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Cohomology of finitely generated Kleinian groups with an invariant component

By Masami NAKADA

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Preliminaries. Let G be a non-elementary finitely generated Kleinian group with the region of discontinuity $\Omega(G)$ and let $B_q(\Omega(G), G)$ be the space of bounded holomorphic automorphic forms of weight $-2q$ for G operating on $\Omega(G)$, where $q(\geqq 2)$ is an integer. We denote by Π_{2q-2} the vector space of complex polynomials in one variable of degree at most $2q-2$. Clearly Π_{2q-2} is a G-module with $(v\cdot\gamma)(z)=v(\gamma(z))\gamma'(z)^{1-q}$ for $v\in\Pi_{2q-2}$ and $\gamma\in G$.

Now we can form the (first) cohomology space $H^1(G, \Pi_{2q-2})$, that is, $H^1(G, \Pi_{2q-2})$ is the space of cocycles $Z^1(G, \Pi_{2q-2})$ factored by the space of coboundaries $B^1(G, \Pi_{2q-2})$. Let p be an element of $Z^1(G, \Pi_{2q-2})$. If p satisfies the condition $p|_{G_0}\in B^1(G_0, \Pi_{2q-2})$ for any parabolic cyclic subgroup G_0 of G, then we say that p belongs to $PZ^1(G, \Pi_{2q-2})$, the space of parabolic cocycles. We denote by $PH^1(G, \Pi_{2q-2})$, the space of parabolic cohomology, that is, the space of parabolic cocycles factored by the space of coboundaries. From this definition, we see

$$\dim PH^1(G, \Pi_{2q-2})=\dim PZ^1(G, \Pi_{2q-2})-\dim B^1(G, \Pi_{2q-2}).$$

Further, for a non-elementary Kleinian group G, the equality

$$\dim B^1(G, \Pi_{2q-2})=2q-1$$

is known (see Bers [1]).

We have the so-called Bers' map

$$\beta^*: B_q(\Omega(G), G) \longrightarrow PH^1(G, \Pi_{2q-2})$$

which is anti-linear and injective (see Bers [1] and Kra [2]).

Throughout this paper, we call the group consisting only of the identity to be trivial. This group is, of course, a cyclic group. Let H be a cyclic subgroup of a Kleinian group G. The interior B of a closed topological disc is called a precisely invariant disc under H if $h(\bar{B}-\Lambda(H))=\bar{B}-\Lambda(H)$ for $h\in H$ and $g(\bar{B}-\Lambda(H))\cap(\bar{B}-\Lambda(H))=\emptyset$ for $g\in G-H$, where $\bar{B}$ is the closure of B, $\Lambda(H)$ is the limit set of H and $\bar{B}-\Lambda(H)\subset\Omega(G)$.

The following Maskit's Combination Theorems play a fundamental role in our discussion.

COMBINATION THEOREM I. *For $i=1, 2$, let B_i be a precisely invariant disc under H, a cyclic subgroup of both Kleinian groups G_1 and G_2. Assume that B_1 and B_2 have the common boundary C and $B_1 \cap B_2 = \emptyset$. Let G be the group generated by G_1 and G_2. Then*

(I.1) *G is Kleinian,*

(I.2) *G is the free product of G_1 and G_2 with the amalgamated subgroup H,*

and

(I.3) $$\Omega(G)/G = (\Omega(G_1)/G_1 - B_1/H) \cup (\Omega(G_2)/G_2 - B_2/H),$$
where $(\Omega(G_1)/G_1 - B_1/H) \cap (\Omega(G_2)/G_2 - B_2/H) = (C \cap \Omega(H))/H$.

COMBINATION THEOREM II. *Let G_1 be a Kleinian group with cyclic subgroups H_1 and H_2. For $i=1, 2$, let B_i be a precisely invariant disc for the cyclic subgroup H_i and let C_i be the boundary of B_i. Assume that $\gamma(\bar{B}_1) \cap \bar{B}_2 = \emptyset$ for all γ in G_1. Let G_2 be the cyclic group generated by f, where $f(C_1) = C_2$, $f(B_1) \cap B_2 = \emptyset$ and $f \circ H_1 \circ f^{-1} = H_2$. Let G be the group generated by G_1 and G_2. Then*

(II.1) *G is Kleinian,*

(II.2) *every relation in G is a consequence of the relations in G_1 and the relation $f \circ H_1 \circ f^{-1} = H_2$,*

and

(II.3) $\Omega(G)/G = \Omega(G_1)/G_1 - (B_1/H_1 \cup B_2/H_2)$, *where in* $\Omega(G)/G$, $(C_1 \cap \Omega(G))/H$ *is identified with* $(C_2 \cap \Omega(G))/H_2$.

In this Combination Theorem II, note that the transformation f is a loxodromic element.

A basic group is by definition a finitely generated Kleinian group which has a simply connected invariant component and contains no accidental parabolic transformations. Hence a basic group is either elementary, degenerate or quasi-Fuchsian (see Maskit [3]).

Let G be a non-elementary finitely generated Kleinian group with an invariant component. In [4], Maskit proved that G can be constructed from basic groups in a finite number of steps by using Combination Theorems I and II, where in each step, the amalgamated subgroups and the conjugated subgroups are trivial or elliptic cyclic or parabolic cyclic.

The purpose of this paper is to prove the following: Let G be a non-elementary finitely generated Kleinian group with an invariant component and let G be constructed from basic groups $G_1, \cdots, G_s$ by using Combination Theorems I and II. Then G_i is an elementary group or a quasi-Fuchsian group for $i=1, \cdots, s$ if and only if $PH^1(G, \Pi_2) = \beta^*(B_2(\Omega(G), G))$.

