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On curvature properties of Kähler *C*-spaces

By Mitsuhiro ITOH

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Introduction.

Hermitian symmetric spaces play an important role in Kähler geometry. These spaces of compact type admit nonnegative sectional curvature (Helgason [6]). It is also known that an operator Q associated with the curvature tensor has at most two eigenvalues for each irreducible hermitian symmetric space of compact type (Calabi and Vesentini [5] and Borel [2]).

An irreducible hermitian symmetric space of compact type is a typical example of Kähler C-spaces. By a C-space we mean a compact simply connected complex homogeneous space (Wang [13]) and by a Kähler C-space a C-space M which admits a Kähler metric such that a group of holomorphic isometries is transitive on M.

The purpose of this paper is to discuss, for a Kähler C-space,

(i) some properties of curvature tensor R,

(ii) the positivity of the holomorphic sectional curvature and

(iii) a relation between the hermitian symmetry and the number of eigenvalues of Q.

Our main results assert that a Kähler C-space of the second Betti number $b_2=1$ is of strictly positive holomorphic curvature under a certain condition (Theorem 3.1) and that a Kähler C-space of $b_2=1$ is hermitian symmetric if and only if the operator Q has at most two eigenvalues (Theorem 5.2).

In addition to irreducible symmetric spaces of compact type, Kähler C-spaces associated with $(\mathfrak{g}, \alpha_i)$, where $\mathfrak{g}$ is a complex simple Lie algebra of classical type, afford countably many typical examples of compact Kähler manifolds of positive holomorphic curvature.

Theorem 5.2 gives a complete characterization for a Kähler C-space of $b_2=1$ to be hermitian symmetric in view of an arithmetical property of the curvature.

Relative to Theorem 5.2, the following problem naturally arises; whether a compact Kähler manifold is hermitian symmetric when the operator Q has at most two different and constant eigenvalues.

§1 treats of a description of C-spaces with the aid of structure of a semi-simple Lie algebra. An invariant Kähler metric is described in Proposition 2.1 in § 2 in terms of a root system.

A connection function Λ associated with an invariant Riemannian connection (cf., Nomizu [**11**]) is computed explicitly in Proposition 2.2, and Proposition 2.4 gives the components of the curvature tensor R.

In § 3, we give Theorem 3.1, which asserts that for any Kähler C-space associated with $(\mathfrak{g}, \alpha_i)$ under the condition $\Delta_i^+(k)=\emptyset$, $k\geqq 3$, the holomorphic sectional curvature is strictly positive.

The operator Q, called the curvature operator, on $\mathfrak{m}^+\cdot\mathfrak{m}^+$ (the symmetric tensor product of $\mathfrak{m}^+$) is defined in § 4. Theorem 4.1 asserts that the Λ-invariance of eigenspaces of Q is equivalent to the hermitian symmetry. Representation theory of complex semi-simple Lie algebras together with the properties of the curvature tensor makes clear relations between eigenvalues of Q and weight vectors of the representation $(\mathfrak{m}^+\cdot\mathfrak{m}^+, ad_{\mathfrak{l}'})$ (cf., Propositions 4.2, 4.3 and 4.4). Proposition 4.5 deals with a key condition on the Λ-invariance of eigenspaces of Q.

§ 5 is devoted to show Theorem 5.2. By using classification of complex simple Lie algebras together with the propositions in § 4, we clarify which is a hermitian symmetric space (Theorem 5.1).

Theorem 5.2 is a consequence by the combination of Theorem 5.1 and of the number of eigenvalues of Q.

§ 1. Description of C-spaces.

In our attempt to study the structures of C-spaces, first we recall some facts about complex semi-simple Lie algebras.

Let $\mathfrak{g}$ and $\mathfrak{h}$ respectively be a complex semi-simple Lie algebra and its Cartan subalgebra. Put $l=\dim_C\mathfrak{h}$. Δ denotes the set of nonzero roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Then we have $\mathfrak{g}=\mathfrak{h}+\sum_{\alpha} \boldsymbol{C}E_\alpha$, where E_α is a root vector of a root α. Let B be the Killing form of $\mathfrak{g}$. It is nondegenerate, since $\mathfrak{g}$ is semi-simple. For $\xi\in\mathfrak{h}^*=\mathrm{Hom}(\mathfrak{h}: \boldsymbol{C})$ we define $H_\xi\in\mathfrak{h}$ by $B(H, H_\xi)=\xi(H)$ for all $H\in\mathfrak{h}$. We define a bilinear form $(\cdot,\cdot)$ on $\mathfrak{h}^*$ by $(\xi, \eta)=B(H_\xi, H_\eta)$, $\xi, \eta\in\mathfrak{h}^*$. Fix a suitable lexicographic ordering among roots of Δ with respect to some fundamental root system $\Pi=\{\alpha_1, \cdots, \alpha_l\}$. By Δ^+ and Δ^- we mean the subsets of positive and negative roots, respectively. Every positive (resp. negative) root is given by a linear combination of $\alpha_1, \cdots, \alpha_l$, whose coefficients are nonnegative (resp. nonpositive) integers.

We can choose root vectors $\{E_\alpha;\ \alpha\in\Delta\}$ so as $\{H_j=H_{\alpha_j},\ j=1, \cdots, l,\ E_\alpha;\ \alpha\in\Delta\}$ to be a so-called Weyl's canonical basis of $\mathfrak{g}$, namely,

$$(1.1)\qquad \begin{aligned} &B(E_\alpha, E_{-\alpha}) = -1\,, \qquad \alpha \in \Delta^+\,, \\ &[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}\,, \qquad N_{\alpha,\beta} = N_{-\alpha,\,-\beta} \in \boldsymbol{R}\,. \end{aligned}$$

Then, $\sum_{j=1}^{l} \boldsymbol{R}\sqrt{-1}H_j + \sum_{\alpha\in\Delta^+} (\boldsymbol{R}A_\alpha + \boldsymbol{R}B_\alpha)$, denoted by $\mathfrak{g}_u$, is a compact real form of $\mathfrak{g}$, where $A_\alpha = E_\alpha + E_{-\alpha}$, $B_\alpha = \sqrt{-1}(E_\alpha - E_{-\alpha})$, $\alpha \in \Delta^+$. The complex conjugation on $\mathfrak{g}$ with respect to $\mathfrak{g}_u$ satisfies the following; $\bar{E}_\alpha = E_{-\alpha}$, $\bar{E}_{-\alpha} = E_\alpha$, $\alpha \in \Delta^+$ and $\bar{H}_j = -H_j$, $j = 1, \cdots, l$.

From now on, $\mathfrak{g}$ is assumed to be a complex simple Lie algebra. Consider a subset Φ of Π; $\Phi = \{\alpha_{i_1}, \cdots, \alpha_{i_r}\}$.

Let $\Delta^+(\Phi)$ be defined by

$$(1.2)\qquad \Delta^+(\Phi) = \{\alpha = \sum_{j=1}^{l} n_j \alpha_j \in \Delta^+\,;\ n_{i_k} > 0 \quad \text{for some} \quad \alpha_{i_k} \in \Phi\}\,.$$

$\mathfrak{l}_\Phi = \mathfrak{h} + \sum_{\alpha\in\Delta^+} \boldsymbol{C}E_{-\alpha} + \sum_{\alpha\in\Delta^+-\Delta^+(\Phi)} \boldsymbol{C}E_\beta$ defines a parabolic subalgebra of $\mathfrak{g}$, and the intersection $\mathfrak{g}_u \cap \mathfrak{l}_\Phi$ which is denoted by $\mathfrak{k}_\Phi$, or simply $\mathfrak{k}$, is a real subalgebra of $\mathfrak{g}_u$;

$$(1.3)\qquad \mathfrak{k}_\Phi = \sum_{j=1}^{l} \boldsymbol{R}\sqrt{-1}H_j + \sum_{\alpha\in\Delta^+-\Delta^+(\Phi)} (\boldsymbol{R}A_\alpha + \boldsymbol{R}B_\alpha)\,.$$

Let G be a simply connected complex Lie group whose Lie algebra is $\mathfrak{g}$ and L_Φ a connected closed complex subgroup of G generated by $\mathfrak{l}_\Phi$.

Let G_u and K_Φ, or simply K, be a simply connected group and its connected closed subgroup which correspond to $\mathfrak{g}_u$ and $\mathfrak{k}_\Phi$ respectively.

The canonical imbedding $G_u \to G$ gives a diffeomorphism of a compact coset space $M = G_u/K_\Phi$ to a simply connected complex coset space G/L_Φ.

We have from Borel and Hirzebruch [**3**],

$H^2(M; R) = H^1(K_\Phi; R) =$ center of $\mathfrak{k}_\Phi = \sum_{k=1}^{r} \boldsymbol{R}\sqrt{-1}H_{\Lambda_k}$, where $\Lambda_k \in \mathfrak{h}^*$ is defined by $2(\Lambda_k, \alpha_j) = (\alpha_j, \alpha_j)\delta_{i_k j}$ for all j. Thus we have the second Betti number $b_2(M) = r$.

Hence we obtain a C-space G_u/K of $b_2 = r$ from a pair $(\mathfrak{g}, \Phi)$, where $\mathfrak{g}$ is a complex simple Lie algebra and Φ is a subset of Π.

$(G_u/K_\Phi, g)$ is, then, a Kähler C-space, when G_u/K_Φ admits a G_u-invariant Kähler metric g.

Conversely, a Kähler C-space of $b_2 = r$ can be described by a coset space G_u/K_Φ for some G_u and K_Φ, with a G_u-invariant Kähler metric g.

We remark that every compact homogeneous Kähler manifold is a Kählerian direct product of a Kähler C-space and a flat complex torus (Matsushima [**9**]). By a homogeneous Kähler manifold we mean a Kähler manifold on which the group of holomorphic isometric transformations is transitive.

Define a linear subspace $\mathfrak{m}$ of $\mathfrak{g}_u$ as follows;

$$\mathfrak{m}=\sum_{\alpha\in\Delta^+(\Phi)}(\boldsymbol{R}A_\alpha+\boldsymbol{R}B_\alpha). \tag{1.4}$$

Then we have $\mathfrak{g}_u=\mathfrak{k}_\Phi+\mathfrak{m}$ as a direct sum and $[\mathfrak{k}_\Phi,\mathfrak{k}_\Phi]\subset\mathfrak{k}_\Phi$, $[\mathfrak{k}_\Phi,\mathfrak{m}]\subset\mathfrak{m}$ and $\mathfrak{k}_\Phi\perp\mathfrak{m}$ with respect to B.

Since K leaves fixed the origin of a C-space $M=G_u/K$, K acts on the tangent space at the origin as the linear isotropy. We can naturally identify the tangent space with $\mathfrak{m}$ such that the identification commutes with the action of K on the tangent space as the linear isotropy and the adjoint representation of K on $\mathfrak{m}$.

Let a complex structure I of $\mathfrak{m}$ be defined by $IA_\alpha=B_\alpha$, $IB_\alpha=-A_\alpha$, $\alpha\in\Delta^+(\Phi)$. This gives a G_u-invariant complex structure on G_u/K and coincides with the canonical structure induced from the complex coset space G/L.

Denote by $\mathfrak{k}^C$ and $\mathfrak{m}^C$ the complexifications of $\mathfrak{k}$ and $\mathfrak{m}$ respectively. These are complex linear subspaces of $\mathfrak{g}=\mathfrak{g}_u{}^C$;

$$\begin{aligned}\mathfrak{h}^C&=\mathfrak{l}+\sum_{\alpha\in\Delta^+-\Delta^+(\Phi)}(\boldsymbol{C}E_\alpha+\boldsymbol{C}E_{-\alpha}),\\ \mathfrak{m}^C&=\sum_{\alpha\in\Delta^+(\Phi)}(\boldsymbol{C}E_\alpha+\boldsymbol{C}E_{-\alpha}).\end{aligned} \tag{1.5}$$

$\mathfrak{m}^C$ is decomposed into subspaces $\mathfrak{m}^+=\sum_{\alpha\in\Delta^+(\Phi)}\boldsymbol{C}E_\alpha$ and $\mathfrak{m}^-=\sum_{\alpha\in\Delta^+(\Phi)}\boldsymbol{C}E_{-\alpha}$. We have, $\mathfrak{m}^+=\{Z\in\mathfrak{m}^C;\ IZ=\sqrt{-1}Z\}$ and $\mathfrak{m}^-=\{Z\in\mathfrak{m}^C;\ IZ=-\sqrt{-1}Z\}$. And we have the following representations which are associated with the adjoint representation; $(\mathfrak{m}^+, ad_{\mathfrak{l}}{}^C)$ and $(\mathfrak{m}^-, ad_{\mathfrak{l}}{}^C)$, since $\alpha+\beta$ belongs to $\Delta^+(\Phi)$ for $\alpha\in\Delta^+(\Phi)$, $\beta\in\Delta-\Delta^+(\Phi)$, if $\alpha+\beta$ is a root. To any element $\xi\in\sum_j Z\alpha_j$ $(\supset\Delta)$, we assign an element $n(\xi)$ of $Z^r=Z\times Z\times\cdots\times Z$ (r-times), by $n(\xi)=(n_{i_1},\cdots,n_{i_r})$ if $\xi=\sum_j n_j\alpha_j$. Then we have, $n(\alpha)\in Z^{+r}=Z^+\times\cdots\times Z^+$ for $\alpha\in\Delta^+(\Phi)$, where Z^+ denotes the set of nonnegative integers. By $\Delta^+(\Phi;n)$ we mean a subset $\{\alpha\in\Delta^+(\Phi);\ n(\alpha)=n\}$ of $\Delta^+(\Phi)$ for $n\in Z^{+r}$. Then $\mathfrak{m}^{+n}=\sum_{\alpha\in\Delta^+(\Phi;\,n)}\boldsymbol{C}E_\alpha$ and $\mathfrak{m}^{-n}=\sum_{\alpha\in\Delta^+(\Phi;\,n)}\boldsymbol{C}E_{-\alpha}=\overline{\mathfrak{m}^{+n}}$ give linear subspaces of $\mathfrak{m}^+$ and $\mathfrak{m}^-$ respectively. They satisfy the following properties, from the linearity of $n(\alpha)$ with respect to $\alpha\in\Delta^+(\Phi)$;

$$\begin{aligned}&[\mathfrak{k}^C,\mathfrak{m}^\pm]\subset\mathfrak{m}^\pm,\\ &[\mathfrak{m}^{\pm n},\mathfrak{m}^{\pm n'}]\subset\mathfrak{m}^{\pm(n+n')},\\ &[\mathfrak{m}^{+n},\mathfrak{m}^{-n'}]\subset\mathfrak{m}^{+(n-n')} && (n>n'),\\ &\qquad\qquad\subset\mathfrak{k}^C && (n=n'),\\ &\qquad\qquad\subset\mathfrak{m}^{-(n'-n)} && (n<n'),\\ &\qquad\qquad=\{0\} && (\text{otherwise}),\end{aligned} \tag{1.6}$$

here $n>n'$ means that $n_k\geqq n'_k$ for all $k=1,\cdots,r$ and $n_j>n'_j$ for some j; $n=(n_j)_{j=1,\cdots,r}$, $n'=(n'_j)_{j=1,\cdots,r}$.

Since $ad(\mathfrak{k}^C)\mathfrak{m}^{+n}\subset\mathfrak{m}^{+n}$, we have a decomposition of the adjoint representation $(\mathfrak{m}^+, ad_{\mathfrak{k}}{}^C)$; $(\mathfrak{m}^+, ad_{\mathfrak{k}}{}^C)=\sum_{n\in Z+r}(\mathfrak{m}^{+n}, ad_{\mathfrak{k}}{}^C)$.

§ 2. Invariant Kähler metric and invariant curvature tensor.

Let G_u/K_Φ be a C-space associated with $(\mathfrak{g}, \Phi)$, where $\mathfrak{g}$ is a complex simple Lie algebra and Φ is a nonempty subset of Π.

Suppose that G_u/K_Φ admits a G_u-invariant Kähler metric g.

Kähler metrics, linear connections and tensors are regarded as ones extended naturally over complex numbers $\boldsymbol{C}$.

