in X$.*

In case that F is such that $f\vee r\in F$ whenever $f\in F$ and $r\in\boldsymbol{R}$ then all this is equivalent to

(iv) *For every sequence $f_n\geqq 0$ in F which is pointwise decreasing to zero we do have $\inf_{n\in N}\mu(f_n)=0$.*

PROOF. (i)$\Rightarrow$(ii) is trivial, (ii)$\Leftrightarrow$(iii) is a direct consequence of [**5**, partial decomposition theorem] and (ii)$\Leftrightarrow$(iv) follows with the same argument as in [**5**, theorem 1].

(ii)$\Rightarrow$(i): Let $Z_n \subset X$ be arbitrary such that $\bigcup_{n=1}^{\infty} Z_n = X$, and let

$$\nu = \sum_{n=1}^{\infty} \lambda_n \nu_n + \lambda_0 \nu_0$$

be a maximal decomposition of $\nu \in \mathrm{Face}(\mu)$. If $\lambda_0 = 0$ then we are done. For $\lambda_0 > 0$ we consider a maximal decomposition

$$\nu_0 = \sum_{n=1}^{\infty} \tilde{\lambda}_n \tilde{\nu}_n + \tilde{\lambda}_0 \tilde{\nu}_0$$

of $\nu_0 \in \mathrm{Face}(\mu)$. Since ν_0 is partially decomposable on X, we know that $\sum_{n=1}^{\infty} \tilde{\lambda}_n > 0$. This is in contradiction to the maximality of the decomposition for ν because this gives the following decomposition:

$$\nu = \sum_{n=1}^{\infty} (\lambda_n \nu_n + \lambda_0 \tilde{\lambda}_n \tilde{\nu}_n) + \lambda_0 \tilde{\lambda}_0 \tilde{\nu}_0 . \qquad \text{q. e. d.}$$

LEMMA 6. *Let μ be monotone dispersable, and let all $\nu \in \mathrm{Face}(\mu)$ be partially decomposable. Then the unique extension of μ to the vector space $F-F$ is monotone decomposable.*

PROOF. Let $Z_n \subset X$ be arbitrary such that $\bigcup_{n=1}^{\infty} Z_n = X$, and let

$$\mu = \sum_{n=1}^{\infty} \lambda_n \nu_n + \lambda_0 \nu_0$$

be a suitable monotone and maximal decomposition. As in lemma 5 we draw the conclusion that $\lambda_0 = 0$ because otherwise ν_0 is partially decomposable. Now, the unique extensions $\tilde{\nu}_n$ of ν_n to $F-F$ are Z_n-monotone [4, lemma 2]; hence we have for $\tilde{\mu}$ (unique extension of μ to $F-F$) that

$$\tilde{\mu} = \sum_{n=1}^{\infty} \lambda_n \tilde{\nu}_n . \qquad \text{q. e. d.}$$

A positive measure m on X with respect to Σ_F, the σ-algebra generated by F, is called a *strict representing measure* for τ if

$$\tau(f) = \int_X f \, dm \quad \forall f \in F .$$

The next theorem can be considered as a local version of [4, theorem 1].

THEOREM 2. *Let F be a vector space and let μ be a Dirichlet state. Then the following are equivalent:*

(i) *Every $\nu \in \mathrm{Face}(\mu)$ has a unique strict representing measure m_ν.*

(ii) *Every* $\nu \in \mathrm{Face}(\mu)$ *is decomposable on* X.

(iii) *For every* $\nu \in \mathrm{Face}(\mu)$ *we have* $\sum_{n=1}^{\infty} \nu(f_n) = -\infty$ *whenever* $0 \geqq f_n \in F$ *such that* $\sum_{n=1}^{\infty} f_n(x) = -\infty \ \forall x \in X$.

PROOF. The state space $\Omega = \{\omega \mid \omega \text{ state of } F\}$ is compact under the coarsest topology which makes the functions $\omega \to \hat{f}(\omega) = \omega(f)$, $f \in F$ continuous. $\beta : X \to \Omega$ shall denote the canonical map $\beta(x)(f) = f(x)$ $\forall f \in F$.

