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Algebraic differential equations of the first order free from parametric singularities from the differential-algebraic standpoint

By Michihiko MATSUDA

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Abstract.

A *differential-algebraic* definition for an algebraic differential equation of the first order to be free from parametric singularities will be given. From this standpoint we shall prove three theorems which are essentially due to Briot and Bouquet, Fuchs, and Poincaré respectively.

§ 0. Introduction.

Let k be an algebraically closed differential field of characteristic 0, and K be a one-dimensional algebraic function field over k. We shall suppose that K is a differential extension of k. Let P be a prime divisor of K, and K_P be the completion of K with respect to P. Then, K_P is a differential extension of K, and the differentiation gives a continuous mapping from K_P to itself (cf. [8]).

We shall say that K is a *differential algebraic function field* over k if there exists an element y of K such that $K=k(y, y')$. Suppose that K is a differential algebraic function field over k. Then, y and y' satisfy an irreducible algebraic equation $F(y, y')=0$ over k. We shall say that K is associated with F in y.

Conversely, let $k\{y\}$ be the differential polynomial algebra in a single indeterminate y over k. We shall take and fix a universal extension Ω of k, the existence of which was proved by Kolchin [3, p. 771]. Let F be an algebraically irreducible element of $k\{y\}$ of the first order, and η be a generic point of the general solution of F in Ω over k. Then, $k(\eta, \eta')$ is a differential algebraic function field over k associated with F in η.

Throughout this note K will denote a differential algebraic function field over k.

Let ν_P be the normalized valuation in K belonging to a prime divisor P of K. If $\nu_P(\tau')\geqq 0$ for a prime element τ in P, then $\nu_P(\sigma')\geqq 0$ for any prime ele-

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ment σ in P.

DEFINITION. K will be said to be free from parametric singularities if we have $\nu_P(\tau')\geqq 0$ for any prime divisor P of K, where τ is a prime element in P.

Comparing it with Fuchs' criterion [2] for an algebraic differential equation of the first order to be free from parametric singularities, we shall see that our definition is a reasonable one (Theorem 1 in § 1).

It will be said that K is a *differential elliptic function field* over k if there exists an element z of K such that $K=k(z, z')$ and

$$(z')^2=\lambda z(z^2-1)(z-\delta)\,; \quad \lambda\neq 0\,; \quad \delta^2\neq 0, 1\,;$$

here, λ, $\delta\in k$, and δ is a constant.

Suppose that K is free from parametric singularities. Then, the following three theorems are essentially due to Briot and Bouquet, Fuchs, and Poincaré respectively (cf. Forsyth [**1**, Chap. s 9, 10], Picard [**6**, Chap. 4]):

THEOREM BB. *Suppose that any element of k is a constant. Then, the genus of K is* 0 *or* 1.

THEOREM F. *Suppose that the genus of K is* 0. *Then, there exists an element t of K such that $K=k(t)$ and*

$$t'=a+bt+ct^2\,; \quad a, b, c\in k\,.$$

THEOREM P. *Suppose that the genus of K is* 1. *Then, K is a differential elliptic function field over k.*

In the last two theorems we do not assume that k consists of constants. From our standpoint we shall reproduce Fuchs' proof [**2**] of Theorems BB, F, and Poincaré's one [**7**] of Theorem P. They discussed the problems in the case where k is the algebraic closure of a field of functions meromorphic throughout a domain in the plane of the complex variable. Three Theorems BB, F, P will be proved respectively in § 2, § 1, § 3.

REMARK 1. Poincaré's proof [**7**] of Theorem P is purely algebraic. Unfortunately, however, careful consideration of transcendental constants over the coefficient field lacks in it (cf. [**1**, Chap. 9], [**4**], [**5**], [**6**, Chap. 4]).

REMARK 2. In § 4 we shall give an example of K in Theorem P such that there exists a transcendental constant of K over k.

REMARK 3. Poincaré [**7**] stated the following theorem: Suppose that an algebraic differential equation of the first order is free from parametric singularities, and that the genus is greater than 1. Then, its general solution can be obtained through an algebraic process (cf. Forsyth [**1**, Chap. 9], Painlevé [**4**], [**5**], and Picard [**6**, Chap. 4]).

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criterion for K to be a differential elliptic function field from the standpoint of Galois theory of differential fields (cf. [**3**, pp. 808-823]).

§ 1. Fuchs' criterion.