Since the Kähler metric g is G_u-invariant, a connection function Λ: $\mathfrak{m}^C\times\mathfrak{m}^C\to\mathfrak{m}^C$, which is associated with the Riemannian connection is given (Kobayashi and Nomizu [8] and Nomizu [**11**]) by

$$\Lambda(X)(Y)=\frac{1}{2}[X, Y]_{\mathfrak{m}^C}+U(X, Y), \qquad X, Y\in\mathfrak{m}^C, \tag{2.1}$$

where $[X, Y]_{\mathfrak{m}^C}$ denotes the $\mathfrak{m}^C$-part of $[X, Y]$ and $U(\cdot, \cdot)$ is a symmetric bilinear mapping, which is defined by

$$2g(U(X, Y), Z)=g([Z, X]_{\mathfrak{m}^C}, Y)+g(X, [Z, Y]_{\mathfrak{m}^C}) \qquad X, Y \text{ and } Z\in\mathfrak{m}^C. \tag{2.2}$$

We use the same symbol g for the Kähler metric g restricted to the origin.

The connection function Λ gives the curvature tensor R as follows (Nomizu [**11**]),

$$\begin{aligned} R(X, Y)Z=&[\Lambda(X), \Lambda(Y)]Z-\Lambda([X, Y]_{\mathfrak{m}^C})Z \\ &-[[X, Y]_{\mathfrak{k}^C}, Z], \qquad X, Y \text{ and } Z\in\mathfrak{m}^C, \end{aligned} \tag{2.3}$$

here $[X, Y]_{\mathfrak{k}^C}$ denotes the $\mathfrak{k}^C$-part of $[X, Y]$.

Since g, Λ and R are $ad_{\mathfrak{k}^C}$-invariant at the origin, we have,

$$g([V, X], Y)+g(X, [V, Y])=0, \tag{2.4}$$

$$[V, \Lambda(X)Y]=\Lambda([V, X])Y+\Lambda(X)[V, Y] \tag{2.5}$$

and

$$\begin{aligned} &[V, R(X, Y)Z]=R([V, X], Y)Z+R(X, [V, Y])Z+R(X, Y)[V, Z], \\ &\qquad V\in\mathfrak{k}^C \text{ and } X, Y \text{ and } Z\in\mathfrak{m}^C. \end{aligned} \tag{2.6}$$

The following theorem is known with respect to the invariance of a Kähler metric g on a C-space G_u/K_Φ.

THEOREM (Borel [**1**]). *Let g be a G_u-invariant Kähler metric on a C-space G_u/K_Φ. Then g is given at the origin by*

$$g=2\sum_{\alpha\in\Delta^+(\Phi)} c_\alpha\omega^\alpha\cdot\omega^{\bar\alpha}\,,\qquad \omega^\alpha\cdot\omega^{\bar\alpha}=\frac{1}{2}(\omega^\alpha\otimes\omega^{\bar\alpha}+\omega^{\bar\alpha}\otimes\omega^\alpha)\,,\quad c_\alpha>0\,, \tag{2.7}$$

$$\alpha\in\Delta^+(\Phi)\,,\quad c_\alpha+c_\beta=c_{\alpha+\beta}\quad \textit{if}\quad \alpha,\beta\quad \textit{and}\quad \alpha+\beta\in\Delta^+(\Phi)\quad \textit{and}$$

$$c_\alpha=c_{\alpha+\gamma}\quad \textit{if}\quad \alpha,\alpha+\gamma\in\Delta^+(\Phi)\quad \textit{and}\quad \gamma\in\Delta-\Delta^+(\Phi)\,.$$

Conversely, if a bilinear form $2\sum_\alpha c_\alpha\omega^\alpha\cdot\omega^{\bar\alpha}$, *which satisfies* (2.7), *then it can be extended to a* G_u*-invariant Kähler metric on* G_u/K.

Here, ω^α *and* $\omega^{\bar\alpha}$ *are the dual of* E_α *and* $E_{-\alpha}$, $\alpha\in\Delta^+(\Phi)$ *respectively.*

Note that the condition $c_\alpha+c_\beta=c_{\alpha+\beta}$ is derived from the d-closedness of the fundamental form, which is associated with g and $c_\alpha=c_{\alpha+\gamma}$ is obtained by the $ad_{\mathfrak{t}^C}$-invariance of the metric g.

From the above theorem we obtain a precise description of the Kähler metric g in terms of roots as follows.

PROPOSITION 2.1. *Under the same notations as in the theorem, g is given at the origin by*

$$g=2\sum_{\alpha\in\Delta^+(\Phi)}\sum_{j=1}^{r} c_j\cdot n_{i_j}(\alpha)\omega^\alpha\cdot\omega^{\bar\alpha},\quad \text{for some}\quad c_j>0\,,\qquad j=1,\cdots,r\,. \tag{2.8}$$

Here $n_k(\alpha)$ *means the* α_k*-coefficient of* α.

Conversely, if there is a bilinear form $2\sum_\alpha c_\alpha\omega^\alpha\cdot\omega^{\bar\alpha}$ *on* $\mathfrak{m}^C\times\mathfrak{m}^C$ *such that* $c_\alpha=\sum c_j\cdot n_{i_j}(\alpha)$, *then it can be extended to an invariant Kähler metric.*

NOTE. By $c\cdot n$ we denote $\sum_{j=1}^r c_j\cdot n_j$ for $c=(c_j)_{j=1,\cdots,r}$ and $n=(n_j)_{j=1,\cdots,r}$. Then (2.8) can be written in the simple form; $g=2\sum_{\alpha\in\Delta^+(\Phi)} c\cdot n(\alpha)\omega^\alpha\cdot\omega^{\bar\alpha}$.

PROOF. If a Kähler metric g is G_u-invariant, then, from the above theorem g can be written as $g=2\sum_\alpha c_\alpha\omega^\alpha\cdot\omega^{\bar\alpha}$ such that the coefficients c_α's satisfy (2.7). We set $c_\alpha=0$ for $\alpha\in\Delta^+-\Delta^+(\Phi)$. Then $c_\alpha+c_\beta=c_{\alpha+\beta}$ for all α and $\beta\in\Delta^+$. Put $c_j=c_{\alpha_{i_j}}$ for each simple root $\alpha_{i_j}\in\Phi$, $j=1,\cdots,r$. Since any root α of $\Delta^+(\Phi)$ which is not simple is a sum of positive roots, the linearity of c_α shows the first part of the proposition inductively with respect to $\sum_j n_{i_j}(\alpha)$, $\alpha\in\Delta^+(\Phi)$. The last part is easily verified by the above theorem. Q.E.D.

From this proposition, we have

$$g|_{\mathfrak{m}^{+n}\times\mathfrak{m}^{-n}}=c\cdot n(-B)|_{\mathfrak{m}^{+n}\times\mathfrak{m}^{-n}}\,,\qquad n\in Z^{+r}\,, \tag{2.9}$$

and the set of G_u-invariant Kähler metrics on a Kähler C-space is parametrized by r parameters, $c_1,\cdots,c_r$.

Now we shall show the following proposition which gives an explicit expression of the connection function Λ in terms of the bracket operation $[\cdot,\cdot]$.

PROPOSITION 2.2. *The connection function* Λ *satisfies that*

(2.10) $$\Lambda(X)Y=\frac{c\cdot n'}{c\cdot(n+n')}[X, Y]_{\mathfrak{m}^+}, \qquad X\in\mathfrak{m}^{+n}, \qquad Y\in\mathfrak{m}^{+n'},$$

$$\Lambda(\bar{X})Y=[\bar{X}, Y]_{\mathfrak{m}^+}, \qquad X \text{ and } Y\in\mathfrak{m}^+ .$$

To prove the proposition we shall verify the following lemma.

LEMMA 2.3. *The symmetric bilinear mapping $U(\cdot, \cdot)$ defined by* (2.2) *satisfies the following, for $X\in\mathfrak{m}^{+n}$ and $Y\in\mathfrak{m}^{+n'}$,*

(2.11) $$U(X, Y)=\frac{c\cdot(n'-n)}{2c\cdot(n+n')}[X, Y]_{\mathfrak{m}^+},$$

$$U(\bar{X}, Y)=\frac{1}{2}[\bar{X}, Y]_{\mathfrak{m}^C}, \qquad n<n',$$

$$=-\frac{1}{2}[\bar{X}, Y]_{\mathfrak{m}^C}, \qquad n>n',$$

$$=0, \qquad \textit{otherwise}.$$

PROOF OF LEMMA. Paying a regard to the relations (2.9), $B(\mathfrak{k}^C, \mathfrak{m}^C)=0$ and $g(\mathfrak{m}^+, \mathfrak{m}^+)=0$, we verify the lemma.

For $X\in\mathfrak{m}^{+n}$, $Y\in\mathfrak{m}^{+n'}$ and $Z\in\mathfrak{m}^C$, we have in (2.2)

$$2g(U(X, Y), Z)=c\cdot(n'-n)(-B)([X, Y]_{\mathfrak{m}^C}, Z).$$

Since $[X, Y]_{\mathfrak{m}^C}\in[\mathfrak{m}^{+n}, \mathfrak{m}^{+n'}]\subset\mathfrak{m}^{+(n+n')}$, it follows that

$$2g(U(X, Y), Z)=\frac{c\cdot(n'-n)}{c\cdot(n+n')}g([X, Y]_{\mathfrak{m}^C}, Z) \quad \text{for all} \quad Z\in\mathfrak{m}^C.$$

To prove the second, fix $X\in\mathfrak{m}^{+n}$ and $Y\in\mathfrak{m}^{+n'}$. Then

$$2g(U(\bar{X}, Y), Z)=c\cdot(n'-n)(-B)([\bar{X}, Y], Z).$$

If $n>n'$, then $[\bar{X}, Y]\in\mathfrak{m}^{-(n-n')}$ $(n-n'>0)$. Therefore

$$2g(U(\bar{X}, Y), Z)=\frac{c\cdot(n'-n)}{c\cdot(n-n')}g([\bar{X}, Y]_{\mathfrak{m}^C}, Z)=-g([\bar{X}, Y]_{\mathfrak{m}^C}, Z).$$

And if $n<n'$, then $[\bar{X}, Y]\in\mathfrak{m}^{+(n'-n)}$ $(n'-n>0)$. It follows then that

$$2g(U(\bar{X}, Y), Z)=\frac{c\cdot(n'-n)}{c\cdot(n'-n)}g([\bar{X}, Y]_{\mathfrak{m}^C}, Z)=g([\bar{X}, Y]_{\mathfrak{m}^C}, Z).$$

For the other cases, we have clearly $[\bar{X}, Y]_{\mathfrak{m}^C}=0$ from (1.6).
Together with these arguments we have the last part of (2.11). Q.E.D.

Now we shall show the proposition. For $X\in\mathfrak{m}^{+n}$ and $Y\in\mathfrak{m}^{+n'}$, we have from (2.11),

$$\Lambda(X)Y=\frac{1}{2}[X, Y]_{\mathfrak{m}^C}+\frac{c\cdot(n'-n)}{2c\cdot(n+n')}[X, Y]_{\mathfrak{m}^C}=\frac{c\cdot n'}{c\cdot(n+n')}[X, Y]_{\mathfrak{m}^C}.$$

Thus, the first part of (2.10) is obtained.

To prove the last part, we set $X=\sum_n X^n$ and $Y=\sum_n Y^n$, $X^n, Y^n \in \mathfrak{m}^{+n}$ for $X, Y \in \mathfrak{m}^+$.

Then it follows from (1.6) and (2.11) that

$$\begin{aligned} \Lambda(\bar{X})Y &= \frac{1}{2}[\bar{X}, Y]_{\mathfrak{m}^C} + U(\bar{X}, Y) \\ &= \frac{1}{2}\Big(\sum_{n>n'} + \sum_{n<n'}\Big)[\bar{X}^n, Y^{n'}]_{\mathfrak{m}^C} \\ &\quad + \Big(\sum_{n>n'} + \sum_{n<n'}\Big)U(\bar{X}^n, Y^{n'}) \\ &= \sum_{n<n'}[\bar{X}^n, Y^{n'}]_{\mathfrak{m}^C} = [\bar{X}, Y]_{\mathfrak{m}^+}. \end{aligned} \qquad \text{Q.E.D.}$$

NOTE. From the proposition, we have, $\Lambda(X)\mathfrak{m}^{\pm} \subset \mathfrak{m}^{\pm}$ and $R(X, Y)\mathfrak{m}^{\pm} \subset \mathfrak{m}^{\pm}$, for X and $Y \in \mathfrak{m}^C$. Moreover it follows from (2.10) together with (2.3) that $R(X, Y)Z=0$ for any $X, Y \in \mathfrak{m}^{\pm}$ and $Z \in \mathfrak{m}^C$.

By using the so-called first Bianchi's identity together with the following relation; $g(R(X, \bar{Y})Z, \bar{W})+g(R(X, \bar{Y})\bar{W}, Z)=0$, we have the symmetric property of the curvature tensor;

(2.12) $\quad R(X, \bar{Y}, Z, \bar{W})=R(Z, \bar{Y}, X, \bar{W})=R(X, \bar{W}, Z, \bar{Y}), \qquad X, Y, Z$ and $W \in \mathfrak{m}^+$,

where $$R(X, \bar{Y}, Z, \bar{W})=g(R(X, \bar{Y})Z, \bar{W}).$$

By the aid of the expression of the connection function Λ, we have the following proposition which is concerned with the curvature tensor R.

PROPOSITION 2.4. *The components* $R_{\alpha\bar{\beta}\gamma\bar{\delta}}=R(E_\alpha, E_{-\beta}, E_\gamma, E_{-\delta})$ of R *with respect to the basis* $\{E_\alpha;\ \alpha \in \Delta^+(\Phi)\}$ *of* $\mathfrak{m}^+$ *satisfy the following.*

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}}=0 \quad \text{if} \quad n(\alpha+\gamma) \neq n(\beta+\delta) \tag{2.13}$$

and

$$R_{\alpha\bar{\alpha}\beta\bar{\beta}}=c\cdot n(\alpha)\Big\{(\alpha, \beta)+\frac{c\cdot n(\alpha)}{c\cdot n(\alpha+\beta)}(N_{\alpha,\beta})^2\Big\}, \qquad n(\alpha) \leqq n(\beta), \tag{2.14}$$

$$=c\cdot n(\beta)\Big\{(\alpha, \beta)+\frac{c\cdot n(\beta)}{c\cdot n(\alpha+\beta)}(N_{\alpha,\beta})^2\Big\}, \qquad \textit{otherwise}.$$

PROOF. We have from Proposition 2.2,

$R(X, Y)Z \in \mathfrak{m}^{+(n-n'+m)}$ for $X \in \mathfrak{m}^{+n}$, $Y \in \mathfrak{m}^{+n'}$ and $Z \in \mathfrak{m}^{+m}$. Hence, (2.13) is obtained.

Now we shall show (2.14). Since $[E_\alpha, E_{-\alpha}] \in \mathfrak{k}^C$, we have

$$\begin{aligned} R_{\alpha\bar{\alpha}\beta\bar{\beta}} &= g(\Lambda(E_\alpha)\Lambda(E_{-\alpha})E_\beta, E_{-\beta}) - g(\Lambda(E_{-\alpha})\Lambda(E_\alpha)E_\beta, E_{-\beta}) \\ &\quad - g([[E_\alpha, E_{-\alpha}], E_\beta], E_{-\beta}). \end{aligned}$$

From (2.11), each term is turned into the following form;

the first term; $c\cdot n(\beta-\alpha)(-B)([E_\alpha, [E_{-\alpha}, E_\beta]_{\mathfrak{m}^+}], E_{-\beta})$,

the second term; $\dfrac{\{c\cdot n(\beta)\}^2}{c\cdot n(\alpha+\beta)}N^2_{\alpha,\beta}$,

the last term; $c\cdot n(\beta)(\alpha, \beta)$.

If $n(\alpha)\leqq n(\beta)$, then $n(\beta-\alpha)\in Z^{+r}$, that is, $[E_{-\alpha}, E_\beta]\in\mathfrak{m}^+$. The first term is made into the following simple form; $c\cdot n(\beta-\alpha)\{-(\alpha, \beta)-N^2_{\alpha,\beta}\}$.

If not $n(\alpha)\leqq n(\beta)$, then $[E_{-\alpha}, E_\beta]\notin\mathfrak{m}^+$. The first term vanishes in this case. Together with these computations, (2.14) is obtained. Q.E.D.

The Ricci curvature tensor S of a Kähler C-space $(G_u/K_\Phi, g)$ is defined by $S(X, Y)$=trace of an endomorphism of $\mathfrak{m}^C$; $Z\to R(X, Y)Z$ for $X, Y\in\mathfrak{m}^C$.