(i)$\Rightarrow$(ii): $\hat{m}_\nu(\Omega_1) = m_\nu(\beta^{-1}(\Omega_1))$, $\Omega_1 \subset \Omega$, defines a Baire probability measure on Ω. Let $\bar{m}_\nu$ denote the corresponding regular Borel measure. Then we have $\int_\Omega \hat{f}\, d\bar{m}_\nu = \nu(f)$ $\forall f \in F$. Now, we consider for $X_n \subset X$ with $\bigcup_{n \in N} X_n = X$ the sets $\bar{X}_n = \text{closure}(\beta(X_n))$ and we put $\lambda_n = \bar{m}_\nu(\bar{X}_n \setminus \bigcup_{k<n} \bar{X}_k)$. Then we have $\sum_{n \in N} \lambda_n = 1$ and $\nu \leqq \sum_{n \in N} \lambda_n \sup_{X_n}$. Hence, ν must be decomposable.

(ii)$\Leftrightarrow$(iii) remark 2 and lemma 5 (iii).

(ii)$\Rightarrow$(i): All the $\tilde{\nu} \in \mathrm{Face}(\mu)$ have unique extensions to the vector lattice $VF - VF$ (theorem 1) and this vector lattice is vector-lattice-isomorphic to the vector lattice on $\bar{X} = \text{closure}(\beta(X))$ generated by the functions $\hat{f}_{|X}$, $f \in F$. So, if $\tilde{\nu} \leqq \sup_Y$ then the Riesz representation theorem tells us that there is a unique Borel probability measure $\bar{m}_{\tilde{\nu}}$ on $\bar{Y} = \text{closure}(\beta(Y)) \subset \bar{X}$ with $\int_{\bar{Y}} \hat{f}\, d\bar{m}_{\tilde{\nu}} = \tilde{\nu}(f)$ $\forall f \in F$. Let $\nu \in \mathrm{Face}(\mu)$ be arbitrary. For compact $Z_n \subset \bar{X}$ with $\bigcup_{n \in N} Z_n \supset \beta X$ we consider $X_n = \beta^{-1}(Z_n) \subset X$. Then $\bigcup_{n \in N} X_n = X$, and since ν is decomposable on X we obtain with [**5**, sum theorem] that $\nu = \sum_{n \in N} \lambda_n \nu_n$, where $\nu_n \leqq \sup_{X_n}$.

According to the preceding statement there are for the ν_n and ν Borel probability measures $\bar{m}_{\nu_n}$ on Z_n and $\bar{m}_\nu$ on $\bar{X}$ such that $\int_{Z_n} \hat{f}\, d\bar{m}_{\nu_n} = \nu_n(f)$ and $\int_{\bar{X}} \hat{f}\, d\bar{m}_\nu = \nu(f)$ $\forall f \in F$. This implies $\bar{m}_\nu(\bigcup_{n \in N} Z_n) = 1$ because of $\bar{m}_\nu = \sum_{n \in N} \lambda_n \bar{m}_{\nu_n}$. Since the Z_n have been arbitrarily chosen, $\bar{m}_\nu$ is supported by every F_σ-set containing βX. Therefore we may conclude from regularity that $\bar{m}_\nu(B) = 0$ for every Baire set B with $B \cap \beta X = \emptyset$. Now, because of $\Sigma_F = \{\beta^{-1}(B) \mid B \text{ Baire set } \subset \bar{X}\}$ we get the desired strict representing measure m_ν for ν by $m_\nu(\beta^{-1}(B)) = \bar{m}_\nu(B)$ $\forall B$ Baire set $\subset \bar{X}$. The uniqueness of m_ν follows from theorem 1.

q. e. d.

EXAMPLE 2. Consider a compact convex set K with point separating $A(K)$. And let k be a *Dirichlet point* in K, which shall mean that $h \to h(k)$ is a Dirichlet state of $A(K)_{|\mathrm{ex}(K)}$. If $Y \subset \mathrm{ex}(K)$ we may ask when the (unique!) boundary measure m representing k is *pseudo-carried* by Y (i. e. $m(B) = 0$ $\forall$ Baire sets B with $B \cap Y = \emptyset$). Theorem 2 tells us that this is the case if and only if $\sum_{n=1}^{\infty} h_n(x)=-\infty$ $\forall x\in\mathrm{Face}(k)$ whenever $0\geqq h_n\in A(K)$ with $\sum_{n=1}^{\infty} h_n(y)=-\infty$ $\forall y\in Y$.