1. First we derive a relation of $\dim PZ^1(G, \Pi_{2q-2})$, $\dim PZ^1(G_1, \Pi_{2q-2})$ and $\dim PZ^1(G_2, \Pi_{2q-2})$ for a group G which is generated by its subgroups G_1 and G_2 by application of Combination Theorem I. For the purpose, we need three lemmas.

LEMMA 1. *Let G be a Kleinian group and let G_0 be an elliptic cyclic subgroup of G. Then the map*

$$\operatorname*{res}_{G,G_0}: Z^1(G, \Pi_{2q-2}) \longrightarrow Z^1(G_0, \Pi_{2q-2})$$

defined by $\operatorname*{res}_{G,G_0}(p)=p|_{G_0}$ is surjective.

PROOF. Let ν be the order of G_0 and let γ be a generator of G_0. By considering conjugation, we may assume $\gamma(z)=\lambda z$, $\lambda^\nu=1$, $\lambda\neq 1$. Let p_0 be an element of $Z^1(G_0, \Pi_{2q-2})$. Set $p_0(\gamma)=\sum_{i=0}^{2q-2} a_i z^i$. Since $\gamma^\nu=id$, we have

$$0=p_0(\gamma^\nu)=\sum_{i=0}^{2q-2} a_i(1+\lambda^{i+1-q}+\cdots+\lambda^{(\nu-1)(i+1-q)})z^i .$$

Hence $p_0(\gamma)=\sum_i{}' a_i z^i$, where $\sum_i{}'$ means summation for indices i satisfying $\lambda^{i+1-q}\neq 1$. Therefore we have $p_0(\gamma)=w\cdot\gamma-w$ for $w(z)=\sum_i{}' \frac{a_i}{\lambda^{i+1-q}-1} z^i$. Now it is clear that p_0 can be extended to an element of $Z^1(G, \Pi_{2q-2})$. Hence the map $\operatorname*{res}_{G,G_0}$ is surjective.

LEMMA 2. *Let G be a Kleinian group and let G_0 be a parabolic cyclic subgroup of G. Then the map*

$$\operatorname*{res}_{G,G_0}: PZ^1(G, \Pi_{2q-2}) \longrightarrow PZ^1(G_0, \Pi_{2q-2})$$

defined by $\operatorname*{res}_{G,G_0}(p)=p|_{G_0}$ is surjective.

PROOF. Since G_0 is parabolic cyclic, we see that $PZ^1(G_0, \Pi_{2q-2})=B^1(G_0, \Pi_{2q-2})$ by definition of $PZ^1(G_0, \Pi_{2q-2})$. Hence, for any $p_0\in PZ^1(G_0, \Pi_{2q-2})$ there exists a polynomial $w\in\Pi_{2q-2}$ such that $p_0(\gamma)=w\cdot\gamma-w$. Therefore we have $\operatorname*{res}_{G,G_0}$ is surjective.

The following lemma is well known.

LEMMA 3. *Let G_1 and G_2 be subgroups of a group and let G be the free product of G_1 and G_2 with the amalgamated subgroup $H=G_1\cap G_2$. Let $G_1=H+\sum_\alpha Ha_\alpha$ and $G_2=H+\sum_\beta Hb_\beta$ be the right coset representations of G_1 and G_2, respectively. Then any element $\gamma\in G$ can be represented uniquely as*

$$\gamma=h\circ\gamma_1\circ\cdots\circ\gamma_t ,$$

where $h\in H$ and γ_i is some a_α or some b_β, and, γ_i and γ_{i+1} are not contained simultaneously in the same G_j ($j=1, 2$).

Now we can prove the following

THEOREM 1. *If G is a non-elementary Kleinian group which is generated by its finitely generated subgroups G_1 and G_2 by application of Combination Theorem I and if $H=G_1\cap G_2$ is elliptic cyclic or parabolic cyclic or trivial, then*

$$\dim PZ^1(G, \Pi_{2q-2})=\dim PZ^1(G_1, \Pi_{2q-2})+\dim PZ^1(G_2, \Pi_{2q-2}) -\dim PZ^1(H, \Pi_{2q-2}).$$

PROOF. Since G is generated by G_1 and G_2, the linear mapping

$$\Phi : PZ^1(G, \Pi_{2q-2}) \longrightarrow PZ^1(G_1, \Pi_{2q-2})\times PZ^1(G_2, \Pi_{2q-2})$$

defined by $\Phi(p)=(p_1, p_2)$, $p_i=\operatorname*{res}_{G,G_i}(p)(=p|_{G_i})$, is injective. We consider the mapping

$$\tilde{\Phi} : [PZ^1(G_1, \Pi_{2q-2})\times PZ^1(G_2, \Pi_{2q-2})]/\Phi(PZ^1(G, \Pi_{2q-2})) \longrightarrow PZ^1(H, \Pi_{2q-2})$$

defined by $\tilde{\Phi}(\{(p_1, p_2)\})=\operatorname*{res}_{G_1,H}(p_1)-\operatorname*{res}_{G_2,H}(p_2)$. It is easily seen that the mapping $\tilde{\Phi}$ is well defined and linear.