Since $\left\{\dfrac{1}{\sqrt{c\cdot n(\alpha)}}E_\alpha, \alpha\in\Delta^+(\Phi)\right\}$ constitutes a unitary basis, S is given by

$$S(E_\alpha, E_{-\alpha})=2\sum_{\beta\in\Delta^+(\Phi)}\frac{1}{c\cdot n(\beta)}R_{\alpha\bar\alpha\beta\bar\beta}.$$

Thus, we have from the proposition,

$$S(E_\alpha, E_{-\alpha})=2\sum_{n(\beta)\leqq n(\alpha)}\left\{(\alpha, \beta)+\frac{c\cdot n(\beta)}{c\cdot n(\alpha+\beta)}N^2_{\alpha,\beta}\right\} \tag{2.15}$$
$$+2\sum_{n(\beta)\geqq n(\alpha)}\frac{c\cdot n(\alpha)}{c\cdot n(\beta)}\left\{(\alpha, \beta)+\frac{c\cdot n(\alpha)}{c\cdot n(\alpha+\beta)}N^2_{\alpha,\beta}\right\}.$$

REMARK. If a Kähler C-space $(G_u/K_\Phi, g)$ associated with $(\mathfrak{g}, \Phi)$ is a normal homogeneous space, that is, g is given by an adjoint invariant bilinear form of $\mathfrak{g}_u$, then Φ consists of a single simple root and $[\mathfrak{m}^+, \mathfrak{m}^+]=0$. Since an adjoint invariant bilinear form is a scalar multiple of the Killing form B, $g=2\sum_n c\cdot n\sum_{\alpha\in\Delta^+(\Phi,n)}\omega^\alpha\cdot\omega^{\bar\alpha}$ coincides with $aB|_{\mathfrak{m}^C\times\mathfrak{m}^C}=-2a\sum_n\sum_{\alpha\in\Delta^+(\Phi,n)}\omega^\alpha\cdot\omega^{\bar\alpha}$ for some constant a. Hence we have $c\cdot n(\alpha)=-a$ for all $\alpha\in\Delta^+(\Phi)$, in particular $c_j=-a$ for each $\alpha_{i_j}\in\Phi, j=1, \cdots, r$. Suppose that $\sharp\Phi\geqq 2$. Then, there exists a root $\alpha\in\Delta^+(\Phi)$ such that $\alpha=\sum_j n_j\alpha_j$, $n_{i_j}>0$ and $n_{i_k}>0$ for some j and k. We have $-a=c\cdot n(\alpha)\geqq c_j+c_k=-2a$. This is a contradiction. Hence $\sharp\Phi=1$; $\Phi=\{\alpha_i\}$. Moreover, by the similar argument any root α of $\Delta^+(\Phi)$ takes one as its α_i-coefficient, that is, $\mathfrak{m}^+=\mathfrak{m}^{+1}$. Thus, $[\mathfrak{m}^+, \mathfrak{m}^+]=0$.

Conversely, a Kähler C-space associated with $(\mathfrak{g}, \Phi)$, $\Phi=\{\alpha_i\}$ such that $\mathfrak{m}^+=\mathfrak{m}^{+1}$, is a normal homogeneous space. Henceforth, it is a hermitian symmetric space, since $\mathfrak{g}_u=\mathfrak{k}+\mathfrak{m}$ gives an effective and orthogonal symmetric Lie algebra $(\mathfrak{g}_u, s)$ (Helgason [6]).

Let (M, g) be an irreducible hermitian symmetric space of compact type. By the help of the argument of Ise [7], (M, g) is given by a Kähler C-space

$(G_u/K, g)$, which is normal.

Since $[\mathfrak{m}^+, \mathfrak{m}^+]=0$ for a Kähler C-space which is normal, we have

$$R(X, Y)Z=[[X, Y]_{\mathfrak{k}^C}, Z], \qquad X, Y \text{ and } Z\in\mathfrak{m}^C. \tag{2.16}$$

(2.16) coincides with the usual expression of the curvature tensor of symmetric spaces (Helgason [6]).

§ 3. Curvature of Kähler C-spaces.

Let $(G_u/K, g)$ be a Kähler C-space associated with $(\mathfrak{g}, \Phi)$.

In this section, Φ is assumed to consist of a single simple root $\alpha_i \in \Pi$; $\Phi=\{\alpha_i\}$. By Δ_i^+ and $\Delta_i^+(k)$ we mean $\Delta^+(\{\alpha_i\})=\{\alpha=\sum_j n_j\alpha_j\in\Delta^+;\ n_i>0\}$ and $\Delta^+(\{\alpha_i\}, k)=\{\alpha=\sum_j n_j\alpha_j\in\Delta^+;\ n_i=k\}$ for positive integer k, respectively.

The invariant Kähler metric g and the connection function Λ can be written in this case as follows (a parameter $c=1$);

$$g=2\sum_k k \sum_{\alpha\in\Delta_i^+(k)} \omega^\alpha\cdot\omega^{\bar{\alpha}}=\sum_k k(-B)|_{\mathfrak{m}^{+k}\times\mathfrak{m}^{-k}}, \tag{3.1}$$

$$\Lambda(Z^j)W^k=\frac{k}{j+k}[Z^j, W^k]_{\mathfrak{m}^C}, \qquad Z^j\in\mathfrak{m}^{+j} \text{ and } W^k\in\mathfrak{m}^{+k}, \tag{3.2}$$

$$\Lambda(\bar{Z})W=[\bar{Z}, W]_{\mathfrak{m}^+}, \qquad Z, W\in\mathfrak{m}^+,$$

where $$\mathfrak{m}^+=\sum_{k\in Z}\mathfrak{m}^{+k}, \qquad \mathfrak{m}^{+k}=\sum_{\alpha\in\Delta_i^+(k)} CE_\alpha.$$

Relative to holomorphic sectional curvature of $(G_u/K, g)$, the following is obtained.

THEOREM 3.1. *Let $(G_u/K, g)$ be a Kähler C-space associated with $(\mathfrak{g}, \alpha_i)$. Suppose that $\Delta_i^+(k)=\emptyset$ for $k\geqq 3$. Then we have,*

(3.3) (*holomorphic bisectional curvature*)

$$H(X\wedge IX, Y\wedge IY)=-\frac{1}{2}\|[Z^1, W^1]\|^2+\|[Z^1, \bar{W}^1]+[Z^2, \bar{W}^2]\|^2+\|[Z^1, \bar{W}^2]\|^2$$
$$+\|[Z^2, \bar{W}^1]\|^2+\|[Z^2, \bar{W}^2]\|^2,$$

(3.4) (*holomorphic sectional curvature*)

$$H(X\wedge IX)=\|[Z^1, \bar{Z}^1]\|^2+2\|[Z^2, \bar{Z}^2]\|^2+4\|[Z^1, \bar{Z}^2]\|^2,$$

for $X, Y\in\mathfrak{m}$, where Z^k, W^k are $\mathfrak{m}^{+k}$-components of $Z=\frac{1}{\sqrt{2}}(X-\sqrt{-1}IX)$ and $W=\frac{1}{\sqrt{2}}(Y-\sqrt{-1}IY)$ respectively, $g(Z, \bar{Z})=g(W, \bar{W})=1$. By $\|Z\|^2$ we mean $-B(Z, \bar{Z})$.

Moreover, the Kähler C-space has strictly positive holomorphic sectional curvature.

In the above, the condition $\Delta_i^+(k)=\emptyset$ means that there is no root α of Δ_i^+, whose α_i-coefficient is equal to k.

By the aid of tables of roots of complex simple Lie algebras (Bourbaki [4]), the following is directly obtained from the theorem.

COROLLARY 3.2. *If a Kähler C-space $(G_u/K, g)$ is associated with one of the following pairs the holomorphic sectional curvature is strictly positive;* $(A_l, \alpha_i)_{i=1,\cdots,l,l\geqq 1}$, $(B_l,\alpha_i)_{i=1,\cdots,l,l\geqq 2}$, $(C_l, \alpha_i)_{i=1,\cdots,l,l\geqq 3}$, $(D_l, \alpha_i)_{i=1,\cdots,l,l\geqq 4}$, $(E_6, \alpha_i)_{i=1,2,3,5,6}$, $(E_7, \alpha_i)_{i=1,2,6,7}$, (E_8, α_1), (E_8, α_8), (F_4, α_1), (F_4, α_4) *and* (G_2, α_2).

PROOF OF THEOREM 3.1. From (3.2), we have, for Z and $W\in\mathfrak{m}^+$

$$\begin{aligned} g([\Lambda(Z), \wedge(\bar{Z})]W, \bar{W}) &= \frac{1}{2} g([Z^1, [\bar{Z}^1, W^2]_{\mathfrak{m}^+}]_{\mathfrak{m}^+}, \bar{W}^1+\bar{W}^2) \\ &\quad -\frac{1}{2} g([\bar{Z}^1, [Z^1. W^1]_{\mathfrak{m}^+}]_{\mathfrak{m}^+}, \bar{W}^1+\bar{W}^2) \\ &= -B([Z^1, [\bar{Z}^1, W^2]]_{\mathfrak{m}^+}, \bar{W}^2) \\ &\quad +\frac{1}{2} B([\bar{Z}^1, [Z^1, W^1]]_{\mathfrak{m}^+}, \bar{W}^1) \\ &= -\|[Z^1, \bar{W}^2]\|^2+\frac{1}{2}\|[Z^1, W^1]\|^2. \end{aligned}$$

Similarly, we get,

$$\begin{aligned} g(\Lambda([Z, \bar{Z}]_{\mathfrak{m}^c})W, \bar{W}) = &-\|[Z^1, \bar{W}^1]+[Z^2, \bar{W}^2]\|^2+\|[Z^1, \bar{W}^1]\|^2 \\ &+\|[Z^2, \bar{W}^2]\|^2, \end{aligned}$$

and

$$\begin{aligned} g([[Z, \bar{Z}]_{\mathfrak{k}^c}, W], \bar{W}) = &-\|[Z^1, \bar{W}^1]\|^2+\|[Z^1, W^1]\|^2-2\|[Z^1, \bar{W}^2]\|^2 \\ &-\|[Z^2, \bar{W}^1]\|^2-2\|[Z^2, \bar{W}^2]\|^2. \end{aligned}$$

Hence, from (2.3), the combination of the above leads to obtain (3.3). To verify (3.4), we set $W=Z$ in (3.3). Then we have,

$$H(X\wedge IX)=2\|[Z^1, \bar{Z}^2]\|^2+\|[Z^2, \bar{Z}^2]\|^2+\|[Z^1, \bar{Z}^1]+[Z^2, \bar{Z}^2]\|^2.$$

The last term is turned into $\|[Z^1, \bar{Z}^1]\|^2+\|[Z^2, \bar{Z}^2]\|^2+2\|[Z^1, \bar{Z}^2]\|^2$, by a slight computation. Therefore, (3.4) is an immediate conclusion.

The positivity of the holomorphic sectional curvature is assured by the following argument. Since $H(X\wedge IX)\geqq 0$, what to show is that $[Z^1, \bar{Z}^1]=[Z^1, \bar{Z}^2]=[Z^2, \bar{Z}^2]=0$ leads to $Z=0$. Put $Z^1=\sum_{\alpha\in\Delta_i^+(1)} \xi^\alpha E_\alpha, \xi^\alpha\in \boldsymbol{C}$. Then, $[Z^1, \bar{Z}^1]=\sum_{\alpha,\beta}\xi^\alpha\bar{\xi}^\beta[E_\alpha, E_{-\beta}]$

$$=\sum_{\alpha}|\xi^{\alpha}|^{2}[E_{\alpha}, E_{-\alpha}]+\sum_{\alpha\neq\beta}\xi^{\alpha}\bar{\xi}^{\beta}[E_{\alpha}, E_{-\beta}].$$

Thus, the $\mathfrak{h}$-component of $[Z^1, \bar{Z}^1]$ is $\sum_{\alpha}|\xi^{\alpha}|^{2}[E_{\alpha}, E_{-\alpha}]=-\sum_{\alpha}|\xi^{\alpha}|^{2}H_{\alpha}$. If $\alpha\in\Delta_i^{+}(1)$ is given by $\alpha=\sum_j n_j(\alpha)\alpha_j$, $n_i(\alpha)=1$, then $H_{\alpha}=\sum_j n_j(\alpha)H_j$. Hence we have, $\sum_{\alpha}|\xi^{\alpha}|^{2}H_{\alpha}=\sum_j(\sum_{\alpha}|\xi^{\alpha}|^{2}n_j(\alpha))H_j$. Since $\{H_j;\ j=1, \cdots, l\}$ is a basis of $\mathfrak{h}$, $[Z^1, \bar{Z}^1]=0$ means $\sum_{\alpha}|\xi^{\alpha}|^{2}n_j(\alpha)=0$ for all j. Then, we have $\xi^{\alpha}=0$ for $\alpha\in\Delta_i^{+}(1)$, from the nonnegativity of $n_j(\alpha)$ together with $n_i(\alpha)=1$.

By the similar argument, we obtain that $Z^2=0$. Thus, we get $Z=Z^1+Z^2=0$.
Q.E.D.

REMARK. Relative to a Kähler C-space $(G_u/K, g)$ which is normal, the holomorphic sectional curvature is given by $H(X\wedge IX)=\|[Z, \bar{Z}]\|^2$.

§4. Curvature operator.

Let $(G_u/K_{\Phi}, g)$ be a Kähler C-space which is associated with $(\mathfrak{g}, \Phi)$, where $\mathfrak{g}$ is a complex simple Lie algebra and $\Phi=\{\alpha_{i_1}, \cdots, \alpha_{i_r}\}\subset\Pi$.

In this section, we shall define a linear operator, called a curvature operator, and investigate properties of this operator.

We denote by $\mathfrak{m}^+\cdot\mathfrak{m}^+$ the symmetric tensor product of $\mathfrak{m}^+$. $\mathfrak{m}^+\cdot\mathfrak{m}^+$ is generated by $X\cdot Y=\frac{1}{2}(X\otimes Y+Y\otimes X)$, $X, Y\in\mathfrak{m}^+$. In particular, $\{E_{\alpha}\cdot E_{\beta};\alpha, \beta\in\Delta^+(\Phi)\}$ constitutes a basis of $\mathfrak{m}^+\cdot\mathfrak{m}^+$.

We introduce a hermitian inner product $\langle\cdot, \cdot\rangle$ on $\mathfrak{m}^+\cdot\mathfrak{m}^+$ from the form g on $\mathfrak{m}^+$;

$$\langle X\cdot Y, Z\cdot W\rangle=\frac{1}{2}\{g(X, \bar{Z})\cdot g(Y, \bar{W})+g(X, \bar{W})g(Y, \bar{Z})\}. \tag{4.1}$$

Since, $\Lambda(X)$, $X\in\mathfrak{m}^C$ leaves the space $\mathfrak{m}^+$ invariant by (2.10), $\Lambda(X)$ induces a linear operator on $\mathfrak{m}^+\cdot\mathfrak{m}^+$, for which we use the same letter;

$$\Lambda(X)(Y\cdot Z)=(\Lambda(X)Y)\cdot Z+Y\cdot(\Lambda(X)Z), \qquad X\in\mathfrak{m}^C,\ Y, Z\in\mathfrak{m}^+. \tag{4.2}$$

Λ on $\mathfrak{m}^+\cdot\mathfrak{m}^+$ is skew symmetric with respect to the inner product $\langle\cdot, \cdot\rangle$; $\langle\Lambda(X)Y\cdot Z, W\cdot T\rangle+\langle Y\cdot Z, \Lambda(\bar{X})W\cdot T\rangle=0$.

And also the adjoint representation $(\mathfrak{m}^+, ad_{\mathfrak{k}^C})$ canonically introduces a representation on $\mathfrak{m}^+\cdot\mathfrak{m}^+$, which is denoted by $(\mathfrak{m}^+\cdot\mathfrak{m}^+, ad_{\cdot C})$;

$$ad(V)(X\cdot Y)=[V, X]\cdot Y+X\cdot[V, Y], \qquad V\in\mathfrak{k}^C, \qquad X, Y\in\mathfrak{m}^+. \tag{4.3}$$

From (2.5), we have the following relation between $\Lambda(X)$ and $ad(V)$;

$$ad(V)\circ\Lambda(X)(Y\cdot Z)=\Lambda([V, X])(Y\cdot Z)+\Lambda(X)\circ ad(V)(Y\cdot Z). \tag{4.4}$$

Now we define a linear endomorphism Q of $\mathfrak{m}^+\cdot\mathfrak{m}^+$, which we call the curvature operator associated with R, by

$$\langle Q(X\cdot Y),\ Z\cdot W\rangle = R(X, \bar{Z}, Y, \bar{W})\,. \tag{4.5}$$

Q is well-defined from the property (2.12). See Calabi and Vesentini [5] and Borel [2] for another definition of Q in terms of a basis of $\mathfrak{m}^+\cdot\mathfrak{m}^+$. It is easily verified that these definitions are equivalent. Q is self-adjoint with respect to $\langle\cdot,\cdot\rangle$ from (2.12).