It is quite easy to reformulate theorem 2 such that it can be applied to convex cones of functions. From lemma 6 we immediately obtain:

COROLLARY 3. *Let μ be a monotone dispersable Dirichlet state of the convex cone $F=F(X)$. Then the following are equivalent:*

(i) *Every $\nu\in\mathrm{Face}(\mu)$ has a unique strict representing measure m_ν on X.*

(ii) *Every $\nu\in\mathrm{Face}(\mu)$ is decomposable on X.*

(iii) *For every $\nu\in\mathrm{Face}(\mu)$ we have $\sum_{n=1}^{\infty}\nu(f_n)=-\infty$ whenever $0\geqq f_n\in F$ such that $\sum_{n=1}^{\infty} f_n(x)=-\infty$ $\forall x\in X$.*

(iv) *Every $\nu\in\mathrm{Face}(\mu)$ is Dini-continuous.*

Here, *Dini-continuous* means that we do have for all pointwise decreasing sequences f_n in F:

$$\inf_{n\in N}\nu(f_n)\leqq\sup_{x\in X}\ \inf_{n\in N} f_n(x).$$

It should be mentioned that (iv)$\Rightarrow$(iii) is trivial and (i)$\Rightarrow$(iv) follows from the monotone convergence theorem.

COROLLARY 4. *Let μ be a maximal state of the max-stable convex cone $F=F(X)$. Then the following are equivalent:*

(i) *Every $\nu\in\mathrm{Face}(\mu)$ has a unique strict representing measure on X.*

(ii) *$\inf_{n\in N}\mu(f_n)=0$ whenever $0\leqq f_n\in F$ is pointwise decreasing to zero.*

PROOF. Corollary 3 and lemma 5 together with remark 2 and Corollary 1. q.e.d.

EXAMPLE 3. Corollary 4 applied to $VA(K)_{|\mathrm{ex}(K)}$ (K=compact convex) leads with the help of Dini's lemma and Bauer's maximum principle (for the upper-semicontinuous convex functions) immediately to: Every maximal measure (in the Choquet ordering) is pseudo-carried by ex(K) (Choquet's theorem) (cf. [**1**]).

§4. Localization of decomposable states.

In certain cases theorem 2 can be used for obtaining representing measures for non-Dirichlet states. This is done via a localization of decomposability.

Let $Z\subset X$. We say that a state μ is *disjoint from* Z if there are Z_n ($n=1, 2, \cdots$) with $\bigcup_{n=1}^{\infty} Z_n=Z$ such that $\lambda=0$ whenever $1\geqq\lambda\geqq 0$ and there is some n with $\mu\leqq\lambda\sup_{Z_n}+(1-\lambda)\sup_X$. Of course, if μ is not partially decomposable on Z, then it is disjoint from Z.

EXAMPLE 4. Let $Z_n\subset X$ and let μ be dispersable such that

$$\mu=\sum_{n=1}^{\infty}\lambda_n\mu_n+\lambda_0\mu_0$$

is a maximal decomposition of μ with respect to $(Z_n)_{n\in N}$ (in the sense of §3). Then maximality implies that when $\lambda_0>0$ then μ_0 is disjoint from $Z=\bigcup_{n=1}^{\infty}Z_n$.

REMARK 3. When μ is disjoint from Z, then every $\nu\in\mathrm{Face}(\mu)$ is disjoint from Z.

LEMMA 7. *Let μ be disjoint from $Z\subset X$ and decomposable on $Y\subset X$. Then μ is decomposable on $Y\backslash Z$.*

PROOF. Let the $Z_n\subset Z$ be as in the definition above and consider $X_n\subset Y\backslash Z$ with $\bigcup_{n=1}^{\infty}X_n=Y\backslash Z$. Then we have

$$\mu\leqq\sum_{n=1}^{\infty}\lambda_n\sup_{Z_n}+\sum_{n=1}^{\infty}\tilde{\lambda}_n\sup_{X_n}$$

because μ is decomposable on Y. Now, all the λ_n have to be equal to zero, since $\mu\leqq\lambda_n\sup_{Z_n}+(1-\lambda_n)\sup_X$. q.e.d.