From Lemma 1 and Lemma 2 we see that, for any $p\in PZ^1(H, \Pi_{2q-2})$, there exist elements $p_i\in PZ^1(G_i, \Pi_{2q-2})$ $(i=1, 2)$ such that $\operatorname*{res}_{G_i,H}(p_i)=p$. Hence $\tilde{\Phi}(\{(2p_1, p_2)\})=\operatorname*{res}_{G_1,H}(2p_1)-\operatorname*{res}_{G_2,H}(p_2)=2p-p=p$. This shows the surjectivity of $\tilde{\Phi}$.

Next we shall show the injectivity of $\tilde{\Phi}$. Let $\tilde{\Phi}(\{(p_1, p_2)\})=0$. Then $\operatorname*{res}_{G_1,H}(p_1)=\operatorname*{res}_{G_2,H}(p_2)$. We set $p=\operatorname*{res}_{G_1,H}(p_1)=\operatorname*{res}_{G_2,H}(p_2)$. For any element $\gamma\in G$ we have a unique representation $\gamma=h\circ\gamma_1\circ\cdots\circ\gamma_t$ by Lemma 3. We define the mapping $\tilde{p} : G\rightarrow\Pi_{2q-2}$ as follows:

$$\tilde{p}(\gamma)=p(h)\cdot(\gamma_1\circ\cdots\circ\gamma_t)+p_{i_1}(\gamma_1)\cdot(\gamma_2\circ\cdots\circ\gamma_t)+p_{i_2}(\gamma_2)\cdot(\gamma_3\circ\cdots\circ\gamma_t)+\cdots+p_{i_t}(\gamma_t),$$

where $i_k=1$ if $\gamma_k\in G_1$ and $i_k=2$ if $\gamma_k\in G_2$. Take one more $\gamma'\in G$ and let $\gamma'=h'\circ\gamma'_1\circ\cdots\circ\gamma'_s$ be a unique representation of γ'. By induction on t, we can verify $\tilde{p}(\gamma\circ\gamma')=\tilde{p}(\gamma)\cdot\gamma'+\tilde{p}(\gamma')$.

In fact, if $t=1$ and if γ_1 and γ'_1 are contained in the same G_j, say G_1, then $h\circ\gamma_1\circ h'\circ\gamma'_1=\tilde{h}\circ a_\alpha$ for some a_α and $\tilde{h}\in H$, so $\gamma\circ\gamma'=\tilde{h}\circ a_\alpha\circ\gamma'_2\circ\cdots\circ\gamma'_s$. Here $G_1=H+\sum_\alpha Ha_\alpha$ is the right coset representation of G_1. Hence, by the definition of $\tilde{p}$, we have $\tilde{p}(\gamma\circ\gamma')=p(\tilde{h})\cdot(a_\alpha\circ\gamma'_2\circ\cdots\circ\gamma'_s)+p_1(a_\alpha)\cdot(\gamma'_2\circ\cdots\circ\gamma'_s)+p_2(\gamma'_2)\cdot(\gamma'_3\circ\cdots\circ\gamma'_s)+\cdots+p_{i_s}(\gamma'_s)$. Since $a_\alpha=\tilde{h}^{-1}\circ h\circ\gamma_1\circ h'\circ\gamma'_1$ and since $p_1\in Z^1(G_1, \Pi_{2q-2})$, we have $p_1(a_\alpha)=-p(\tilde{h})\cdot(\tilde{h}^{-1}\circ h\circ\gamma_1\circ h'\circ\gamma'_1)+p(h)\cdot(\gamma_1\circ h'\circ\gamma'_1)+p_1(\gamma_1)\cdot(h'\circ\gamma'_1)+p(h')\cdot\gamma'_1+p_1(\gamma'_1)$. Therefore $\tilde{p}(\gamma\circ\gamma')=\tilde{p}(\gamma)\cdot\gamma'+\tilde{p}(\gamma')$. In a similar way, we can also prove $\tilde{p}(\gamma\circ\gamma')=\tilde{p}(\gamma)\cdot\gamma'+\tilde{p}(\gamma')$ when γ_1 and γ'_1 are not contained simultaneously in the same G_j. Now assume that $\tilde{p}(\gamma\circ\gamma')=\tilde{p}(\gamma)\cdot\gamma'+\tilde{p}(\gamma')$ holds for $\gamma=h\circ\gamma_1\circ\cdots\gamma_t$ and $\gamma'=h'\circ\gamma'_1\circ\cdots\circ\gamma'_s$. Let $\tilde{\gamma}=h\circ\gamma_1\circ\cdots\circ\gamma_{t+1}$ be a unique representation of $\tilde{\gamma}\in G$ by Lemma 3. If γ_{t+1} and γ'_1 are contained in the same G_j, say G_1, then $\gamma_{t+1}\circ h'\circ\gamma'_1=\tilde{h}\circ a_\alpha$ for some a_α and $\tilde{h}\in H$, so $\tilde{\gamma}\circ\gamma'=h\circ\gamma_1\circ\cdots\circ\gamma_t\circ\tilde{h}\circ a_\alpha\circ\gamma_2'\circ\cdots\circ\gamma_s'$. Hence, by the induction hypothesis, we have $\tilde{p}(\tilde{\gamma}\circ\gamma')=\tilde{p}(h\circ\gamma_1\circ\cdots\circ\gamma_t)\cdot(\tilde{h}\circ a_\alpha\circ\gamma_2'\circ\cdots\circ\gamma_s')+\tilde{p}(\tilde{h}\circ a_\alpha\circ\gamma_2'\circ\cdots\circ\gamma_s')$. So, by the definition of $\tilde{p}$, we have $\tilde{p}(\tilde{\gamma}\circ\gamma')=\{\tilde{p}(h\circ\gamma_1\circ\cdots\circ\gamma_{t+1})\cdot\gamma_{t+1}^{-1}-p_1(\gamma_{t+1})\cdot\gamma_{t+1}^{-1}\}\cdot(\tilde{h}\circ a_\alpha\circ\gamma_2'\circ\cdots\circ\gamma_s')+p(\tilde{h})\cdot(a_\alpha\circ\gamma_2'\circ\cdots\circ\gamma_s')+p_1(a_\alpha)\cdot(\gamma_2'\circ\cdots\circ\gamma_s')+p_2(\gamma_2')\cdot(\gamma_3'\circ\cdots\circ\gamma_s')+\cdots+p_{i_s}(\gamma_s')$. Since $a_\alpha=\tilde{h}^{-1}\circ\gamma_{t+1}\circ h'\circ\gamma_1'$ and since $p_1\in Z^1(G_1, \Pi_{2q-2})$, we have $p_1(a_\alpha)=-p(\tilde{h})\cdot(\tilde{h}^{-1}\circ\gamma_{t+1}\circ h'\circ\gamma_1')+p_1(\gamma_{t+1})\cdot(h'\circ\gamma_1')+p(h')\cdot\gamma_1'+p_1(\gamma_1')$. Therefore $\tilde{p}(\tilde{\gamma}\circ\gamma')=\tilde{p}(\tilde{\gamma})\cdot\gamma'+p(\gamma')$. We can also prove this equality when γ_{t+1} and γ_1' are not contained simultaneously in the same G_j. Therefore, $\tilde{p}(\gamma\circ\gamma')=\tilde{p}(\gamma)\cdot\gamma'+\tilde{p}(\gamma')$ for any γ and γ' in G.