Let K be a differential algebraic function field over k associated with F in y. Let P be a prime divisor of K satisfying $\nu_P(y)\geqq 0$. Suppose that $\nu_P(y')\geqq 0$. Then, in K_P we have

$$y=\eta+\tau^{\alpha} \qquad (\eta\in k),$$

$$y'=\zeta+g_0\tau^{\kappa}+g_1\tau^{\kappa+1}+\cdots \qquad (\zeta, g_0, g_1, \cdots\in k, g_0\neq 0),$$

where τ is a prime element in P, and α, κ are positive integers. Let P be a prime divisor of K satisfying $\nu_P(y)<0$. Then, in K_P we have

$$y=\tau^{-\alpha} \qquad (\alpha>0),$$

$$y'=g_0\tau^{\kappa}+g_1\tau^{\kappa+1}+\cdots \qquad (g_0, g_1, \cdots\in k, g_0\neq 0),$$

where τ is a prime element in P, and α, κ are integers. Fuchs' condition necessary and sufficient for $F=0$ to be free from parametric singularities is as follows (cf. Forsyth [**1**, pp. 277-285]):

Fuchs' criterion. For any P satisfying $\nu_P(y)\geqq 0$, we have $\nu_P(y')\geqq 0$. Suppose that $\nu_P(y)\geqq 0$ and $\alpha>1$. Then, $\eta'=\zeta$ and $\kappa\geqq\alpha-1$. Suppose that $\nu_P(y)<0$. Then, $\kappa\geqq-\alpha-1$.

THEOREM 1. *K is free from parametric singularities in the sense of our Definition if and only if Fuchs' criterion is satisfied by K.*

PROOF. First we shall suppose that $\nu_P(y)\geqq 0$. Then, in K_P we have $y'=\eta'+\alpha\tau^{\alpha-1}\tau'$ by $y=\eta+\tau^{\alpha}$. Hence, $\nu_P(\tau')\geqq 0$ if and only if $\nu_P(y'-\eta')\geqq\alpha-1$. Secondly we shall suppose that $\nu_P(y)<0$. Then, in K_P we have $y'=-\alpha\tau^{-\alpha-1}\tau'$ by $y=\tau^{-\alpha}$. Hence, $\nu_P(\tau')\geqq 0$ if and only if $\nu_P(y')\geqq-\alpha-1$.

The following theorem is due to Fuchs [**2**]:

THEOREM 2. *Suppose that K is free from parametric singularities. Let $F(X_1, X_2)$ take the form*

$$A_0X_2^m+A_1X_2^{m-1}+\cdots+A_m, \tag{1}$$

where $A_0, A_1, \cdots, A_m\in k[X_1]$, and $A_0\neq 0$. Then, for each j $(0\leqq j\leqq m)$ we have $\deg A_j\leqq 2j$ *unless $A_j=0$.*

PROOF. For any prime divisor P satisfying $\nu_P(y)\geqq 0$, we have $\nu_P(y')\geqq 0$ by Theorem 1. Hence, $A_0\in k$. Let w denote $1/y$. Then, $w'=-y'/y^2$. We shall define $G(X_1, X_2)$ by

$$G=X_1^lF(1/X_1, -X_2/X_1^2),$$

where

(2) $$l=\max\{\deg A_j+2(m-j)\,;\ A_j\neq 0,\ 0\leqq j\leqq m\}.$$

Then, K is associated with G in w. The leading coefficient of G is $A_0(-1)^m X_1^{l-2m}$. Hence, $l=2m$.

PROOF OF THEOREM F. Since the genus of K is 0, there exists an element t of K such that $K=k(t)$. We have $t'=A/B$, where $A, B\in k[t]$, and $(A, B)=1$. Let us define G by

$$G(X_1, X_2)=A(X_1)-X_2B(X_1)\,.$$

Then, K is associated with G in t. By Theorem 2, $\deg B=0$, and $\deg A\leqq 2$ unless $A=0$.

§ 2. Briot-Bouquet's Theorem.

Suppose that K is associated with F in y, and that F takes the form (1). Then, we have

$$2(g-1)=\sum_P (e_P-1)-2m \qquad \text{(Riemann's formula)},$$

where g is the genus of K, and e_P is the ramification exponent of P with respect to $k(y)$.

There exists an element ξ of k such that $e_P=1$ for any P satisfying $\nu_P(y-\xi)>0$. Let z denote $1/(y-\xi)$, and H be the polynomial defined by

$$H(X_1, X_2)=X_1^l F(\xi+1/X_1,\ \xi'-X_2/X_1^2)\,,$$

where l is the number defined by (2). Then, K is associated with H in z. For any P satisfying $\nu_P(z)<0$, we have $e_P=1$.