The following theorem gives a condition on the hermitian symmetry of a Kähler C-space $(G_u/K, g)$ in terms of the operator Q.

THEOREM 4.1. *Let $(G_u/K, g)$ be a Kähler C-space associated with $(\mathfrak{g}, \Phi)$. The following conditions are equivalent;*

(i) *$(G_u/K, g)$ is a hermitian symmetric space.*

(ii) *R is Λ-invariant, that is,*

$$[\Lambda(X), R(Y, Z)]W = R(\Lambda(X)Y, Z)W + R(Y, \Lambda(X)Z)W\,, \quad X, Y, Z \text{ and } W \in \mathfrak{m}^C. \tag{4.6}$$

(iii) *Q is Λ-invariant, $\Lambda(X)\circ Q = Q\circ\Lambda(X)$, $X\in\mathfrak{m}^C$.*

(iv) *every eigenspace of Q is Λ-invariant, that is,*

$$\Lambda(X)\mathfrak{n}\subset\mathfrak{n}\,,\quad X\in\mathfrak{m}^C \text{ for each eigenspace } \mathfrak{n} \text{ of } Q\,.$$

PROOF. (i) $\Leftrightarrow$ (ii); this is obvious from Nomizu [11], since G_u/K is simply connected.

(iii) $\Leftrightarrow$ (iv); this is easily verified.

That (ii) implies (iii) will be shown by the aid of the following formula;

$$\begin{aligned} &\langle Q\circ\Lambda(X)(Y\cdot W), Z\cdot T\rangle - \langle\Lambda(X)\circ Q(Y\cdot W), Z\cdot T\rangle \\ &= g(R(\Lambda(X)Y, \bar{Z})W, \bar{T}) + g(R(Y, \Lambda(X)\bar{Z})W, \bar{T}) - g([\Lambda(X), R(Y, \bar{Z})]W, \bar{T})\,, \\ &\qquad X\in\mathfrak{m}^C,\ Y, Z, W, T\in\mathfrak{m}^+. \end{aligned} \tag{4.7}$$

To prove (iii) $\Rightarrow$ (ii), it is sufficient to show that, under the invariance of Q,

$$g([\Lambda(X), R(Y, \bar{Z})]W, \bar{T}) - g(R(\Lambda(X)\bar{Y}, Z)W, \bar{T}) - g(R(Y, \Lambda(X)\bar{Z})W, \bar{T}) = 0 \tag{4.8}$$

only for $X\in\mathfrak{m}^C$, $Y, Z, W, T\in\mathfrak{m}^+$, since $\mathfrak{m}^C = \mathfrak{m}^+ + \mathfrak{m}^-$, $g(\mathfrak{m}^+, \mathfrak{m}^+) = 0$, $\Lambda(X)\mathfrak{m}^+\subset\mathfrak{m}^+$, $R(\mathfrak{m}^+, \mathfrak{m}^+)Z = 0$ and g, Λ and R are all real.

(4.8) is an immediate consequence of (4.7). Q.E.D.

Since R is $ad_{\mathfrak{l}^C}$-invariant, so is Q, that is,

$$Q\circ ad(V) = ad(V)\circ Q\,, \qquad V\in\mathfrak{l}^C\,. \tag{4.9}$$

If the representation $(\mathfrak{m}^+\cdot\mathfrak{m}^+, ad_{\mathfrak{l}^C})$ has an irreducible decomposition; $(\mathfrak{m}^+\cdot\mathfrak{m}^+, ad_{\mathfrak{l}^C}) = \sum((\mathfrak{m}^+\cdot\mathfrak{m}^+)_a, ad_{\mathfrak{l}^C})$, and $(\mathfrak{m}^+\cdot\mathfrak{m}^+)_a$ satisfies that $Q((\mathfrak{m}^+\cdot\mathfrak{m}^+)_a)\subset(\mathfrak{m}^+\cdot\mathfrak{m}^+)_a$, then each irreducible subspace $(\mathfrak{m}^+\cdot\mathfrak{m}^+)_a$ belongs to some eigenspace of Q from (4.9) together with Schur's lemma.

We shall clarify relations between such an irreducible decomposition of $\mathfrak{m}^+\cdot\mathfrak{m}^+$ and the operation of Q, as follows.

$\mathfrak{k}^C$ is given by a sum of the center $\mathfrak{c}$ and the maximal semi-simple part $\mathfrak{l}'$; $\mathfrak{k}^C=\mathfrak{c}+\mathfrak{l}'$, $\mathfrak{c}=\sum_{j=1}^{r} CH_{\Lambda_j}$, where Λ_j is given by $2(\Lambda_j, \alpha_k)=(\alpha_k, \alpha_k)\delta_{i_j k}$, $j=1, \cdots, r$, $k=1, \cdots, l$, $\mathfrak{l}'=\sum_{\alpha\in\Delta^+-\Delta^+(\Phi)}\{C[E_\alpha, E_{-\alpha}]+CE_\alpha+CE_{-\alpha}\}$. Therefore, the restriction $(\mathfrak{m}^+\cdot\mathfrak{m}^+, ad_{\mathfrak{l}'})$ of $(\mathfrak{m}^+\cdot\mathfrak{m}^+, ad_{\mathfrak{k}^C})$ to $\mathfrak{l}'$ gives a representation of the semi-simple Lie algebra $\mathfrak{l}'$.

Since every root $\alpha\in\Delta^+(\Phi)$, more precisely $\alpha|_{\mathfrak{h}'}$ gives a weight of $(\mathfrak{m}^+, ad_{\mathfrak{l}'})$, where $\mathfrak{h}'$ is the Cartan subalgebra of $\mathfrak{l}'$, defined by $\mathfrak{h}'=\sum_{\alpha\in\Delta^+-\Delta^+(\Phi)} C[E_\alpha, E_{-\alpha}]$, $\{\alpha+\beta;\ \alpha, \beta\in\Delta^+(\Phi)\}$ constitutes the set of all weights of $(\mathfrak{m}^+\cdot\mathfrak{m}^+, ad_{\mathfrak{l}'})$. $E_\alpha\cdot E_\beta$, denoted simply by $E_{\alpha\cdot\beta}$, is a weight vector corresponding to a weight $\alpha+\beta$. Then, the weight space W_Σ which corresponds to Σ in $\mathfrak{m}^+\cdot\mathfrak{m}^+$ is given by $W_\Sigma=\sum_{\substack{\alpha+\beta=\Sigma\\ \alpha,\beta\in\Delta^+(\Phi)}} CE_{\alpha\cdot\beta}$.

We have the following proposition which asserts that Q leaves W_Σ invariant.

PROPOSITION 4.2. *Let* W_Σ *be a weight space of* $(\mathfrak{m}^+\cdot\mathfrak{m}^+, ad_{\mathfrak{l}'})$. *Then* $Q(W_\Sigma)\subset W_\Sigma$ *for any weight* Σ.

PROOF. By (4.9) we have, for $Z\in W_\Sigma$, $ad(H)Q(Z)=Q(ad(H)Z)=\Sigma(H)Q(Z)$, $H\in\mathfrak{h}$, hence $Q(Z)\in W_\Sigma$ by definition of a weight space. Q.E.D.

Suppose that there exist a weight Σ and a weight vector Z_Σ of W_Σ such that $ad(E_\beta)Z_\Sigma=0$ for any $\beta\in\Pi-\Phi$ and $Q(Z_\Sigma)=\nu Z_\Sigma$. Then, with the aid of the representation theory of semi-simple Lie algebras, Σ satisfies that $2(\Sigma, \alpha_j)/(\alpha_j, \alpha_j)\in Z^+$ for all $\alpha_j\in\Pi-\Phi$, that is, Σ is a so-called dominant integral weight, and the subspace of $\mathfrak{m}^+\cdot\mathfrak{m}^+$ which is generated by Z_Σ and $ad(E_{-\beta_k})ad(E_{-\beta_{k-1}})\cdots ad(E_{-\beta_1})Z_\Sigma$, $\beta_1, \cdots, \beta_k\in\Pi-\Phi$, $k>0$, is an irreducible subspace whose highest weight is Σ. Moreover ν is an eigenvalue of Q on the irreducible subspace, since Q and $ad_{\mathfrak{c}}$ are commutative by (4.9).

$(\mathfrak{m}^+\cdot\mathfrak{m}^+)_{\Sigma,Z}$, or simply $(\mathfrak{m}^+\cdot\mathfrak{m}^+)_\Sigma$ denotes this irreducible subspace. Hence, we have the following

PROPOSITION 4.3. *If a weight vector* $Z\in W_\Sigma$ *satisfies that* $ad(E_\beta)Z=0$ *for all* $\beta\in\Pi-\Phi$, *and* $Q(Z)=\nu Z$. *Then* Σ *is a dominant integral weight and* $(\mathfrak{m}^+\cdot\mathfrak{m}^+)_{\Sigma,Z}$ *gives a linear subspace of an eigenspace corresponding to* ν.

Since $R(\mathfrak{m}^{+n}, \mathfrak{m}^{-n'})\mathfrak{m}^{+m}\subset\mathfrak{m}^{+(n-n'+m)}$ from (2.13), and $ad(\mathfrak{k}^C)\mathfrak{m}^{+n}\subset\mathfrak{m}^{+n}$, we have the following decomposition of $(\mathfrak{m}^+\cdot\mathfrak{m}^+, ad_{\mathfrak{l}'})$; $(\mathfrak{m}^+\cdot\mathfrak{m}^+, ad_{\mathfrak{l}'})=\sum_n((\mathfrak{m}^+\cdot\mathfrak{m}^+)^n, ad_{\mathfrak{l}'})$, $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^n=\sum_{n_1+n_2=n}\mathfrak{m}^{+n_1}\cdot\mathfrak{m}^{+n_2}$, and $Q((\mathfrak{m}^+\cdot\mathfrak{m}^+)^n)\subset(\mathfrak{m}^+\cdot\mathfrak{m}^+)^n$.

We note that this decomposition is not always irreducible.

With respect to eigenvalues of Q, we have the following.

PROPOSITION 4.4. (i) *If a weight vector* $E_{\alpha\cdot\beta}$ of $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^n$ $(\alpha\leqq\beta)$ *is an eigenvector of* Q; $Q(E_{\alpha\cdot\beta})=\nu E_{\alpha\cdot\beta}$, *then the eigenvalue* ν *is given by*

$$\begin{aligned}(4.10)\qquad \nu&=\frac{1}{c\cdot n(\alpha)}(\alpha,\alpha), \qquad \alpha=\beta,\\ &=\frac{2}{c\cdot n(\beta)}\left\{(\alpha,\beta)+\frac{c\cdot n(\alpha)}{c\cdot n(\alpha+\beta)}N^2_{\alpha,\beta}\right\}, \qquad n(\alpha)\leqq n(\beta),\end{aligned}$$

(ii) *Relative to a weight space* W_Σ *in* $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^n$, *the trace of* $Q|_{W_\Sigma}$ *is written by*

$$\begin{aligned}(4.11)\qquad \mathrm{Tr}\,(Q|_{W_\Sigma})=\sum_\nu m_\nu\cdot\nu=\sum_{\substack{\alpha+\beta=\Sigma\\ \alpha<\beta}}\frac{2}{c\cdot n(\beta)}\left\{(\alpha,\beta)+\frac{c\cdot n(\alpha)}{c\cdot n(\alpha+\beta)}N^2_{\alpha,\beta}\right\}\\ +\frac{1}{c\cdot n(\gamma)}(\gamma,\gamma),\end{aligned}$$

where ν *is an eigenvalue of* Q *in* W_Σ *with multiplicity* m_ν, *and* $\gamma=\Sigma/2$. *The summation of the left hand side is taken over all the eigenvalues of* Q *in* W_Σ. *The second term of the right hand side is excluded if* $\gamma=\Sigma/2$ *is not in* $\Delta^+(\Phi)$.

PROOF. Suppose that Q satisfies, $Q(E_{\alpha\cdot\beta})=\nu E_{\alpha\cdot\beta}$, $\alpha,\beta\in\Delta^+(\Phi)$, $n(\alpha)+n(\beta)=n$, $n(\alpha)\leqq n(\beta)$. Then, we have $\nu\langle E_{\alpha\cdot\beta}, E_{\alpha\cdot\beta}\rangle=R_{\alpha\bar\alpha\beta\bar\beta}$. Hence, (4.10) is verified directly from Proposition 2.4.

The second part is shown by the following consideration.

Since $Q(W_\Sigma)\subset W_\Sigma$, Q is given on W_Σ by

$$Q(E_{\alpha,\beta})=\sum_{\substack{\gamma+\delta=\Sigma\\ \gamma\leqq\delta}}Q_{\alpha\beta}{}^{\gamma\delta}E_{\gamma\cdot\delta}\ \text{ for any } \alpha,\beta\in\Delta^+(\Phi),\ \alpha+\beta=\Sigma,\ \alpha\leqq\beta.$$

We have $\mathrm{Tr}\,(Q|_{W_\Sigma})=\sum_{\substack{\alpha+\beta=\Sigma\\ \alpha\leqq\beta}}Q_{\alpha\beta}{}^{\alpha\beta}$. Since

$$Q_{\alpha\beta}{}^{\alpha\beta}=\frac{2}{(c\cdot n(\alpha))\cdot(c\cdot n(\beta))}R_{\alpha\bar\alpha\beta\bar\beta}\qquad(\alpha<\beta),$$

$$Q_{\gamma\gamma}{}^{\gamma\gamma}=\frac{1}{(c\cdot n(\gamma))^2}R_{\gamma\bar\gamma\gamma\bar\gamma}\qquad(\gamma=\Sigma/2\in\Delta^+(\Phi)),$$

(4.11) is immediately obtained from Proposition 2.4. Q.E.D.

Let Θ be the highest weight among $\{\alpha+\beta;\ n(\alpha)+n(\beta)=n\}$ with respect to the lexicographic ordering. Then, Θ is given by a sum of some roots; $\Theta=\alpha^{n_1}+\alpha^{n_2}$, where α^{n_i} is the maximal root in $\Delta^+(\Phi;\ n_i)$, $i=1,2$ respectively and $n_1+n_2=n$. Of course, Θ is a dominant integral weight of $((\mathfrak{m}^+\cdot\mathfrak{m}^+)^n, ad_{\mathfrak{l}})$. If $\dim_C W_\Theta=1$, then $E_{\alpha^{n_1}\cdot\alpha^{n_2}}$ is a weight vector of Θ such that $Q(E_{\alpha^{n_1}\cdot\alpha^{n_2}})=\nu E_{\alpha^{n_1}\cdot\alpha^{n_2}}$. Thus, the irreducible subspace associated with Θ and $E_{\alpha^{n_1}\cdot\alpha^{n_2}}$ gives a subspace of an eigenspace corresponding to ν from Proposition 4.3.

On the other hand, we have, relative to the operation of Λ, $\Lambda(X)(\mathfrak{m}^+\cdot\mathfrak{m}^+)^n \subset(\mathfrak{m}^+\cdot\mathfrak{m}^+)^{n+n'}$, $\Lambda(\bar{X})(\mathfrak{m}^+\cdot\mathfrak{m}^+)^n\subset(\mathfrak{m}^+\cdot\mathfrak{m}^+)^{n-n'}$, $X\in\mathfrak{m}^{+n'}$, if $n>n'$ and $\Lambda(\bar{X})(\mathfrak{m}^+\cdot\mathfrak{m}^+)^n=\{0\}$ for $X\in\mathfrak{m}^{+n'}$, if $n\not> n'$.

We shall show the following proposition, which can be applied to the Λ-invariance of eigenspaces of Q.