Thus, $\tilde{p}$ defined as above belongs to $Z^1(G, \Pi_{2q-2})$. Let $\gamma\in G$ be any parabolic element. Then there exist a parabolic element $\gamma_i\in G_i$ ($i=1$ or 2) and an element $\alpha\in G$ such that $\gamma=\alpha\circ\gamma_i\circ\alpha^{-1}$ (see Maskit [4]). From the definition of $\tilde{p}$, we see $\operatorname{res}_{G,G_i}(\tilde{p})=p_i\in PZ^1(G_i, \Pi_{2q-2})$ and $\tilde{p}(\gamma_i)=v\cdot\gamma_i-v$ for some $v\in\Pi_{2q-2}$. Hence we have $\tilde{p}(\gamma)=\tilde{p}(\alpha)\cdot(\gamma_i\circ\alpha^{-1})+(v\cdot\gamma_i-v)\cdot\alpha^{-1}+\tilde{p}(\alpha^{-1})=-\tilde{p}(\alpha^{-1})\cdot(\alpha\circ\gamma_i\circ\alpha^{-1})+(v\cdot\alpha^{-1})\cdot(\alpha\circ\gamma_i\circ\alpha^{-1})-v\cdot\alpha^{-1}+\tilde{p}(\alpha^{-1})=w\cdot\gamma-w$ for $w=v\cdot\alpha^{-1}-\tilde{p}(\alpha^{-1})\in\Pi_{2q-2}$. Therefore, we obtain $\tilde{p}\in PZ^1(G, \Pi_{2q-2})$, which shows $(p_1, p_2)=\Phi(\tilde{p})\in\Phi(PZ^1(G, \Pi_{2q-2}))$, that is $\{(p_1, p_2)\}=0$. Thus the mapping $\tilde{\Phi}$ is injective.

Therefore, $\tilde{\Phi}$ is bijective and consequently we have

$$\dim([PZ^1(G_1, \Pi_{2q-2})\times PZ^1(G_2, \Pi_{2q-2})]/\Phi(PZ^1(G, \Pi_{2q-2})))$$
$$=\dim PZ^1(H, \Pi_{2q-2})\,.$$

From the injectivity of Φ, we have the desired equality.

2. Next we derive a relation between $\dim PZ^1(G, \Pi_{2q-2})$ and $\dim PZ^1(G_1, \Pi_{2q-2})$ for the group G which is generated by its subgroup G_1 and an element f by application of Combination Theorem II.

First we shall prove the following

LEMMA 4. *Let G be a non-elementary Kleinian group which is generated by its finitely generated subgroup G_1 and an element f by application of Combination Theorem II. Assume that a group H_1 (or H_2) be elliptic cyclic or parabolic cyclic or trivial and let G_2 be the cyclic group generated by f. Then for $(p_1, p_2)\in PZ^1(G_1, \Pi_{2q-2})\times PZ^1(G_2, \Pi_{2q-2})$, there exists an element $p\in PZ^1(G, \Pi_{2q-2})$ such that $p|_{G_i}=p_i$ for $i=1, 2$, if and only if*

$$p_2(f)\cdot(h_1\circ f^{-1}\circ h_2^{-1})+p_1(h_1)\cdot(f^{-1}\circ h_2^{-1})+p_2(f^{-1})\cdot h_2^{-1}+p_1(h_2^{-1})=0\,,$$

where h_i is a generator of H_i satisfying $f\circ h_1\circ f^{-1}=h_2$.