PROOF OF THEOREM BB. Let us assume that $e_P=1$ for any P satisfying $\nu_P(y)<0$. We do not lose the generality by this assumption. If $y'=0$, then $K=k(y)$. We shall assume that $y'\neq 0$. Then, $A_m\neq 0$. Suppose that $e_P>1$ for a prime divisor P of K. Then, there exists an element η of k such that $0<\nu_P(y-\eta)$. By Theorem 1, we have $\nu_P(y')\geqq 0$, and there exists an element ζ of k such that $\nu_P(y'-\zeta)>0$. Since $e_P>1$, we have $\zeta=\eta'$ by Theorem 1. Hence, $\zeta=\eta'=0$, because k is a constant field by our assumption. We have $F(\eta, 0)=0$, and η is a root of A_m. Let ρ be the multiplicity of η in A_m, and $\{P_1, \cdots, P_r\}$ be the totality of prime divisors of K for which $e_P>1$ and $\nu_P(y-\eta)>0$. Then,

$$\rho\geqq\sum_{i=1}^{r}\nu_{P_i}(y')\,.$$

By Theorem 1, we have

$$\nu_{P_i}(y')\geqq e_{P_i}-1$$

for each i $(1\leqq i\leqq r)$. Let P run through all prime divisors of K. Then, we have

$$\deg A_m\geqq \sum_P (e_P-1).$$

By Theorem 2, $\deg A_m\leqq 2m$. Hence,

$$2m\geqq\deg A_m\geqq \sum_P (e_P-1).$$

By Riemann's formula, we have $g-1\leqq 0$.

§ 3. Poincaré's Theorem.

PROOF OF THEOREM P. Since the genus of K is 1, there exists an element u of K such that $K=k(u, v)$ and

$$v^2=R(u)=u(u^2-1)(u-\delta)\,;\ \delta\in k\,;\ \delta^2\neq 0,\ 1\,.$$

First let us assume that u is not a constant. Then, $u'\notin k(u)$. In fact, to the contrary suppose that $u'\in k(u)$. Then, we have $u'=A/B\neq 0$, where A, $B\in k[u]$, and $(A, B)=1$. Suppose that $\deg B>0$. Then, there exists a root ξ of B in k. Take a prime divisor P of K such that $\nu_P(u-\xi)>0$. Then, $u=\xi+\tau^\alpha$ in K_P, where τ is a prime element in P. Hence, $u'=\xi'+\alpha\tau^{\alpha-1}\tau'$. Since $\nu_P(\tau')\geqq 0$, we have $\nu_P(u')\geqq 0$. This is a contradiction, because $A(\xi)\neq 0$. Hence, $u'=C\in k[u]$. Take a prime divisor P such that $\nu_P(u)<0$. In K_P, $u=\tau^{-\alpha}$, where τ is a prime element in P. Hence, $u'=-\alpha\tau^{-\alpha-1}\tau'$. Since $\nu_P(\tau')\geqq 0$, we have

$$\alpha(1-\deg C)+1\geqq 0\,.$$

Hence, $\deg C\leqq 2$. Let η be a root of R, and $P(\eta)$ denote the prime divisor determined by the condition that $\nu_P(u-\eta)>0$. Then, $u=\eta+\tau(\eta)^2$ in $K_{P(\eta)}$, where $\tau(\eta)$ is a prime element in $P(\eta)$. Set $\eta=0$, ± 1. Then, $u'=2\tau\tau'$, where $\tau=\tau(\eta)$. By $\nu_{P(\eta)}(\tau')\geqq 0$, we have $C(\eta)=0$. This is a contradiction, because $\deg C\leqq 2$. Hence, $u'\notin k(u)$, and $K=k(u, u')$. Let $G(u, u')=0$ be an irreducible algebraic equation over k satisfied by u and u'. Then, K is associated with G in u. By Theorem 2, we have

$$G(X_1, X_2)=(X_2-E)^2-D \qquad (D,\ E\in k[X_1])\,,$$

where $\deg D\leqq 4$, and $\deg E\leqq 2$ if $E\neq 0$. Unless an element ξ of k is a root of D, we have $\nu_P(u-\xi)=1$ for each P satisfying $\nu_P(u-\xi)>0$. Hence, every root η of R satisfies $D(\eta)=0$. Since $\deg D\leqq 4$, we have $D=\lambda R$, where $\lambda\in k$. For each root η of R, $\nu_{P(\eta)}(u-\eta)=2$. Hence, by Theorem 1, $E(\eta)=\eta'$. Set $\eta=0$, ± 1. Then, $E(\eta)=\eta'=0$. We have $E=0$, because $\deg E\leqq 2$ if $E\neq 0$. Set $\eta=\delta$. Then, $0=E(\delta)=\delta'$.