PROPOSITION 4.5. *Let* $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^{n_i}_{\Sigma_i}$ *be an irreducible subspace associated with the highest weight* Σ_i, $i=1, 2$ *respectively.*

Suppose that $n_1<n_2$ *(respectively* $n_1>n_2$*), and*

$$\Lambda(E_\alpha)Z_{\Sigma_1}\in(\mathfrak{m}^+\cdot\mathfrak{m}^+)^{n_2}_{\Sigma_2} \ (\textit{resp.}, \ \Lambda(E_{-\alpha})Z_{\Sigma_1}\in(\mathfrak{m}^+\cdot\mathfrak{m}^+)^{n_2}_{\Sigma_2})$$
$$\textit{for a weight vector } Z_{\Sigma_1}$$

corresponding to Σ_1 *and any root* α, $n(\alpha)=n_2-n_1$ *(resp.,* $n(\alpha)=n_1-n_2$*). Then* $\Lambda(X)(\mathfrak{m}^+\cdot\mathfrak{m}^+)^{n_1}_{\Sigma_1}\subset(\mathfrak{m}^+\cdot\mathfrak{m}^+)^{n_2}_{\Sigma_2}$ *for all* $X\in\mathfrak{m}^{+(n_2-n_1)}$ *(resp.* $\Lambda(\bar{X})(\mathfrak{m}^+\cdot\mathfrak{m}^+)^{n_1}_{\Sigma_1}\subset(\mathfrak{m}^+\cdot\mathfrak{m}^+)^{n_2}_{\Sigma_2}$ *for* $X\in\mathfrak{m}^{+(n_1-n_2)}$*).*

PROOF. Let $n_2>n_1$. Then, the subspace $\mathfrak{m}^{+(n_2-n_1)}$ is spanned by E_α's, $n(\alpha)=n_2-n_1$. Since the irreducible subspace $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^{n_1}_{\Sigma_1}$ is generated by the vector Z_{Σ_1} and $ad(E_{-\beta_k})\cdots ad(E_{-\beta_1})Z_{\Sigma_1}$, $\beta_1, \cdots, \beta_k\in\Pi-\Phi$, $k>0$, it is sufficient to show that $\Lambda(E_\alpha)\,ad(E_{-\beta_k})\cdots ad(E_{-\beta_1})Z_{\Sigma_1}$ belongs to $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^{n_2}_{\Sigma_2}$ for α, $n(\alpha)=n_2-n_1$ and $k>0$. It follows from (4.4) that

$$\Lambda(E_\alpha)\,ad(E_{-\beta})Z_{\Sigma_1}=ad(E_{-\beta})\Lambda(E_\alpha)Z_{\Sigma_1}-\Lambda([E_{-\beta}, E_\alpha])Z_{\Sigma_1}\,.$$

Hence, from the assumption of the proposition, $\Lambda(E_\alpha)\,ad(E_{-\beta})Z_{\Sigma_1}$ is in $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^{n_2}_{\Sigma_2}$. Thus, the assertion is inductively verified on k.

The statement for $n_1>n_2$ is similarly verified. Q.E.D.

§ 5. Hermitian symmetry and eigenvalues of curvature operator.

We shall show, in this section, the following theorems which are concerned with the hermitian symmetry of Kähler C-spaces.

Let $(G_u/K, g)$ be a Kähler C-space associated with $(\mathfrak{g}, \Phi)$. Suppose that Φ consists of a single root α_i of Π; $\Phi=\{\alpha_i\}$. This is equivalent to the second Betti number $b_2=1$. The G_u-invariant Kähler metric g is written in the following form; $g=2\sum_k k\sum_{\alpha\in\Delta_i^+(k)}\omega^\alpha\cdot\omega^{\bar{\alpha}}$ (a parameter $c=1$).

Then, we have the following.

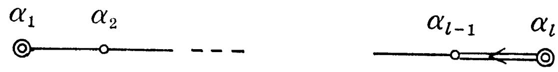
THEOREM 5.1. *A Kähler* C*-space* $(G_u/K, g)$ of $b_2=1$ *is hermitian symmetric if and only if it is associated with one of the following pairs;*

$$(A_l, \alpha_i)_{i=1,\cdots,l}, \quad (B_l, \alpha_i)_{i=1,l}, \quad (C_l, \alpha_i)_{i=1,l},$$
$$(D_l, \alpha_i)_{i=1,l-1,l}, \quad (E_6, \alpha_i)_{i=1,6}, \quad (E_7, \alpha_7) \text{ and } (G_2, \alpha_1)\,.$$

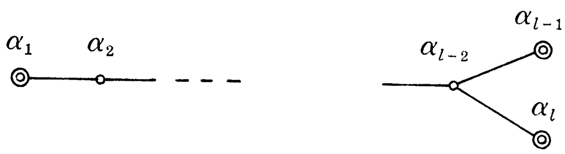
A_l
$(l \geqq 1)$

B_l
$(l \geqq 2)$

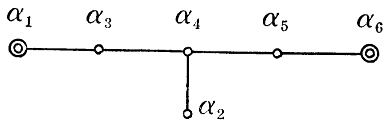
C_l
$(l \geqq 2)$



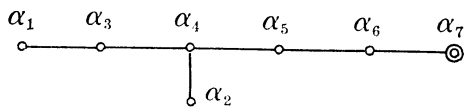
D_l
$(l \geqq 3)$



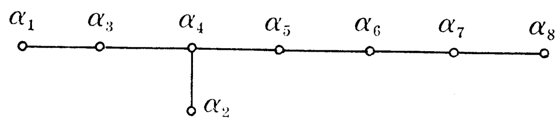
E_6



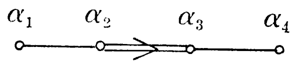
E_7



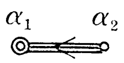
E_8



F_4



G_2



(*A Kähler C-space associated with* $(\mathfrak{g}, \alpha_i)$ *is hermitian symmetric if* α_i *is enclosed with double circles in the above Dynkin diagrams.*)

The following theorem gives an equivalent condition on the hermitian symmetry of Kähler C-spaces of $b_2=1$ with the aid of numbers of eigenvalues of Q.

THEOREM 5.2. *A Kähler C-space of* $b_2=1$ *is hermitian symmetric if and only* *if its curvature operator has at most two distinct eigenvalues.*

REMARK. A Kähler C-space $(G_u/K, g)$ of $b_2=1$ is a normal homogeneous space if it is associated with one of the following pairs;

$$(A_l, \alpha_i)_{i=1,\cdots,l}, \quad (B_l, \alpha_1), \quad (C_l, \alpha_l), \quad (D_{l,\alpha_1}), \quad (D_l, \alpha_{l-1}),$$

$$(D_l, \alpha_l), \quad (E_6, \alpha_1), \quad (E_6, \alpha_6), \quad (E_7, \alpha_7).$$

On the other hand, a Kähler C-space, associated with each of (B_l, α_l), (C_l, α_1) and (G_2, α_1), admits a proper subgroup of $I_0(G_u/K, g)$ (the identity component of the isometry group) which acts transitively on G_u/K, since $G_u/C\cap K$, where C is the center of G_u, is such a proper subgroup.

It has been shown in Calabi and Vesentini [5] and Borel [2] that every irreducible hermitian symmetric space of compact type, and consequently of noncompact type, has at most two eigenvalues.

We shall show these theorems by investigating the Λ-invariance of eigenspaces of Q together with computations of eigenvalues.

Relative to the Dynkin diagrams and the root systems of complex simple Lie algebras, we refer to Bourbaki [4].

In order to prepare for the proof of the theorems, we assume first that $\Delta_i^+(k)=\emptyset$ for $k\geqq 3$, that is $\mathfrak{m}^+=\mathfrak{m}^{+1}+\mathfrak{m}^{+2}$. This assumption is satisfied by almost pairs $(\mathfrak{g}, \alpha_i)$'s exclusive of some exceptional Lie algebras $\mathfrak{g}$'s and some roots α_i's (cf., Corollary 3.2 in § 3). Then, we have, $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2=\mathfrak{m}^{+1}\cdot\mathfrak{m}^{+1}$, $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^3=\mathfrak{m}^{+1}\cdot\mathfrak{m}^{+2}$ and $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4=(\mathfrak{m}^{+2}\cdot\mathfrak{m}^{+2})$, for which $2\alpha^1$, $\alpha^1+\alpha^2$ and $2\alpha^2$ give the highest weights respectively, and $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^k=\{0\}$ for $k\geqq 5$.

The weight spaces $W_{2\alpha^1}$, $W_{\alpha^1+\alpha^2}$, and $W_{2\alpha^2}$ consist of $\boldsymbol{C}E_{\alpha^1\cdot\alpha^1}$, $\boldsymbol{C}E_{\alpha^1\cdot\alpha^2}$ and $\boldsymbol{C}E_{\alpha^2\cdot\alpha^2}$ respectively. $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(1)}$, $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^3_{(1)}$, and $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(1)}$ denote the irreducible subspaces $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{2\alpha^1}$, $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^3_{\alpha^1+\alpha^2}$ and $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{2\alpha^2}$. They have $E_{\alpha^1\cdot\alpha^1}$, $E_{\alpha^1\cdot\alpha^2}$ and $E_{\alpha^2\cdot\alpha^2}$ as weight vectors corresponding to $2\alpha^1$, $\alpha^1+\alpha^2$ and $2\alpha^2$ respectively.

We obtain the following table from Proposition 4.4;

Table 1.

	eigenvalue of Q	highest weight
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(1)}$	(α^1, α^1)	$2\alpha^1$
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^3_{(1)}$	(α^1, α^2)	$\alpha^1+\alpha^2$
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(1)}$	$(\alpha^2, \alpha^2)/2$	$2\alpha^2$

The dimensions of these irreducible subspaces can be computed by using Weyl's dimension formula. For example, $\dim_C(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(1)}$ is given by

$$\dim_C(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(1)} = \prod_{\alpha\in\Delta^+-\Delta_i^+} \frac{(\delta+2\alpha^1, \alpha)}{(\delta, \alpha)},$$

where,
$$\delta = \{\sum_{\alpha\in\Delta^+-\Delta_i^+} \alpha\}/2.$$

As a concrete example of the irreducible decomposition of $(\mathfrak{m}^+\cdot\mathfrak{m}^+, ad_{\mathfrak{l}'})$, we shall investigate the complete decomposition of $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2$, for instance, associated with $(B_l, \alpha_i)_{1<i<l}$ and compute the eigenvalues of the curvature operator Q.

(a) *the root system of type* B_l $(l\geqq 2)$ (Bourbaki [4]);

Let $(\varepsilon_i)_{1\leqq i\leqq l}$ be an orthonormal basis of a l-dim euclidean space $(\boldsymbol{R}^l, (\cdot,\cdot))$. A fundamental root system $\Pi=\{\alpha_1, \cdots, \alpha_l\}$ is defined by $\alpha_i=\varepsilon_i-\varepsilon_{i+1}$, $i=1, \cdots, l-1$ and $\alpha_l=\varepsilon_l$ and positive roots are given as follows;

positive roots: $\varepsilon_j=\alpha_j+\alpha_{j+1}+\cdots+\alpha_l$, $\quad 1\leqq j\leqq l$,

$\varepsilon_j-\varepsilon_k=\alpha_j+\alpha_{j+1}+\cdots+\alpha_{k-1}$, $\quad 1\leqq j<k\leqq l$,

$\varepsilon_j+\varepsilon_k=(\alpha_j+\cdots+\alpha_l)+(\alpha_k+\cdots+\alpha_l)$, $\quad 1\leqq j<k\leqq l$.

(b) $\Delta_i^+=\Delta_i^+(1)\cup\Delta_i^+(2)$ *for the pair* $(B_l, \alpha_i)_{1<i<l}$;

$\Delta_i^+(1)$: $\alpha_j+\cdots+\alpha_i+\cdots+\alpha_k=\varepsilon_j-\varepsilon_{k+1}$, $\quad 1\leqq j\leqq i\leqq k<l$,

$\alpha_j+\cdots+\alpha_l=\varepsilon_j$, $\quad 1\leqq j\leqq i$,

$\alpha_j+\cdots+\alpha_i+\cdots+\alpha_{k-1}+2\alpha_k+\cdots+2\alpha_l=\varepsilon_j+\varepsilon_k$, $\quad 1\leqq j\leqq i<k\leqq l$,

$\Delta_i^+(2)$: $\alpha_j+\cdots+\alpha_{k-1}+2\alpha_k+\cdots+2\alpha_i+\cdots+2\alpha_l=\varepsilon_j+\varepsilon_k$, $\quad 1\leqq j<k\leqq i$.

$\sharp\Delta_i^+(1)=\dim_C\mathfrak{m}^{+1}=i(2l-2i+1)$, $\quad \sharp\Delta_i^+(2)=\dim_C\mathfrak{m}^{+2}=i(i-1)/2$.

(c) *dominant integral weights;*

$\Sigma_1=2(\alpha_1+\cdots+\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_l)=2\varepsilon_1+2\varepsilon_{i+1}$,

$\Sigma_2=\alpha_1+2\alpha_2+\cdots+2\alpha_i+3\alpha_{i+1}+4\alpha_{i+2}+\cdots+4\alpha_l=\varepsilon_1+\varepsilon_2+\varepsilon_{i+1}+\varepsilon_{i+2}$

and

$\Sigma_3=2(\alpha_1+\cdots+\alpha_l)=2\varepsilon_1$.

These weights are dominant integral, since $\Sigma_1/2$ is the maximal root in $\Delta_i^+(1)$, Σ_2 and Σ_3 satisfy that

$$\frac{2(\Sigma_2, \alpha_2)}{(\alpha_2, \alpha_2)}=\frac{2(\Sigma_2, \alpha_{i+1})}{(\alpha_{i+1}, \alpha_{i+1})}=1, \qquad \frac{2(\Sigma_2, \alpha_j)}{(\alpha_j, \alpha_j)}=0, \qquad j\neq 2, i+1$$

and

$$\frac{2(\Sigma_3, \alpha_1)}{(\alpha_1, \alpha_1)}=2, \qquad \frac{2(\Sigma_3, \alpha_j)}{(\alpha_j, \alpha_j)}=0, \qquad j\neq 1.$$

$$\dim_C W_{\Sigma_1}=1, \quad \dim_C W_{\Sigma_2}=2 \quad \text{and} \quad \dim_C W_{\Sigma_3}=l-i+1,$$

since $\Sigma_1=(\varepsilon_1+\varepsilon_{i+1})+(\varepsilon_1+\varepsilon_{i+1})$, $\Sigma_2=(\varepsilon_1+\varepsilon_{i+1})+(\varepsilon_2+\varepsilon_{i+2})=(\varepsilon_1+\varepsilon_{i+2})+(\varepsilon_2+\varepsilon_{i+1})$ and

$\Sigma_3=(\varepsilon_1-\varepsilon_{i+1})+(\varepsilon_1+\varepsilon_{i+1})=(\varepsilon_1-\varepsilon_{i+2})+(\varepsilon_1+\varepsilon_{i+2})=\cdots=(\varepsilon_1-\varepsilon_l)+(\varepsilon_1+\varepsilon_l)=\varepsilon_1+\varepsilon_1$. Σ_i, $i=1, 2, 3$ are not roots. Then coefficients $N_{\alpha,\beta}=0$ for any α, $\beta\in\Delta_i^+(1)$, $\alpha+\beta=\Sigma_i$, $i=1, 2, 3$. Hence from Proposition 4.4, $\mathrm{Tr}\,(Q\,|\,_{W_{\Sigma_1}})=(\Sigma_1/2, \Sigma_1/2)=(\varepsilon_1+\varepsilon_{i+1}, \varepsilon_1+\varepsilon_{i+1})=2$, $\mathrm{Tr}\,(Q\,|\,_{W_{\Sigma_2}})=2\{(\varepsilon_1+\varepsilon_{i+1}, \varepsilon_2+\varepsilon_{i+2})+(\varepsilon_1+\varepsilon_{i+2}, \varepsilon_2+\varepsilon_{i+1})\}=0$ and $\mathrm{Tr}\,(Q\,|\,_{W_{\Sigma_3}})=(\varepsilon_1, \varepsilon_1)+2\{(\varepsilon_1-\varepsilon_{i+1}, \varepsilon_1+\varepsilon_{i+1})+\cdots+(\varepsilon_1-\varepsilon_l, \varepsilon_1+\varepsilon_l)\}=1$.

(d) *decomposition of* $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2$ *and eigenvalues of* Q;

$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(1)}$ denotes the irreducible subspace of $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2$ whose highest weight is Σ_1; $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(1)}=(\mathfrak{m}^+\cdot\mathfrak{m}^+)_{\Sigma_1,E_{\alpha}\cdot_{\alpha}}$, $\alpha=\Sigma_1/2$. Because of $\dim_C W_{\Sigma_1}=1$, $\mathrm{Tr}\,(Q\,|\,_{W_{\Sigma_1}})=2$ gives the eigenvalue of $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(1)}$.