PROOF. It is sufficient to show only the if part. Let $\{\alpha_1, \cdots, \alpha_n, h_1, h_2\}$ be a system of generators of G_1. For $(p_1, p_2)\in PZ^1(G_1, \Pi_{2q-2})\times PZ^1(G_2, \Pi_{2q-2})$ we define a mapping p: $\{\alpha_1, \cdots, \alpha_n, h_1, h_2, f\}\to\Pi_{2q-2}$, defined on a system of generators of G, as follows;

$$p(\alpha_i)=p_1(\alpha_i), \quad p(h_j)=p_1(h_j), \quad p(f)=p_2(f), \quad i=1, \cdots, n, \quad j=1, 2.$$

By using (II.2), we see that if $p_2(f)\cdot(h_1\circ f^{-1}\circ h_2^{-1})+p_1(h_1)\cdot(f^{-1}\circ h_2^{-1})+p_2(f^{-1})\cdot h_2^{-1}+p_1(h_2^{-1})=0$, then p can be extended to an element of $Z^1(G, \Pi_{2q-2})$ (see Weil [**5**]). For the extended p it is obvious that $p|_{G_i}=p_i$ ($i=1, 2$). Moreover, for any parabolic element $\gamma\in G$ there exists a parabolic element $\gamma_1\in G_1$ and an element $\alpha\in G$ such that $\gamma=\alpha\circ\gamma_1\circ\alpha^{-1}$ (see Maskit [**4**]). Since $p|_{G_1}\in PZ^1(G_1, \Pi_{2q-2})$, in the same way as in the proof of Theorem 1, we see that $p\in PZ^1(G, \Pi_{2q-2})$. This completes the proof of our lemma.

THEOREM 2. *If G is a non-elementary Kleinian group which is generated by its finitely generated subgroup G_1 and an element f by application of Combination Theorem II and if a group H_1 (or H_2) is elliptic cyclic or parabolic cyclic or trivial, then*

$$\dim PZ^1(G, \Pi_{2q-2})=\dim PZ^1(G_1, \Pi_{2q-2})+\dim PZ^1(G_2, \Pi_{2q-2}) \\ -\dim PZ^1(H_2, \Pi_{2q-2}),$$

where G_2 is the cyclic group generated by f.

PROOF. Using the mapping Φ defined in the proof of Theorem I, we consider the mapping

$$\tilde{\Psi}: [PZ^1(G_1, \Pi_{2q-2})\times PZ^1(G_2, \Pi_{2q-2})]/\Phi(PZ^1(G, \Pi_{2q-2})) \\ \longrightarrow PZ^1(H_2, \Pi_{2q-2})$$

defined by $\tilde{\Psi}(\{(p_1, p_2)\})=p$, where $p(h_2)=p_2(f)\cdot(h_1\circ f^{-1})-p_2(f)\cdot f^{-1}+p_1(h_1)\cdot f^{-1}-p_1(h_2)$ for $h_1=f^{-1}\circ h_2\circ f\in H_1$. It is easy to see $p\in PZ^1(H_2, \Pi_{2q-2})$. The well-definedness and the linearity of $\tilde{\Psi}$ is obvious.

To show that the mapping $\tilde{\Psi}$ is injective, we assume that $\tilde{\Psi}(\{(p_1, p_2)\})=0$. Then we have $p_2(f)\cdot(h_1\circ f^{-1})-p_2(f)\cdot f^{-1}+p_1(h_1)\cdot f^{-1}-p_1(h_2)=0$. Therefore

$$p_2(f)\cdot(h_1\circ f^{-1}\circ h_2^{-1})+p_1(h_1)\cdot(f^{-1}\circ h_2^{-1})+p_2(f^{-1})\cdot h_2^{-1}+p_1(h_2^{-1})=0.$$

Hence, by Lemma 4, there exists an element $p\in PZ^1(G, \Pi_{2q-2})$ such that $p|_{G_i}=p_i$ for $i=1, 2$, which shows $\{(p_1, p_2)\}=0$, that is, $\tilde{\Psi}$ is injective.

Next we shall show the surjectivity of $\tilde{\Psi}$. Let p and p_1 be arbitrary elements of $PZ^1(H_2, \Pi_{2q-2})$ and $PZ^1(G, \Pi_{2q-2})$, respectively. For $h_i\in H_i$ ($i=1, 2$), we see by the proofs of Lemma 1 and Lemma 2 that

$$p(h_2)=v\cdot h_2-v, \qquad p_1(h_1)=w\cdot h_1-w, \qquad p_1(h_2)=u\cdot h_2-u$$

for some polynomials $v, w, u\in\Pi_{2q-2}$. Now, for $h_1\in H_1$ and $h_2\in H_2$ such that $f\circ h_1\circ f^{-1}=h_2$, we have

$$\begin{aligned}p(h_2)\cdot f-p_1(h_1)+p_1(h_2)\cdot f&=v\cdot(h_2\circ f)-v\cdot f-\{w\cdot h_1-w\}+u\cdot(h_2\circ f)-u\cdot f\\&=v\cdot(f\circ h_1)-v\cdot f-\{w\cdot h_1-w\}+u\cdot(f\circ h_1)-u\cdot f\\&=(v\cdot f-w+u\cdot f)\cdot h_1-(v\cdot f-w+u\cdot f)\,.\end{aligned}$$

Since G_2 is a loxodromic cyclic group, we see that an element of $PZ^1(G_2, \Pi_{2q-2})$ can be uniquely determined by determing its image at the generator f of G_2. So we define $p_2 \in PZ^1(G_2, \Pi_{2q-2})$ by

$$p_2(f)=v\cdot f-w+u\cdot f\,.$$

Then we have

$$p(h_2)\cdot f-p_1(h_1)+p_1(h_2)\cdot f=p_2(f)\cdot h_1-p_2(f)\,.$$

Hence $p(h_2)=p_2(f)\cdot(h_1\circ f^{-1})-p_2(f)\cdot f^{-1}+p_1(h_1)\cdot f^{-1}-p_1(h_2)$. Therefore we have $\tilde{\Psi}(\{(p_1, p_2)\})=p$, that is, $\tilde{\Psi}$ is surjective.