Secondly, assume that u is a constant. Then, $\delta'=0$. In fact, there exists a prime divisor P of K such that $\nu_P(n-\delta)=2$, where ν_P is the normalized valuation belonging to P. We have $u-\delta=\tau^2$ with a prime element τ in P, and $-\delta'=2\tau\tau'$. Since K is free from parametric singularities, $\nu_P(\tau')\geqq 0$. Hence, $\delta'=0$. There exists an element of k which is not a constant. For, to the contrary, suppose that any element of k is a constant. Then, every element of K is a constant, because the constant u is transcendental over k. Hence, $y'=0$, and $K=k(y)$. This contradicts the assumption that the genus of K is one. Take an element ξ of k which is not a constant. Let us define an element λ of k by

$$\lambda=(\xi')^2/R(\xi),$$

and a new differentiation signed by the dot in K by

$$\dot{x}=\mu x', \qquad \mu^2=\lambda^{-1}(2/\delta).$$

With respect to this differentiation, ξ is a nonsingular solution of

$$\delta(\dot{\xi})^2=2R(\xi),$$

and u remains to be a constant. We shall define two transcendental constants a, b over k and an element w of k by

$$a=2u/(1+u), \qquad b=\varepsilon v/(1+u)^2,$$

$$\varepsilon^2=2/\delta, \qquad w=2\xi/(1+\xi).$$

Then, w is a nonsingular solution of

$$(\dot{w})^2/4=S(w)=w(1-w)(1-\kappa^2 w), \qquad \kappa^2=(1+\delta)/(2\delta),$$

and $b^2=S(a)$. We have $\ddot{w}=2S_w$. Let us define an element z of K by the following formula (cf. Remark 4 at the end of this section):

(3) $$z=\{a(1-w)(1-\kappa^2 w)+b\dot{w}+w(1-a)(1-\kappa^2 a)\}/(1-\kappa^2 aw)^2.$$

Then,

(4) $$(1-\kappa^2 wz)^2a^2-2\{z(1-w)(1-\kappa^2 w)+w(1-z)(1-\kappa^2 z)\}a+(z-w)^2=0.$$

Because of $\dot{w}\neq 0$, $K=k(a, b)=k(a, z)$ by (3). Hence, z is transcendental over k. We have

(5) $$[K:k(z)]=2$$

from (4). Let us prove that z is a solution of $(\dot{z})^2=4S(z)$. Set

$$A=1-\kappa^2 aw, \qquad B=w(1-a)(1-\kappa^2 a)+a(1-w)(1-\kappa^2 w).$$

Then, $z=(B+b\dot{w})/A^2$, and

$$A^3\dot{z}=(AB_w-2A_wB)\dot{w}+\{A\ddot{w}-2A_w(\dot{w})^2\}b\,.$$

We have

$$AB_w-2A_wB=\kappa^2aB-A^2+2S(a)/a\,.$$

Let $C(a, w)$ denote the right hand member. Then,

$$A\ddot{w}-2A_w(\dot{w})^2=2C(w, a)=2\{\kappa^2wB-A^2+2S(w)/w\}\,.$$

Hence,

$$A^6(\dot{z})^2=4\{S(w)C(a, w)^2+S(a)C(w, a)^2+C(a, w)C(w, a)b\dot{w}\}\,.$$

On the other hand,

$$A^6S(z)=D(a, w)+b\dot{w}E(a, w)\,,$$

$$D=B(B-A^2)(\kappa^2B-A^2)+4\{3\kappa^2B-(1+\kappa^2)A^2\}S(a)S(w)\,,$$

$$E=3\kappa^2B^2-2(1+\kappa^2)BA^2+A^4+4\kappa^2S(a)S(w)\,.$$

We obtain the two identities:

$$S(w)C(a, w)^2+S(a)C(w, a)^2=D(a, w)\,;$$

$$C(a, w)C(w, a)=E(a, w)\,.$$

Hence, $(\dot{z})^2=4S(z)$. Because of (5), $K=k(z, \dot{z})$. Let Z denote $z/(2-z)$. Then, $K=k(Z, Z')$ and $(Z')^2=\lambda R(Z)$.