Since $\Sigma_2=\Sigma_1-(\alpha_1+\alpha_{i+1})$, $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(1)}$ contains a weight vector corresponding to Σ_2, that is, $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(1)}\cap W_{\Sigma_2}\neq\{0\}$. From the inequality, $\mathrm{Tr}\,(Q\,|\,_{W_{\Sigma_2}})<\mathrm{Tr}\,(Q\,|\,_{W_{\Sigma_1}})=$the eigenvalue of $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(1)}$ together with $\dim_C W_{\Sigma_2}=2$, there is a nonzero vector Z_2 in W_{Σ_2}, orthogonal to $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(1)}$. Then we have another irreducible subspace $(\mathfrak{m}^+\cdot\mathfrak{m}^+)_{\Sigma_2,Z_2}$, denoted by $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(2)}$, whose highest weight vector is Z_2.

Because of $\Sigma_3=\Sigma_1-2(\alpha_{i+1}+\cdots+\alpha_l)$, $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(1)}$ has also a non-trivial vector of W_{Σ_3}; $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(1)}\cap W_{\Sigma_3}\neq\{0\}$. Since $\Sigma_2-\Sigma_3=-\alpha_1+\alpha_{i+1}+2\alpha_{i+2}+\cdots+2\alpha_l$, by representation theory of complex semisimple Lie algebras it is concluded that Σ_3 never appears among weights of $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(2)}$, hence $W_{\Sigma_3}\cap(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(2)}=\{0\}$. And $W_{\Sigma_3}\not\subset(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(1)}$, for $\mathrm{Tr}\,(Q\,|\,_{W_{\Sigma_3}})<2$. Hence there is a weight vector Z_3 in W_{Σ_3}, orthogonal to $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(1)}$. Z_3 is also orthogonal to $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(2)}$. Therefore, we have the third irreducible subspace $(\mathfrak{m}^+\cdot\mathfrak{m}^+)_{\Sigma_3,Z_3}$, denoted by $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(3)}$, of highest weight vector Z_3.

Since $(\Sigma_1, \alpha_1)=(\Sigma_1, \alpha_{i+1})=2$ and $(\Sigma_1, \alpha_j)=0$, $j\neq1$, $i+1$, $\dim_C(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(1)}=\prod_\alpha\frac{(\Sigma_1+\delta, \alpha)}{(\delta, \alpha)}$, where the product $\prod_\alpha$ is taken over only roots of $\mathfrak{l}'$ whose α_1- or α_i-coefficient is positive. Then by a slight calculation,

$$\dim_C(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(1)}=\frac{1}{2}i(i+1)(l-i)(2l-2i+3)\,.$$

By similar calculations,

$$\dim_C(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(2)}=\frac{1}{2}i(i-1)(l-i)(2l-2i+1)$$

and

$$\dim_C(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(3)}=\frac{1}{2}i(i+1)\,.$$

Since the sum of these dimensions is equal to the dimension of the whole space $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2=\mathfrak{m}^{+1}\cdot\mathfrak{m}^{+1}$, $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2$ is completely decomposed into irreducible spaces, $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2=(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(1)}+(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(2)}+(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(3)}$.

The eigenvalues ν_j of Q on $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(j)}$, $j=1, 2, 3$ are given as follows.

$\nu_1=2$, this is already obtained.

$W_{\Sigma_2}=(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(1)}\cap W_{\Sigma_2}+(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(2)}\cap W_{\Sigma_2}$ gives a direct sum of W_{Σ_2}. Then, from (4.11) $\mathrm{Tr}\,(Q\,|_{W_{\Sigma_2}})=\nu_1+\nu_2=0$, or $\nu_2=-2$.

Since $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(3)}\cap W_{\Sigma_3}$ is one-dimensional and $W_{\Sigma_3}\cap(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(2)}=\{0\}$, $\mathrm{Tr}(Q\,|_{W_{\Sigma_3}})=(l-i)\nu_1+\nu_3=1$, hence $\nu_3=-2(l-i)+1$.

For each classical type, by a similar argument we can list the irreducible decomposition of $(\mathfrak{m}^+\cdot\mathfrak{m}^+, ad_{\mathfrak{l}'})$ together with eigenvalues of Q, the highest weights and the dimensions of irreducible subspaces (Tables 2~11).

Table 2. $(\mathrm{A}_l, \alpha_i)_{1\leq i\leq l}$

	eigenvalue	highest weight	dimension
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(1)}$	2	$2\alpha^1$	$i(i+1)(l-i+1)(l-i+2)/4$
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(2)}$	-2	$\alpha_1+2\alpha_2+\cdots+2\alpha_{l-1}+\alpha_l$	$i(i-1)(l-i)(l-i+1)/4$

(i) $\dim_C\mathfrak{m}^+=i(l-i+1)$,

(ii) $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(2)}=\{0\}$, $i=1$ or l,

(iii) $\alpha^1=\alpha_1+\cdots+\alpha_l$

Table 3. $(B_l, \alpha_1)_{2\leq l}$

	eigenvalue	highest weight	dimension
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(1)}$	2	$2\alpha^1$	$(l-1)(2l+1)$
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(2)}$	$-2l+3$	$2(\alpha_1+\cdots+\alpha_l)$	1

(i) $\dim_C\mathfrak{m}^+=2l-1$,

(ii) $\alpha^1=\alpha_1+2\alpha_2+\cdots+2\alpha_l$

Table 4. $(B_l, \alpha_i)_{1<i<l,\ 2\leq l}$

	eigenvalue	highest weight	dimension
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(1)}$	2	$2\alpha^1$	$i(i+1)(l-i)(2l-2i+3)/2$
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(2)}$	-2	$\alpha_1+2\alpha_2+\cdots+2\alpha_i$ $+3\alpha_{i+1}+4\alpha_{i+2}+\cdots$ $+4\alpha_l$	$i(i-1)(l-i)(2l-2i+1)/2$
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(3)}$	$-2(l-i)+1$	$2(\alpha_1+\cdots+\alpha_l)$	$i(i+1)/2$
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^3_{(1)}$	1	$\alpha^1+\alpha^2$	$i(i-1)(i+1)(2l-2i+1)/3$
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^3_{(2)}$	-2	$\alpha_1+2\alpha_2+3\alpha_3$ $+\cdots+3\alpha_i+4\alpha_{i+1}$ $+\cdots+4\alpha_l$	$i(i-1)(i-2)(2l-2i+1)/6$
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(1)}$	1	$2\alpha^2$	$i(i-1)(i^2-i+2)/8$

(i) $\dim_C \mathfrak{m}^+ = i(4l-3i+1)/2$, $\dim_C \mathfrak{m}^{+1} = i(2l-2i+1)$,
$\dim_C \mathfrak{m}^{+2} = i(i-1)/2$

(ii) $\alpha^1 = \alpha_1 + \cdots + \alpha_i + 2\alpha_{i+1} + \cdots + 2\alpha_l$,
$\alpha^2 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_l$

Table 5. $(B_l, \alpha_l)_{2 \leq l}$

	eigenvalue	highest weight	dimension
$(\mathfrak{m}^+ \cdot \mathfrak{m}^+)^2_{(1)}$	1	$2\alpha^1$	$l(l+1)/2$
$(\mathfrak{m}^+ \cdot \mathfrak{m}^+)^3_{(1)}$	1	$\alpha^1 + \alpha^2$	$(l-1)l(l+1)/3$
$(\mathfrak{m}^+ \cdot \mathfrak{m}^+)^3_{(2)}$	-2	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + \cdots + 3\alpha_l$	$(l-2)(l-1)l/6$
$(\mathfrak{m}^+ \cdot \mathfrak{m}^+)^4_{(1)}$	1	$2\alpha^2$	$l^2(l-1)(l+1)/12$
$(\mathfrak{m}^+ \cdot \mathfrak{m}^+)^4_{(2)}$	-2	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + \cdots + 4\alpha_l$	$l(l-1)(l-2)(l-3)/24$

(i) $\dim_C \mathfrak{m}^+ = l(l+1)/2$, $\dim_C \mathfrak{m}^{+1} = l$, $\dim_C \mathfrak{m}^{+2} = l(l-1)/2$,

(ii) $\alpha^1 = \alpha_1 + \cdots + \alpha_l$,
$\alpha^2 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_l$

Table 6. $(C_l, \alpha_1)_{2 \leq l}$

	eigenvalue	highest weight	dimension
$(\mathfrak{m}^+ \cdot \mathfrak{m}^+)^2_{(1)}$	2	$2\alpha^1$	$(l-1)(2l-1)$
$(\mathfrak{m}^+ \cdot \mathfrak{m}^+)^3_{(1)}$	2	$\alpha^1 + \alpha^2$	$2l-2$
$(\mathfrak{m}^+ \cdot \mathfrak{m}^+)^4_{(1)}$	2	$2\alpha^2$	1

(i) $\dim_C \mathfrak{m}^+ = 2l-1$, $\dim_C \mathfrak{m}^{+1} = 2l-2$, $\dim_C \mathfrak{m}^{+2} = 1$,

(ii) $\alpha^1 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{l-1} + \alpha_l$,
$\alpha^2 = 2\alpha_1 + \cdots + 2\alpha_{l-1} + \alpha_l$

Table 7. $(C_l, \alpha_i)_{1<i<l,\ 2 \leq l}$

	eigenvalue	highest weight	dimension
$(\mathfrak{m}^+ \cdot \mathfrak{m}^+)^2_{(1)}$	2	$2\alpha^1$	$i(i+1)(l-i)(2l-2i+1)/2$
$(\mathfrak{m}^+ \cdot \mathfrak{m}^+)^2_{(2)}$	-2	$\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_i + 3\alpha_{i+1} + 4\alpha_{i+2} + \cdots + 4\alpha_{l-1} + 2\alpha_l$	$i(i-1)(l-i-1)(2l-2i+1)/2$
$(\mathfrak{m}^+ \cdot \mathfrak{m}^+)^2_{(3)}$	$-2(l-i+1)$	$\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{l-1} + \alpha_l$	$i(i-1)/2$

$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^3_{(1)}$ $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^3_{(1)}$	2 -1	$\alpha^1+\alpha^2$ $2\alpha_1+3\alpha_2+\cdots+3\alpha_i$ $+4\alpha_{i+1}$ $+\cdots+4\alpha_{l-1}+2\alpha_l$	$i(i+1)(i+2)(l-i)/3$ $2i(i+1)(i-1)(l-i)/3$
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(1)}$ $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(2)}$	2 -1	$2\alpha^2$ $2\alpha_1+4\alpha_2+\cdots+4\alpha_{l-1}$ $+2\alpha_l$	$i(i+1)(i+2)(i+3)/24$ $i^2(i+1)(i-1)/12$

(i) $\dim_C\mathfrak{m}^+=i(4l-3i+1)/2,\quad \dim_C\mathfrak{m}^{+1}=2i(l-i),\quad \dim_C\mathfrak{m}^{+2}=i(i+1)/2$

(ii) $\alpha^1=\alpha_1+\cdots+\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_{l-1}+\alpha_l,$
$\alpha^2=2\alpha_1+\cdots+2\alpha_i+\cdots+2\alpha_{l-1}+\alpha_l$

Table 8. $(C_l, \alpha_l)_{2\leq l}$

	eigenvalue	highest weight	dimension
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(1)}$ $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(2)}$	4 -2	$2\alpha^1$ $2\alpha_1+4\alpha_2+\cdots+4\alpha_{l-1}$ $+2\alpha_l$	$(l+1)(l+2)(l+3)/24$ $l^2(l^2-1)/12$

(i) $\dim_C\mathfrak{m}^+=l(l+1)/2$

(ii) $\alpha^1=2\alpha_1+\cdots+2\alpha_{l-1}+\alpha_l.$

Table 9. $(D_l, \alpha_1)_{3\leq l}$

	eigenvalue	highest weight	dimension
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(1)}$ $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(2)}$	2 $-2(l-2)$	$2\alpha^1$ $2\alpha_1+\cdots+2\alpha_{l-2}$ $+\alpha_{l-1}+\alpha_l$	$l(2l-3)$ 1

(i) $\dim_C\mathfrak{m}^+=2l-2$

(ii) $\alpha^1=\alpha_1+2\alpha_2+\cdots+2\alpha_{l-2}+\alpha_{l-1}+\alpha_l$

Table 10. $(D_l, \alpha_i)_{1<i<l,\ 3\leq l}$

	eigenvalue	highest weight	dimension
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(1)}$ $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(2)}$	2 -2	$2\alpha^1$ $\alpha_1+2\alpha_2+\cdots+2\alpha_i$ $+3\alpha_{i+1}+4\alpha_{i+2}$ $+\cdots+4\alpha_{l-2}+2\alpha_{l-1}$ $+2\alpha_l$	$i(i+1)(l-i+1)(2l-2i-1)/2$ $i(i-1)(l-i)(2l-2i-1)/2$
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(3)}$	$-2(l-i)+2$	$2\alpha_1+\cdots+2\alpha_{l-2}$ $+\alpha_{l-1}+\alpha_l$	$i(i+1)/2$

$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^3_{(1)}$ $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^3_{(2)}$	1 -2	$\alpha^1+\alpha^2$ $\alpha_1+2\alpha_2+3\alpha_3+\cdots$ $+3\alpha_i+4\alpha_{i+1}+\cdots$ $+4\alpha_{l-2}+2\alpha_{l-1}+2\alpha_l$	$(i-1)(i+1)(l-i)/2$ $i(i-1)(i-2)(l-i)/3$
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(1)}$ $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(2)}$	1 -1	$2\alpha^2$ $\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4$ $+\cdots+4\alpha_{l-2}+2\alpha_{l-1}$ $+2\alpha_l$	$i^2(i^2-1)/12$ $i(i-1)(i^2-i+2)/24$

(i) $\dim_C\mathfrak{m}^+=i(4l-3i-1)/2$, $\dim_C\mathfrak{m}^{+1}=2i(l-i)$, $\dim_C\mathfrak{m}^{+2}=i(i-1)/2$

(ii) $\alpha^1=\alpha_1+\cdots+\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_{l-2}+\alpha_{l-1}+\alpha_l$, $\alpha^2=\alpha_1+2\alpha_2+\cdots+2\alpha_{l-2}+\alpha_{l-1}+\alpha_l$

Table 11. $(D_l, \alpha_l)_{3\leq l}$

	eigenvalue	highest weight	dimension
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(1)}$ $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(2)}$	2 -4	$2\alpha^1$ *	$l^2(l^2-1)/12$ $(l-1)(l-2)(l-3)/24$

(i) $\dim_C\mathfrak{m}^+=l(l-1)/2$

(ii) $\alpha^1=\alpha_1+2\alpha_2+\cdots+2\alpha_{l-2}+\alpha_{l-1}+\alpha_l$,
the highest weight of $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(2)}$:

$$\alpha_1+2\alpha_2+\alpha_3+2\alpha_4 \qquad (l=4)$$
$$\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4+2\alpha_5 \qquad (l=5)$$
$$\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+\cdots+4\alpha_{l-2}+2\alpha_{l-1}+\alpha_l \qquad (l\geqq 6)$$

(iii) the eigenvalues with multiplicities of (D_l, α_{l-1}) are the same as of (D_l, α_l)

We shall show the assertion of Theorem 5.1 by the aid of tables 2~11 together with Theorem 4.1.

Proof of Theorem 5.1 *for classical types.*

From tables 2~11, a Kähler C-space $(G_u/K, g)$ associated with one of the following pairs; $(B_l, \alpha_i)_{1<i<l}$, $(C_l, \alpha_i)_{1<i<l}$, $(D_l, \alpha_i)_{1<i<l}$, can not be hermitian symmetric.

$(B_l, \alpha_i)_{1<i<l}$ *and* $(D_l, \alpha_i)_{1<i<l}$: The weight vectors $E_{\alpha^1\cdot\alpha^1}$ and $E_{\alpha^1\cdot\alpha^2}$ corresponding to the highest weights $2\alpha^1$ and $\alpha^1+\alpha^2$ give eigenvectors of Q which belong to the eigenvalues 2 and 1 respectively. Moreover we have,

$$\Lambda(E_\beta)E_{\alpha^1\cdot\alpha^1}=2N_{\beta,\alpha^1}\cdot E_{\alpha^1\cdot\alpha^2} \text{ for some } \beta=\alpha^2-\alpha^1\in\Delta_i^+(1),\ N_{\beta,\alpha^1}\neq 0.$$

Thus, the Λ-invariance of eigenspaces of Q does not hold.