Consequently we have

$$\begin{aligned}\dim([PZ^1(G_1, \Pi_{2q-2})\times PZ^1(G_2, \Pi_{2q-2})]/\Phi(PZ^1(G, \Pi_{2q-2})))\\=\dim PZ^1(H_2, \Pi_{2q-2})\,.\end{aligned}$$

From the injectivity of Φ, we have the required equality.

3. We shall prove some lemmas for the later use.

LEMMA 5. *Let G be an elliptic cyclic group of order ν or a parabolic cyclic group. Then*

$$\dim PZ^1(G, \Pi_{2q-2})=2\left[q-\frac{q}{\nu}\right],$$

where for a parabolic cyclic group G, ν is regarded as ∞ and $\left[q-\frac{q}{\infty}\right]$ is regarded as to be equal to $q-1$.

PROOF. Let γ be a generator of G. Considering conjugation by a linear transformation, we may assume that $\gamma(z)=\lambda z$ or $\gamma(z)=z+1$ according to $\nu<\infty$ or $\nu=\infty$. Let p be an element of $PZ^1(G, \Pi_{2q-2})$. If $\gamma(z)=\lambda z$, then by the proof of Lemma 1 we have $p(\gamma)=\sum_i{}' a_i z^i$ and p is uniquely determined by $2\left[q-\frac{q}{\nu}\right]$ parameters and these parameters can be chosen arbitrary. If $\gamma(z)=z+1$, then $p(\gamma)=v(z+1)-v(z)$ $(\in \Pi_{2q-2})$ for some $v\in\Pi_{2q-2}$, whence p is uniquely determined by $2q-2$ parameters and these parameters can be chosen arbitrary. Thus, in both cases $\nu<\infty$ and $\nu=\infty$, we have $\dim PZ^1(G, \Pi_{2q-2})=2\left[q-\frac{q}{\nu}\right]$.

REMARK. When G is trivial, it is clear that $\dim PZ^1(G, \Pi_{2q-2})=0$. In this case, if we define that the order of G is 1, Lemma 5 also holds for G.

LEMMA 6. *If G is a loxodromic cyclic group, then the dimension of $PZ^1(G, \Pi_{2q-2})$ is equal to $2q-1$.*

PROOF. Since an element p of $PZ^1(G, \Pi_{2q-2})$ can be uniquely determined by an arbitrary choice of $p(\gamma)$, where γ is a generator of G, we have $\dim PZ^1(G, \Pi_{2q-2})=2q-1$.

Let G be a non-elementary finitely generated Kleinian group with $\Omega(G)/G=S_1+\cdots+S_k$. Let $(g_i; \nu_{i1}, \cdots, \nu_{in_i})$ be the signature of S_i. Then it is well known that $\dim B_q(\Omega(G), G)=\sum_{i=1}^{k}\left\{(2q-1)(g_i-1)+\sum_{j=1}^{n_i}\left[q-\frac{q}{\nu_{ij}}\right]\right\}$. For an elementary group G with the signature $(g; \nu_1, \cdots, \nu_n)$ we define formally the dimension of $B_q(\Omega(G), G)$ by $\dim B_q(\Omega(G), G)=(2q-1)(g-1)+\sum_{i=1}^{n}\left[q-\frac{q}{\nu_i}\right]$. Under this convention we have the following lemmas.

LEMMA 7. *If G is a Kleinian group which is generated by its finitely generated subgroups G_1 and G_2 by application of Combination Theorem I and if $H=G_1\cap G_2$ is elliptic cyclic or parabolic cyclic or trivial, then*

$$\dim B_q(\Omega(G), G)=\dim B_q(\Omega(G_1), G_1)+\dim B_q(\Omega(G_2), G_2)$$
$$+(2q-1)-2\left[q-\frac{q}{\nu}\right],$$

where ν is the order of H.

PROOF. We set $\Omega(G_1)/G_1=S_{11}+\cdots+S_{1n}$ and $\Omega(G_2)/G_2=S_{21}+\cdots+S_{2m}$. Let $(g_{1i}; \nu_{i1}, \cdots, \nu_{ik_i})$ be the signature of S_{1i} $(i=1, \cdots, n)$ and let $(g_{2i}; \mu_{i1}, \cdots, \mu_{ik'_i})$ be the signature of S_{2i} $(i=1, \cdots, m)$. Since the precisely invariant disc B_i $(i=1, 2)$ under H is contained in a component of $\Omega(G_i)$, we may assume that $B_1/H\subset S_{11}$ and $B_2/H\subset S_{21}$. Let H be an elliptic cyclic group of order ν. Then $\nu=\nu_{1t}=\mu_{1s}$ for some t $(1\leqq t\leqq k_1)$ and s $(1\leqq s\leqq k'_1)$. We may assume that $\nu=\nu_{11}=\mu_{11}$. From (I.3) we have