REMARK 4. For Jacobi's elliptic functions with modulus κ we have the following addition formula (cf. [**10**, p. 50]):

$$\mathrm{sn}(\alpha+\beta)=(\mathrm{sn}\,\alpha\,\mathrm{cn}\,\beta\,\mathrm{dn}\,\beta+\mathrm{sn}\,\beta\,\mathrm{cn}\,\alpha\,\mathrm{dn}\,\alpha)/(1-\kappa^2\,\mathrm{sn}^2\,\alpha\,\mathrm{sn}^2\beta)\,. \tag{6}$$

Let us set

$$a=\mathrm{sn}^2\alpha\,,\qquad b=\mathrm{sn}\,\alpha\,\mathrm{cn}\,\alpha\,\mathrm{dn}\,\alpha\,,\qquad w=\mathrm{sn}^2\beta\,,\qquad z=\mathrm{sn}^2(\alpha+\beta)\,,$$

and suppose that $\dot{x}=dx/d\beta$. Then, squaring the members of (6) on both sides, we have the formula (3).

§ 4. Transcendental constants.

Let λ be an element of k different from 0, and δ be a constant of k different from 0, ± 1. Then, we have the following:

PROPOSITION. *Assume that any nonsingular solution of $(y')^2=\lambda R(y)$ is transcendental over k, and that $K=k(z, z')$ with $(z')^2=\lambda R(z)$. Then, any transcendental element of K over k is not a constant.*

PROOF. Without losing the generality we may assume that $\lambda=1$. Let us prove that $\nu_P(\tau')=0$ for each prime divisor P of K, where τ is a prime element in P. If we suppose that $z=\tau^{-1}$, then

$$z'=\pm\tau^{-2}(1-\delta\tau/2+\cdots)=-\tau^{-2}\tau'.$$

If we suppose that $z=\tau^2+d$ with a root d of R, then

$$z'=h\tau\{1+R''(d)\tau^2/(4h^2)+\cdots\}=2\tau\tau', \qquad h^2=R'(d).$$

If we suppose that $z=\tau+\rho$ with an element ρ of k which is not a root of R, then

$$z'=\pm\zeta\{1+R'(\rho)\tau/(2\zeta^2)+\cdots\}=\tau'+\rho', \qquad \zeta^2=R(\rho).$$

By our assumption, $(\rho')^2\neq\zeta^2$. Hence, $\nu_P(\tau')=0$ for every prime divisor P of K. Take a transcendental element u of K over k. Then, we have $u=\tau^e$, $e\neq 0$ with a prime element τ in some prime divisor P of K. Suppose that u is a constant. Then, $e\tau^{e-1}\tau'=0$, and $\nu_P(\tau')>0$. This is a contradiction. Hence, any transcendental element of K over k is not a constant.

EXAMPLE. For k take the algebraic closure of the one-dimensional rational function field $k_0(X)$ over an algebraically closed field k_0 of characteristic zero. We set $X'=1$, and $c'=0$ for all elements c of k_0. Suppose that

$$F(y, y')=(y')^2-R(y-Xy')\,;\ \delta^2\neq 0,\ 1\,;\ \delta\in k_0.$$

Let η be a generic point of the general solution of F over k, and u denote $\eta-X\eta'$. Since the degree of F in y' is 4, u is not contained in k. We have $K=k(\eta, \eta')=k(u, \eta')$, and the genus of K is one. Since η is a nonsingular solution of $F=0$, $2\eta'+XR_u(u)\neq 0$. Because of $u'=X\eta''$, $\eta''=0$ and $u'=0$. Hence, K is free from parametric singularities. Suppose that θ and γ are constants of k, and that $\theta\neq 0$, $\gamma^2\neq 0$, 1. Then, any nonsingular solution of

$$(w')^2=\theta R(w\,;\gamma)=\theta w(w^2-1)(w-\gamma)$$

is not an element of k (cf. [**9**]). From our Theorem P it follows that we have $K=k(z, z')$ and $(z')^2=\lambda R(z\,;\gamma)$ for some γ with a certain multiplier λ. By Proposition, the multiplier λ can not be a constant for any z, because the constant u is transcendental over k.

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Michihiko MATSUDA
Department of Mathematics
Osaka University
Toyonaka, Osaka 560
Japan