$(C_l, \alpha_i)_{1<i<l}$: Let Z_Σ be a vector which belongs to the eigenvalue -2, $\Sigma=\alpha_1+2\alpha_2+\cdots+2\alpha_i+3\alpha_{i+1}+4\alpha_{i+2}+\cdots+4\alpha_{l-1}+2\alpha_l$. Then, $\Lambda(E_\alpha)Z_\Sigma$ is a nonzero vector in W_Θ $(\subset(\mathfrak{m}^+\cdot\mathfrak{m}^+)^3)$ for $\alpha=\alpha_1+\alpha_2+\cdots+\alpha_i+\alpha_{i+1}$ and $\Theta=2\alpha_1+3\alpha_2+\cdots+3\alpha_i+4\alpha_{i+1}+\cdots+4\alpha_{l-1}+2\alpha_l$.

Since Q has 2 and -1 as its eigenvalues on $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^3$ (cf., table 7), it follows that the eigenspaces are not Λ-invariant.

$(A_l, \alpha_i)_{1\leqq i\leqq l}$, (B_l, α_1), (C_l, α_l), (D_l, α_1), (D_l, α_{l-1}) and (D_l, α_l): In these cases, we have $\mathfrak{m}^+=\mathfrak{m}^{+1}$, that is, a Kähler C-space associated with each of the pairs is normal. It follows that the operation $\Lambda(X)$ is trivial. Hence, the eigenspaces of Q are Λ-invariant.

(C_l, α_1): Q satisfies that $Q=2\cdot$identity on the whole $\mathfrak{m}^+\cdot\mathfrak{m}^+$.

(B_2, α_2): In this case, Q satisfies also that $Q=$identity on the whole space.

$(B_l, \alpha_l)_{l\geqq 3}$: The eigenspaces corresponding to the eigenvalues 1 and -2 are given by $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(1)}+(\mathfrak{m}^+\cdot\mathfrak{m}^+)^3_{(1)}+(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(1)}$ and $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^3_{(2)}+(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(2)}$ respectively. We can easily conclude that

$$\Lambda(X)(\mathfrak{m}^+\cdot\mathfrak{m}^+)^i_{(1)}\subset(\mathfrak{m}^+\cdot\mathfrak{m}^+)^{i+1}_{(1)} \quad\text{and}\quad \Lambda(\bar{X})(\mathfrak{m}^+\cdot\mathfrak{m}^+)^{i+1}_{(1)}\subset(\mathfrak{m}^+\cdot\mathfrak{m}^+)^i_{(1)},$$
$$X\in\mathfrak{m}^{+1},\ i=2 \text{ and } 3,$$

from the properties of the root system of type B_l.

Let $l\geqq 4$. Let W_{Σ_1} and W_{Σ_2} be the weight spaces corresponding to the dominant integral weights $\Sigma_1=\alpha_1+2\alpha_2+3\alpha_3+\cdots+3\alpha_l$ and $\Sigma_2=\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+\cdots+4\alpha_l$ respectively;

$$W_{\Sigma_1}=\sum_{i=1}^{3} CE_{\gamma_i\cdot\delta_i} \text{ and } W_{\Sigma_2}=\sum_{i=1}^{3} CE_{\sigma_i\cdot\tau_i},$$

where, $\gamma_1=\alpha_1+\cdots+\alpha_l$, $\delta_1=\alpha_2+2\alpha_3+\cdots+2\alpha_l$, $\gamma_i=\gamma_{i-1}-\alpha_{i-1}$, $\delta_i=\delta_{i-1}+\alpha_{i-1}$, $i=2$ and 3 respectively, and $\sigma_1=\alpha_1+2\alpha_2+\cdots+2\alpha_l$, $\tau_1=\alpha_3+2\alpha_4+\cdots+2\alpha_l$, $\sigma_i=\sigma_{i-1}-\alpha_i$, $\tau_i=\tau_{i-1}+\alpha_i$, $i=2$ and 3, respectively.

Let $E_{-\alpha}$ be the root vector of $-\alpha=-(\alpha_4+\cdots+\alpha_l)$. Then, the connection function $\Lambda(E_{-\alpha})$ gives a linear isomorphism of W_{Σ_2} onto W_{Σ_1}. Let Z be a weight vector which corresponds to the highest weight of $(\mathfrak{m}^+\mathfrak{m}^+)^4_{(2)}$, that is, $Z\in W_{\Sigma_2}$ and $ad(E_\beta)Z=0$ for each $\beta\in\Pi-\{\alpha_l\}$. Since $ad(E_{-\alpha_2})E_{\sigma_1\cdot\tau_2}=N_{-\alpha,\tau_2}E_{\sigma_1\cdot\tau_1}$ and $ad(E_{-\alpha_3})E_{\sigma_2\cdot\tau_3}=N_{-\alpha_3,\tau_3}E_{\sigma_2\cdot\tau_2}$ belong to $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^3_{(1)}\cap W_{\Sigma_1}$, we have

$$\langle Z, ad(E_{-\alpha_2})E_{\sigma_1\cdot\tau_2}\rangle=\langle Z, ad(E_{-\alpha_3})E_{\sigma_2\cdot\tau_3}\rangle=0.$$

Since $\Lambda(E_\alpha)\Lambda(E_{-\alpha})\,ad(E_{-\alpha_2})E_{\sigma_1\cdot\tau_2}$ is proportional to $ad(E_{-\alpha_2})E_{\sigma_1\cdot\tau_2}$, we get

$$\langle\Lambda(E_{-\alpha})Z, \Lambda(E_{-\alpha})\,ad(E_{-\alpha_2})E_{\sigma_1\cdot\tau_2}\rangle=-\langle Z, \Lambda(E_\alpha)\Lambda(E_{-\alpha})\,ad(E_{-\alpha_2})E_{\sigma_1\cdot\tau_2}\rangle=0.$$

Similarly, $\langle\Lambda(E_{-\alpha})Z, \Lambda(E_{-\alpha})\,ad(E_{-\alpha_3})E_{\sigma_2\cdot\tau_3}\rangle=0$. $\Lambda(E_{-\alpha})\,ad(E_{-\alpha_2})E_{\sigma_1\cdot\tau_2}$ and $\Lambda(E_{-\alpha})\,ad(E_{-\alpha_3})E_{\sigma_2\cdot\tau_3}$ are linearly independent in $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^3_{(1)}\cap W_{\Sigma_1}$. Hence, $\Lambda(E_{-\alpha})Z$ belongs to $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^3_{(2)}\cap W_{\Sigma_1}$. Thus, we have that

$$\Lambda(E_{-\alpha}): (\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(2)}\cap W_{\Sigma_2}\longrightarrow(\mathfrak{m}^+\cdot\mathfrak{m}^+)^3_{(2)}\cap W_{\Sigma_1}$$

is an isomorphism. We can also conclude that

$$\Lambda(E_\alpha):\ (\mathfrak{m}^+\cdot\mathfrak{m}^+)^3_{(2)}\cap W_{\Sigma_1}\longrightarrow(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(2)}\cap W_{\Sigma_2}$$

gives an isomorphism.

In order to verify the Λ-invariance of the eigenspace $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^3_{(2)}+(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(2)}$, we need to show that

$$\Lambda(E_\beta)\Lambda(E_{-\alpha})Z\in(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(2)}\,,\qquad \Lambda(E_{-\beta})Z\in(\mathfrak{m}^+\cdot\mathfrak{m}^+)^3_{(2)}$$

and $\Lambda(E_{-\beta})\Lambda(E_{-\alpha})Z=0$, $\beta\in\Delta_l^+(1)$, from Proposition 4.5.

If $\beta<\alpha$, then, neither $\beta+\gamma_i$ nor $\beta+\delta_i$ is in Δ_l^+, $i=1, 2, 3$. Hence, we have $\Lambda(E_\beta)\Lambda(E_{-\alpha})Z=0$. On the other hand, there is a $\beta\in\Delta^+-\Delta_l^+$ such that $\beta=\alpha+\beta'$, when $\beta>\alpha$. Then, we have,

$$\begin{aligned}\Lambda(E_\beta)\Lambda(E_{-\alpha})Z&=\frac{1}{N_{\beta',\alpha}}\Lambda([E_{\beta'}, E_\alpha])\Lambda(E_{-\alpha})Z\\&=\frac{1}{N_{\beta',\alpha}}\left\{ad\,(E_{\beta'})\Lambda(E_\alpha)\Lambda(E_{-\alpha})Z-\Lambda(E_\alpha)\,ad\,(E_{\beta'})\Lambda(E_{-\alpha})Z\right\}\end{aligned}$$

which is proportional to $ad\,(E_{\beta'})Z$, by a slight calculation. Thus, $\Lambda(E_\beta)\Lambda(E_{-\alpha})Z\in(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(2)}$.

Similarly, we have $\Lambda(E_{-\beta})Z\in(\mathfrak{m}^+\cdot\mathfrak{m}^+)^3_{(2)}$ and $\Lambda(E_{-\beta})\Lambda(E_{-\alpha})Z=0$ for $\beta\in\Delta_l^+(1)$.

If $l=3$, we have $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(2)}=\{0\}$ from table 5. We can easily show that $\Lambda(E_\beta)(\mathfrak{m}^+\cdot\mathfrak{m}^+)^3_{(2)}=\Lambda(E_{-\beta})(\mathfrak{m}^+\cdot\mathfrak{m}^+)^3_{(2)}=\{0\}$ for $\beta\in\Delta_3^+(1)$. Q.E.D.

PROOF OF THEOREM 5.1 *for exceptional types.* Let $(G_u/K, g)$ be a Kähler C-space associated with $(\mathfrak{g}, \alpha_i)$, where $\mathfrak{g}$ is a complex simple Lie algebra of exceptional type and α_i is a fundamental root.

The irreducible decomposition of $\mathfrak{m}^+\cdot\mathfrak{m}^+$ is so complicated for the exceptional Lie algebra $\mathfrak{g}$ that we shall partially compute the eigenvalues of Q and investigate the Λ-invariance of some main irreducible subspaces $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(1)}$ and $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^3_{(1)}$ (cf., table 12).

Table 12.

	E_6		E_7	
	α_1, α_6	$\alpha_2, \alpha_3, \alpha_4, \alpha_5$	$\alpha_i,\ i=1, \cdots, 6$	α_7
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(1)}$	2	2	2	2
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^3_{(1)}$		1	1	

E_8	F_4			G_2	
$\alpha_i,\ i=1, \cdots, 8$	α_1, α_2	α_3	α_4	α_1	α_2
2	2	1	1	2	6
1	1	1/2	1	2	3

With the aid of table 12, a Kähler C-space associated with each of the following pairs can not be hermitian symmetric; $(E_6, \alpha_i)_{i=2,3,4,5}$, $(E_7, \alpha_i)_{i=1,\dots,6}$, $(E_8, \alpha_i)_{i=1,\dots,8}$, $(F_4, \alpha_i)_{i=1,2,3}$ and (G_2, α_2), since there exist nonzero vectors Z^2 and Z^3 ($Z^i \in (\mathfrak{m}^+\cdot\mathfrak{m}^+)^i_{(1)}$, $i=2, 3$ respectively), and $X \in \mathfrak{m}^{+1}$ such that $\Lambda(X)Z^2=Z^3$, for each pairs.

We have $\mathfrak{m}^+=\mathfrak{m}^{+1}$ for a Kähler C-space associated with each of (E_6, α_1), (E_6, α_6) and (E_7, α_7). Hence, each Kähler C-space is hermitian symmetric.

Now we shall show that the Kähler C-space associated with (F_4, α_4) can not be hermitian symmetric, and that the C-space associated with (G_2, α_1) is hermitian symmetric.

(F_4, α_4): The irreducible decomposition of $\mathfrak{m}^+\cdot\mathfrak{m}^+$ for the Kähler C-space associated with (F_4, α_4) is obtained by a slight computation as the following table.

Table 13. (F_4, α_4)

	eigenvalue	highest weight	dimension
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(1)}$	1	$2\alpha^1$	35
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(2)}$	-5	$\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4$	1
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^3_{(1)}$	1	$\alpha^1+\alpha^2$	48
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^3_{(2)}$	$-5/2$	$2\alpha_1+4\alpha_2+6\alpha_3+3\alpha_4$	8
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(1)}$	1	$2\alpha^2$	27
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(2)}$	$-5/2$	$2\alpha_1+4\alpha_2+6\alpha_3+4\alpha_4$	1

i) $\dim_C \mathfrak{m}^+=15$,

ii) $\alpha^1=\alpha_1+2\alpha_2+3\alpha_3+\alpha_4$,

iii) $\alpha^2=2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4$

By the argument which is similar to the case (B_l, α_l), we can verify that $\Lambda(E_{\alpha^1})Z$ is a weight vector of the highest weight of $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^3_{(2)}$, where Z is some weight vector of the highest weight of $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(2)}$. Hence, the eigenspaces of Q are not Λ-invariant.

(G_2, α_1): Let α_1 and α_2 be the fundamental roots of simple Lie algebra of type G_2; $\Pi=\{\alpha_1, \alpha_2\}$. We have, $(\alpha_1, \alpha_1)=2$, $(\alpha_1, \alpha_2)=-3$ and $(\alpha_2, \alpha_2)=6$, and the set Δ^+ of positive roots is given by

$$\Delta^+=\{\alpha_2, \alpha_1, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2, 3\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2\}\ .$$

The subset Δ_1^+ is obtained by $\Delta_1^+=\Delta^+-\{\alpha_2\}$; $\sharp\Delta_1^+=5$. We have, then, $\Delta_1^+=\Delta_1^+(1)\cup\Delta_1^+(2)\cup\Delta_1^+(3)$; $\Delta_1^+(1)=\{\alpha_1, \alpha_1+\alpha_2\}$, $\Delta_1^+(2)=\{2\alpha_1+\alpha_2\}$ and $\Delta_1^+(3)=\{3\alpha_1+$

$\alpha_2, 3\alpha_1+2\alpha_2\}$ and $\mathfrak{m}^+=\sum_{\alpha\in\Delta^+} CE_\alpha$ is decomposed into $\mathfrak{m}^{+1}, \mathfrak{m}^{+2}$ and $\mathfrak{m}^{+3}$. Hence, $\mathfrak{m}^+\cdot\mathfrak{m}^+$ has the following decomposition; $\mathfrak{m}^+\cdot\mathfrak{m}^+=\sum_n(\mathfrak{m}^+\cdot\mathfrak{m}^+)^n$, $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2=\mathfrak{m}^{+1}\cdot\mathfrak{m}^{+1}$, $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^3=\mathfrak{m}^{+1}\cdot\mathfrak{m}^{+2}$, $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4=\mathfrak{m}^{+1}\cdot\mathfrak{m}^{+3}+\mathfrak{m}^{+2}\cdot\mathfrak{m}^{+2}$, $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^5=\mathfrak{m}^{+2}\cdot\mathfrak{m}^{+3}$ and $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^6=\mathfrak{m}^{+3}\cdot\mathfrak{m}^{+3}$, and the sets of weights with respect to $((\mathfrak{m}^+\cdot\mathfrak{m}^+)^n, ad_t c)$ are given in the following forms respectively, $n=2,\cdots,6$; $\{2\alpha_1, 2\alpha_1+\alpha_2, 2\alpha_1+2\alpha_2\}$, $\{3\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2\}$, $\{4\alpha_1+\alpha_2, 4\alpha_1+2\alpha_2, 4\alpha_1+3\alpha_2\}$, $\{5\alpha_1+2\alpha_2, 5\alpha_1+3\alpha_2\}$ and $\{6\alpha_1+2\alpha_2, 6\alpha_1+3\alpha_2, 6\alpha_1+4\alpha_2\}$.

We note that $4\alpha_1+2\alpha_2$ is an only weight whose degree is greater than 1, in fact, its degree is equal to 3;

$$W_\Sigma=CE_{\alpha_1\cdot(3\alpha_1+2\alpha_2)}+CE_{(\alpha_1+\alpha_2)\cdot(3\alpha_1+\alpha_2)}+CE_{(2\alpha_1+\alpha_2)\cdot(2\alpha_1+\alpha_2)}, \quad (\Sigma=4\alpha_1+2\alpha_2).$$

We shall show that W_Σ can be splitted into two eigenspaces of Q, one corresponds to eigenvalue 2 with multiplicity 2 and the other corresponds to -3, and moreover, the eigenspace corresponding to 2 is decomposed into two subspaces, one is contained in $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(1)}$, the other gives the one dimensional irreducible subspace $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(2)}$.