$$\Omega(G)/G=S+S_{12}+\cdots+S_{1n}+S_{22}+\cdots+S_{2m},$$

where $S=(S_{11}-B_1/H)\cup(S_{21}-B_2/H)$ with $(S_{11}-B_1/H)\cap(S_{21}-B_2/H)=(C\cap\Omega(H))/H$. Hence we see that the signature of S is $(g_{11}+g_{21}; \nu_{12}, \cdots, \nu_{1k_1}, \mu_{12}, \cdots, \mu_{1k'_1})$. We have

$$\dim B_q(\Omega(G), G)=(2q-1)(g_{11}+g_{21}-1)+\sum_{j=2}^{k_1}\left[q-\frac{q}{\nu_{1j}}\right]+\sum_{j=2}^{k'_1}\left[q-\frac{q}{\mu_{1j}}\right]$$
$$+\sum_{i=2}^{n}\left\{(2q-1)(g_{1i}-1)+\sum_{j=1}^{k_i}\left[q-\frac{q}{\nu_{ij}}\right]\right\}$$
$$+\sum_{i=2}^{m}\left\{(2q-1)(g_{2i}-1)+\sum_{j=1}^{k'_i}\left[q-\frac{q}{\mu_{ij}}\right]\right\}$$
$$=\sum_{i=1}^{n}\left\{(2q-1)(g_{1i}-1)+\sum_{j=1}^{k_i}\left[q-\frac{q}{\nu_{ij}}\right]\right\}$$

$$+\sum_{i=1}^{m}\left\{(2q-1)(g_{2i}-1)+\sum_{j=1}^{k_i'}\left[q-\frac{q}{\mu_{ij}}\right]\right\}+(2q-1)-2\left[q-\frac{q}{\nu}\right]$$

$$=\dim B_q(\Omega(G_1), G_1)+\dim B_q(\Omega(G_2), G_2)+(2q-1)-2\left[q-\frac{q}{\nu}\right].$$

The other cases are obtained in the same way as above.

LEMMA 8. *If G is a Kleinian group which is generated by its finitely generated subgroup G_1 and an element f by application of Combination Theorem II and if H_1 (or H_2) is elliptic cyclic or parabolic cyclic or trivial, then*

$$\dim B_q(\Omega(G), G)=\dim B_q(\Omega(G_1), G_1)+(2q-1)-2\left[q-\frac{q}{\nu}\right],$$

where ν is the order of H_1.

PROOF. We can prove the lemma in a similar way to that of Lemma 7. We use (II.3) instead of (I.3).

4. In this section we shall show two theorems. These are essential parts of the proof of Theorem stated in preliminaries.

THEOREM 3. *Let G be a Kleinian group which is generated by its finitely generated non-elementary subgroups G_1 and G_2 by application of Combination Theorem I. Assume that $H=G_1\cap G_2$ be elliptic cyclic or parabolic cyclic or trivial. Then $PH^1(G, \Pi_{2q-2})=\beta^*(B_q(\Omega(G), G))$ if and only if $PH^1(G_i, \Pi_{2q-2})=\beta^*(B_q(\Omega(G_i), G_i))$ for $i=1, 2$.*

PROOF. Let ν be the order of H. First we assume that $PH^1(G_i, \Pi_{2q-2})=\beta^*(B_q(\Omega(G_i), G_i))$ for $i=1, 2$. As stated in preliminaries we have $\dim PH^1(G_i, \Pi_{2q-2})=\dim PZ^1(G_i, \Pi_{2q-2})-\dim B^1(G_i, \Pi_{2q-2})$ and $\dim B^1(G_i, \Pi_{2q-2})=2q-1$. Using this fact, Theorem 1 and Lemma 5, we have

$$\dim PH^1(G, \Pi_{2q-2})=\dim PH^1(G_1, \Pi_{2q-2})+\dim PH^1(G_2, \Pi_{2q-2})$$
$$+(2q-1)-2\left[q-\frac{q}{\nu}\right].$$

Since $\dim PH^1(G_i, \Pi_{2q-2})=\dim B_q(\Omega(G_i), G_i)$, we have

$$\dim PH^1(G, \Pi_{2q-2})=\dim B_q(\Omega(G_1), G_1)+\dim B_q(\Omega(G_2), G_2)$$
$$+(2q-1)-2\left[q-\frac{q}{\nu}\right].$$

Hence, from Lemma 7, we have $\dim PH^1(G, \Pi_{2q-2})=\dim B_q(\Omega(G), G)$, that is, $PH^1(G, \Pi_{2q-2})=\beta^*(B_q(\Omega(G), G))$.

Conversely we assume that $PH^1(G, \Pi_{2q-2})=\beta^*(B_q(\Omega(G), G))$. If $PH^1(G_i, \Pi_{2q-2})\supsetneqq\beta^*(B_q(\Omega(G_i), G_i))$ for some i, then

$$\dim PH^1(G_i, \Pi_{2q-2}) > \dim B_q(\Omega(G_i), G_i).$$

Therefore, from Theorem 1, Lemma 5 and Lemma 7, we have

$$\dim PH^1(G, \Pi_{2q-2}) > \dim B_q(\Omega(G), G).$$

This contradicts our hypothesis. Hence $PH^1(G_i, \Pi_{2q-2})=\beta^*(B_q(\Omega(G_i), G_i))$ for $i=1, 2$. Thus we have our Theorem.