LEMMA 8. *Let every $\nu\in\mathrm{Face}(\mu)$ be dispersable and decomposable on $Y=Y_0\cup Y_1$. Then $\nu\in\mathrm{Face}(\mu)$ is either decomposable on Y_1 or there are $1\geqq\lambda>0$ and states ν_i $(i=0, 1)$ with $\nu=\lambda\nu_0+(1-\lambda)\nu_1$ such that ν_0 is disjoint from Y_1 and decomposable on Y_0.*

PROOF. If $\nu\in\mathrm{Face}(\mu)$ is not decomposable on Y_1, then there are $Z_n\subset Y_1$ with $\bigcup_{n=1}^{\infty}Z_n=Y_1$ such that $\lambda_0>0$ for every maximal decomposition

$$\nu=\sum_{n=1}^{\infty}\lambda_n\nu_n+\lambda_0\nu_0$$

of ν with respect to $(Z_n)_{n\in N}$. Now, example 4 shows that ν_0 is disjoint from Y_1, and from lemma 7 we know that ν_0 is decomposable on Y_0. q.e.d.

THEOREM 3. *Let every $\nu\in\mathrm{Face}(\mu)$ be dispersable and decomposable on $Y=Y_1\cup Y_2$; $Y_1, Y_2\neq\emptyset$. Then there are states μ_i $(i=1, 2)$ and $1\geqq\lambda\geqq0$ with $\mu=\lambda\mu_1+(1-\lambda)\mu_2$ such that μ_2 is decomposable on Y_2 and μ_1 is disjoint from Y_2 and decomposable on Y_1.*

PROOF. If μ is decomposable on Y_2 we put $\lambda=0$, and we are done. In case that μ is not decomposable on Y_2 we can write (according to lemma 8) $\mu=\lambda\mu_1+(1-\lambda)\mu_2$ $(1\geqq\lambda>0)$ where μ_1 is disjoint from Y_2. By Zorn's lemma we may assume that $\lambda\mu_1$ is "maximal" in the following sense:

whenever $\mu_2=\tilde{\lambda}\nu_1+(1-\tilde{\lambda})\nu_2$ $(1\geqq\tilde{\lambda}\geqq0)$, where
ν_1 is disjoint from Y_2, then $\tilde{\lambda}=0$.

Then, from lemma 8 (applied to μ_2) we conclude that μ_2 is decomposable on Y_2.

The assertion that μ_1 is decomposable on Y_1 follows immediately from lemma 7.
q. e. d.

Induction leads immediately to:

COROLLARY 5. *Let every* $\nu \in \mathrm{Face}(\mu)$ *be dispersable and decomposable on* $X=\bigcup_{n=1}^{N} Y_n$; $Y_1, \cdots, Y_N \neq \emptyset$. *Then there are* $\lambda_n \geqq 0$ *and states* μ_n *with* $\mu=\sum_{n=1}^{N} \lambda_n \mu_n$ *such that the* μ_n *are decomposable on* Y_n.

This localization procedure leads—for example—to integral representations in the following:

SITUATION. Let $X=\bigcup_{n=1}^{N} Y_n$, $Y_1, \cdots, Y_N \neq \emptyset$, and let $F=F(X)$ be such that the restrictions $F_{|Y_n}=\{f_{|Y_n} \mid f \in F\}$ are max-stable for $n=1, \cdots, N$. And assume μ to be a maximal state of F.

THEOREM 4. *Then the following are equivalent:*

(i) *Every* $\nu \in \mathrm{Face}(\mu)$ *has a strict representing measure on* X.

(ii) *For every* $\nu \in \mathrm{Face}(\mu)$ *we have* $\sum_{n=1}^{\infty} \nu(f_n)=-\infty$ *whenever* $0 \geqq f_n \in F$ *such that* $\sum_{n=1}^{\infty} f_n(x)=-\infty$ $\forall x \in X$.

PROOF. (i)$\Rightarrow$(ii) is a consequence of the monotone convergence theorem. (ii)$\Rightarrow$(i) μ is monotone dispersable (remark 2) and every $\nu \in \mathrm{Face}(\mu)$ is decomposable (lemma 5). Now, we localize according to corollary 5. Corollary 3 together with corollary 1 gives the desired representing measure. q. e. d.

§ 5. Problems.

PROBLEM 1. Under what kind of conditions does a countable analogon of Corollary 5 hold?

An answer to this problem would certainly lead to very powerful integral representation theorem.

PROBLEM 2. Do the results of § 3 and 4 hold for the case that V-valued states (V a Dedekind complete vector lattice) are considered?

Some of the theorems may be true in this case (cf. [6]), but there will be very many problems when V is not weakly σ-distributive [7].

PROBLEM 3. Which results of § 3 remain true in case that the states under consideration are not Dirichlet states?

ACKNOWLEDGEMENT. I am indebted to Thomas Landes and the referee for many helpful remarks.

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