The operator Q is described on W_Σ by

$$Q\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}=\begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

for a unitary basis $\{X, Y, Z\}$ of W_Σ, $X=\sqrt{\frac{2}{3}}E_{\alpha_1\cdot(3\alpha_1+2\alpha_2)}$. $Y=\sqrt{\frac{2}{3}}E_{(\alpha_1+\alpha_2)\cdot(3\alpha_1+\alpha_2)}$ and $Z=\frac{1}{2}E_{(2\alpha_1+\alpha_2)\cdot(2\alpha_1+\alpha_2)}$. Then, by the followingtable, which is concerned with the coefficients $N_{\alpha,\beta}$'s, $\alpha, \beta\in\Delta^+$; we obtain

$$\begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}=\begin{pmatrix} 0 & 2 & -\sqrt{2} \\ 2 & 0 & \sqrt{2} \\ -\sqrt{2} & \sqrt{2} & 1 \end{pmatrix}.$$

Table 14. The coefficients $N_{\alpha,\beta}$'s for G_2

	α_2	α_1	$\alpha_1+\alpha_2$	$2\alpha_1+\alpha_2$	$3\alpha_1+\alpha_2$	$3\alpha_1+2\alpha_2$
α_2	$\times$	$\sqrt{3}$	0	0	$\sqrt{3}$	0
α_1	$-\sqrt{3}$	$\times$	2	$\sqrt{3}$	0	0
$\alpha_1+\alpha_2$	0	-2	$\times$	$\sqrt{3}$	0	0
$2\alpha_1+\alpha_2$	0	$-\sqrt{3}$	$-\sqrt{3}$	$\times$	0	0
$3\alpha_1+\alpha_2$	$-\sqrt{3}$	0	0	0	$\times$	0
$3\alpha_1+2\alpha_2$	0	0	0	0	0	$\times$

Thus, $E_{\alpha_1\cdot(3\alpha_1+2\alpha_2)}+E_{(\alpha_1+\alpha_2)\cdot(3\alpha_1+\alpha_2)}=\sqrt{\frac{3}{2}}(X+Y)$ and $2E_{\alpha_1\cdot(3\alpha_1+2\alpha_2)}-\sqrt{3}\,E_{(2\alpha_1+\alpha_2)\cdot(2\alpha_1+\alpha_2)}=\sqrt{6}(X-\sqrt{2}Z)$ give eigenvectors to the eigenvalue 2, and $4E_{\alpha_1\cdot(3\alpha_1+2\alpha_2)}-4E_{(\alpha_1+\alpha_2)\cdot(3\alpha_1+\alpha_2)}+\sqrt{3}\,E_{(2\alpha_1+\alpha_2)\cdot(2\alpha_1+\alpha_2)}=\frac{\sqrt{2}}{4\sqrt{3}}\left(X-Y+\frac{1}{\sqrt{2}}Z\right)$ belongs to the eigenvalue -3.

Since $ad\,(E_{\alpha_2})(X-\sqrt{2}Z)=ad\,(E_{-\alpha_2})(X-\sqrt{2}Z)=0$,

the weight vectors $2E_{\alpha_1\cdot(3\alpha_1+2\alpha_2)}-\sqrt{3}\,E_{(2\alpha_1+\alpha_2)\cdot(2\alpha_1+\alpha_2)}$ and $4E_{\alpha_1\cdot(3\alpha_1+2\alpha_2)}-4E_{(\alpha_1+\alpha_2)\cdot(\alpha_1+\alpha_2)}+\sqrt{3}\,E_{(2\alpha_1+\alpha_2)\cdot(2\alpha_1+\alpha_2)}$ give the irreducible subspaces, denoted by $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(2)}$ and $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(3)}$ respectively. That is,

$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(2)}=\boldsymbol{C}(2E_{\alpha_1\cdot(3\alpha_1+2\alpha_2)}-\sqrt{3}\,E_{(2\alpha_1+\alpha_2)\cdot(2\alpha_1+\alpha_2)})$ and $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(3)}=\boldsymbol{C}(4E_{\alpha_1\cdot(3\alpha_1+2\alpha_2)}-4E_{(\alpha_1+\alpha_2)\cdot(3\alpha_1+\alpha_2)}+\sqrt{3}E_{(2\alpha_1+\alpha_2)\cdot(2\alpha_1+\alpha_2)}$. On the other hand, we have, $E_{\alpha_1\cdot(3\alpha_1+2\alpha_2)}+E_{(\alpha_1+\alpha_2)\cdot(3\alpha_1+\alpha_2)}=\frac{1}{\sqrt{3}}ad(E_{-\alpha_2})E_{(\alpha_1+\alpha_2)\cdot(3\alpha_1+2\alpha_2)}$, hence $E_{\alpha_1\cdot(3\alpha_1+\alpha_2)}+E_{(\alpha_1+\alpha_2)\cdot(3\alpha_1+\alpha_2)}$ belongs to $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(1)}$.

Therefore, we have the following table which gives the complete irreducible decomposition of $\mathfrak{m}^+\cdot\mathfrak{m}^+$.

Table 15. (G_2, α_1)

	eigenvalue	highest weight	dimension
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(1)}$	2	$2\alpha_1+2\alpha_2$	3
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^3_{(1)}$	2	$3\alpha_1+2\alpha_2$	2
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(1)}$	2	$4\alpha_1+3\alpha_2$	3
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(2)}$	2	$4\alpha_1+2\alpha_2$	1
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(3)}$	-3	$4\alpha_1+2\alpha_2$	1
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^5_{(1)}$	2	$5\alpha_1+3\alpha_2$	2
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^6_{(1)}$	2	$6\alpha_1+4\alpha_2$	3

It is easily verified that $\Lambda(X)(\mathfrak{m}^+\cdot\mathfrak{m}^+)^j_{(1)}\subset(\mathfrak{m}^+\cdot\mathfrak{m}^+)^{j+1}_{(1)}$, and $\Lambda(\bar{X})(\mathfrak{m}^+\cdot\mathfrak{m}^+)^{j+1}_{(1)}\subset(\mathfrak{m}^+\cdot\mathfrak{m}^+)^j_{(1)}$, $X\in\mathfrak{m}^{+1}$, $j=2$ and 5. Hence, in order to show that each eigenspace of Q is Λ-invariant, it is sufficient to verify that

$$\Lambda(X)(\mathfrak{m}^+\cdot\mathfrak{m}^+)^3_{(1)}\subset(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(1)}+(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(2)},$$

$$\Lambda(\bar{X})\{(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(1)}+(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(2)}\}\subset(\mathfrak{m}^+\cdot\mathfrak{m}^+)^3_{(1)},\quad X\in\mathfrak{m}^{+1},$$

and

$$\Lambda(X)(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(3)}=\Lambda(\bar{X})(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(3)}=\{0\},\qquad X\in\mathfrak{m}^+.$$

With respect to the vector $E_{(\alpha_1+\alpha_2)\cdot(2\alpha_1+\alpha_2)}$ corresponding to the highest weight $3\alpha_1+2\alpha_2$ of $(\mathfrak{m}^+\cdot\mathfrak{m}^+)^3_{(1)}$, we have

$\Lambda(E_{\alpha_1})E_{(\alpha_1+\alpha_2)\cdot(2\alpha_1+\alpha_2)}=\frac{2}{\sqrt{3}}E_{(\alpha_1+\alpha_2)\cdot(3\alpha_1+\alpha_2)}+E_{(2\alpha_1+\alpha_2)\cdot(2\alpha_1+\alpha_2)}$ and

$\Lambda(E_{\alpha_1+\alpha_2})E_{(\alpha_1+\alpha_2)\cdot(2\alpha_1+\alpha_2)}=\frac{2}{\sqrt{3}}E_{(\alpha_1+\alpha_2)\cdot(3\alpha_1+2\alpha_2)}$. Thus, $\Lambda(X)E_{(\alpha_1+\alpha_2)\cdot(2\alpha_1+\alpha_2)}\in(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(1)}+(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(2)}$, $X\in\mathfrak{m}^{+1}$. Therefore, from Proposition 4.5, we have

$$\Lambda(X)(\mathfrak{m}^+\cdot\mathfrak{m}^+)^3_{(1)}\subset(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(1)}+(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(2)},\ X\in\mathfrak{m}^{+1}.$$

Similarly, we can verify that

$$\Lambda(\bar{X})\{(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(1)}+(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(2)}\}\subset(\mathfrak{m}^+\cdot\mathfrak{m}^+)^3_{(1)},\ X\in\mathfrak{m}^{+1}.$$

We have also,

$$\begin{aligned}&\Lambda(E_\alpha)(4E_{\alpha_1\cdot(3\alpha_1+2\alpha_2)}-4E_{(\alpha_1+\alpha_2)\cdot(3\alpha_1+\alpha_2)}+\sqrt{3}E_{(2\alpha_1+\alpha_2)\cdot(2\alpha_1+\alpha_2)})\\&=\Lambda(E_{-\alpha})(4E_{\alpha_1\cdot(3\alpha_1+2\alpha_2)}-4E_{(\alpha_1+\alpha_2)\cdot(3\alpha_1+\alpha_2)}+\sqrt{3}E_{(2\alpha_1+\alpha_2)\cdot(2\alpha_1+\alpha_2)})\\&=0,\quad \alpha\in\Delta_1^+.\end{aligned}$$

It follows that $\Lambda(X)(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(3)}=\Lambda(\bar{X})(\mathfrak{m}^+\cdot\mathfrak{m}^+)^4_{(3)}=\{0\}$, $X\in\mathfrak{m}^+$. Thus, we can verify the Λ-invariance of the eigenspaces of Q. Q.E.D.

Remark. It is known that a Kähler C-space associated with (G_2, α_1) is a hermitian symmetric space (Nakagawa and Takagi [**10**]). It can be imbedded as a 5-dim Kähler submanifold in a 6-dim complex projective space, by using the representation of $\mathfrak{g}$ of type G_2, whose highest weight is Λ_1 (see §1 for the definition of Λ_1). It is an Einstein Kähler manifold. Hence, it is hermitian symmetric by the result of Smyth [**12**].

Proof of Theorem 5.2. To verify Theorem 5.2, it is sufficient from Theorem 5.1 to show that the operator Q has at most two different eigenvalues for each of pairs listed at Theorem 5.1, and that Q has at least three eigenvalues for a Kähler C-space associated with each of the others.

This is an immediate conclusion for classical type by tables 2~11.

For exceptional type, we have the following tables;

Table 16. (E_6, α_1)

	eigenvalue	highest weight	dimesion
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(1)}$	2	2α	126
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(2)}$	-6	$\begin{smallmatrix}2&3&4&3&2\\&&2&&\end{smallmatrix}$	10

(i) $\dim_C\mathfrak{m}^+=16$,

(ii) by $\begin{smallmatrix}a&c&d&e&f\\&&b&&\end{smallmatrix}$ we mean a form $a\alpha_1+b\alpha_2+c\alpha_3+d\alpha_4+e\alpha_5+f\alpha_6$,

(iii) $\alpha=\begin{smallmatrix}1&2&3&2&1\\&&2&&\end{smallmatrix}$

(iv) the eigenvalues of (E_6, α_6) with multiplicities are the same as of (E_6, α_1).

Table 17. (E_7, α_7)

	eigenvalue	highest weight	dimension
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(1)}$	2	2α	351
$(\mathfrak{m}^+\cdot\mathfrak{m}^+)^2_{(2)}$	-8	$\begin{smallmatrix}2&4&6&5&4&2\\&&3&&&\end{smallmatrix}$	27

(i) $\dim_C \mathfrak{m}^+=27$,

(ii) by $\begin{smallmatrix}a&c&d&e&f&g\\&&b&&&\end{smallmatrix}$ we mean a form $a\alpha_1+b\alpha_2+c\alpha_3+d\alpha_4+e\alpha_5+f\alpha_6+g\alpha_7$,

(iii) $\alpha=\begin{smallmatrix}2&3&4&3&2&1\\&&2&&&\end{smallmatrix}$

Therefore, Q has just two distinct eigenvalues for each of pairs; (E_6, α_1) (E_6, α_6), (E_7, α_7) and (G_2, α_1).

For a Kähler C-space associated with (F_4, α_4), Q has 1, $-\frac{5}{2}$ and -5 as its eigenvalues (cf., table 13).

With respect to Kähler C-spaces associated with the other pairs, we have shown in table 12 that Q has at least two eigenvalues, which are positive.

If there exists a negative eigenvalue, then, Q has at least three different eigenvalues. Therefore, it is sufficient to show that there is a weight space W_Σ on which the trace of Q is nonpositive.

(E_6, α_2): Relative to a Kähler C-space associated with this pair, we have a weight space W_Σ, $\Sigma=\begin{smallmatrix}2&3&4&3&2\\&&2&&\end{smallmatrix}$; $W_\Sigma=\sum_{j=1}^{3} CE_{\gamma_i\cdot\delta_i}$, where $\gamma_1=\begin{smallmatrix}1&1&1&1&1\\&&1&&\end{smallmatrix}$, $\gamma_i=\gamma_{i-1}+\alpha_{i+2}$, $i=2$ and 3, and $\delta_1=\begin{smallmatrix}1&2&3&2&1\\&&1&&\end{smallmatrix}$, $\delta_i=\delta_{i-1}-\alpha_{i+2}$, $i=2$ and 3. The trace of Q on W_Σ is given by (4.11); $\mathrm{Tr}(Q|_{W_\Sigma})=2\sum_i(\gamma_i, \delta_i)$. By the way, $(\gamma_i, \delta_i)=0$, $i=1$, 2 and 3 from Dynkin diagram of type E_6. Hence, we have $\mathrm{Tr}(Q|_{W_\Sigma})=0$.

$(E_6, \alpha_i)_{i=3,4,5}$, $(E_7, \alpha_i)_{i=1,\dots,6}$, $(E_8, \alpha_i)_{i=1,\dots,8}$, $(F_4, \alpha_i)_{i=1,2,3}$ and (G_2, α_2): By the aid of the similar arguments, we have some weight space W_Σ for these pairs such that $\mathrm{Tr}(Q|_{W_\Sigma})\leqq 0$.

Thus, Theorem 5.2 is proved. Q.E.D.

REMARK. The following table shows the eigenvalues with multiplicities of Q for all Kähler C-spaces of $b_2=1$, which is hermitian symmetric. A Kähler C-space $(G_u/K, g)$ associated with each pair $(\mathfrak{g}, \alpha_i)$ written at the left end in the table is holomorphically isometric (or homothetic) to an irreducible hermitian symmetric space of the type given at the right on the same line (Calabi and Vesentini [5]).

Table 18.

	dim	ν_1	m_1	ν_2	m_2	
(A_l, α_1) (A_l, α_l) $1\leqq l$	l	2	$l(l+1)/2$	$(=\nu_1)$		$B\text{-}I_{1,l}$
(A_l, α_i) $1<i<l$	$i(l-i+1)$	2	$\binom{i+1}{2}\binom{l-i+2}{2}$	-2	$\binom{i}{2}\binom{l-i+1}{2}$	$B\text{-}I_{i,l-i+1}$
(B_l, α_1) $2\leqq l$	$2l-1$	2	$(l-1)(2l+1)$	$-2l+3$	1	$B\text{-}IV_{2l-1}$
(B_2, α_2)	3	1	6	$(=\nu_1)$		$B\text{-}I_{1,2}$
(B_l, α_l) $3\leqq l$	$l(l+1)/2$	1	$(l+1)^2((l+1)^2-1)/12$	-2	$\binom{l+1}{4}$	$B\text{-}II_{l+1}$
(C_l, α_1) $2\leqq l$	$2l-1$	2	$l(2l-1)$	$(=\nu_1)$		$B\text{-}I_{1,2l-1}$
(C_l, α_l) $2\leqq l$	$l(l+1)/2$	4	$\binom{l+3}{4}$	-2	$l^2(l^2-1)/12$	$B\text{-}III_l$
(D_l, α_1) $3\leqq l$	$2l-2$	2	$l(2l-3)$	$-2(l-2)$	1	$B\text{-}IV_{2l-2}$
(D_l, α_{l-1}) (D_l, α_l) $3\leqq l$	$l(l-1)/2$	2	$l^2(l^2-1)/12$	-4	$\binom{l}{4}$	$B\text{-}II_l$
(E_6, α_1) (E_6, α_6)	16	2	126	-6	10	$B\text{-}V$
(E_7, α_7)	27	2	351	-8	27	$B\text{-}VI$
(G_2, α_1)	5	2	14	-3	1	$B\text{-}IV_5$

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Mitsuhiro ITOH
Department of Mathematics
University of Tsukuba
Sakura-mura, Niihari-gun,
Ibaraki, Japan