THEOREM 4. *Let G be a Kleinian group which is generated by its finitely generated non-elementary subgroup G_1 and an element f by application of Combination Theorem II. Assume that H_1 (or H_2) be elliptic cyclic or parabolic cyclic or trivial. Then $PH^1(G, \Pi_{2q-2})=\beta^*(B_q(\Omega(G), G))$ if and only if $PH^1(G_1, \Pi_{2q-2})=\beta^*(B_q(\Omega(G_1), G_1))$.*

PROOF. Let ν be the order of H_1. We assume that $PH^1(G_1, \Pi_{2q-2})=\beta^*(B_q(\Omega(G_1), G_1))$. From Theorem 2, Lemma 5 and Lemma 6, we have

$$\dim PH^1(G, \Pi_{2q-2}) = \dim PH^1(G_1, \Pi_{2q-2}) + (2q-1) - 2\left[q - \frac{q}{\nu}\right],$$

by the same argument as in the proof of Theorem 3. Since $\dim PH^1(G_1, \Pi_{2q-2}) = \dim B_q(\Omega(G_1), G_1)$, we have

$$\dim PH^1(G, \Pi_{2q-2}) = \dim B_q(\Omega(G_1), G_1) + (2q-1) - 2\left[q - \frac{q}{\nu}\right].$$

Hence, from Lemma 8, we have $\dim PH^1(G, \Pi_{2q-2}) = \dim B_q(\Omega(G), G)$, that is, $PH^1(G, \Pi_{2q-2})=\beta^*(B_q(\Omega(G), G))$.

Next we assume that $PH^1(G, \Pi_{2q-2})=\beta^*(B_q(\Omega(G), G))$. If $PH^1(G_1, \Pi_{2q-2}) \supsetneqq \beta^*(B_q(\Omega(G_1), G_1))$, then

$$\dim PH^1(G_1, \Pi_{2q-2}) > \dim B_q(\Omega(G_1), G_1).$$

Therefore, from Theorem 2, Lemma 5, Lemma 6 and Lemma 8, we have $\dim PH^1(G, \Pi_{2q-2}) > \dim B_q(\Omega(G), G)$. This contradicts our hypothesis. Hence $PH^1(G_1, \Pi_{2q-2})=\beta^*(B_q(\Omega(G_1), G_1))$. Thus we have our Theorem.

5. In what follows, we always assume that $q=2$.

LEMMA 9. *Let G_0 be an elementary group. Then*

(1) $\dim PZ^1(G_0, \Pi_2)=2$ *if $\Omega(G_0)/G_0$ has the signature* $(0; \nu, \nu)$,

(2) $\dim PZ^1(G_0, \Pi_2)=3$ *if $\Omega(G_0)/G_0$ has the signature* $(0; \nu_1, \nu_2, \nu_3)$ *or* $(1; -)$

and

(3) $\dim PZ^1(G_0, \Pi_2)=4$ *if $\Omega(G_0)/G_0$ has the signature* $(0; 2, 2, 2, 2)$.

PROOF. The first statement (1) is obvious from Lemma 5. Now we assume that $\Omega(G_0)/G_0$ has the signature $(0;\nu_1,\nu_2,\nu_3)$. There exists a finitely generated quasi-Fuchsian group G_1 of the first kind such that isometric circles of all elements of G_1 and their interiors lie inside the Ford fundamental region of G_0. It is clear that $H=G_1\cap G_0=\{id\}$. For H, G_1 and G_0, we can take a precisely invariant disc B_i $(i=1, 2)$ under H satisfying conditions in Combination Theorem I. So we can construct a Kleinian group G generated by G_1 and G_0 by application of Combination Theorem I.

Therefore, by Theorem 1, we have

$$\dim PH^1(G, \Pi_2)=\dim PH^1(G_1, \Pi_2)+\dim PZ^1(G_0, \Pi_2).$$

Since β^* is injective, the inequality $\dim PH^1(G, \Pi_2)\geqq\dim B_2(\Omega(G), G)$ holds. For the quasi-Fuchsian group G_1, the equality $\dim PH^1(G_1, \Pi_2)=\dim B_2(\Omega(G_1), G_1)$ is known (see [2]). Hence, by using Lemma 7, we have

$$\dim PZ^1(G_0, \Pi_2)\geqq\dim B_2(\Omega(G), G)-\dim B_2(\Omega(G_1), G_1)=3.$$

The inequality in the opposite direction is obtained by direct estimate of $\dim PZ^1(G_0, \Pi_2)$. For instance, let $(0;2,4,4)$ be the signature of $\Omega(G_0)/G_0$. We may assume that G_0 is generated by $\gamma_1(z)=z+1$ and $\gamma_2(z)=iz$. For $p\in PZ^1(G_0, \Pi_2)$, set $p(\gamma_1)=a_2z^2+a_1z+a_0$ and $p(\gamma_2)=b_2z^2+b_1z+b_0$. Since $p(\gamma_1)=v\cdot\gamma_1-v$ for some $v\in\Pi_2$, we have $a_2=0$. Since $p(\gamma_2^4)=0$, we have $b_1=0$. Hence $p(\gamma_1)=a_1z+a_0$ and $p(\gamma_2)=b_2z^2+b_0$. Moreover $p((\gamma_2\circ\gamma_1)^4)=0$, whence $a_1+(1+i)b_2=0$. Therefore we see that p can be uniquely determined by three parameters a_0, a_1, b_0, which shows $\dim PZ^1(G_0, \Pi_2)\leqq3$. For all elementary groups with the signature $(0;\nu_1,\nu_2,\nu_3)$, we can also obtain $\dim PZ^1(G_0, \Pi_2)\leqq3$ in the same way. Thus we see (2).

We can also prove (3) in a similar manner.

LEMMA 10. *If G is a non-elementary Kleinian group which is generated by its elementary subgroups G_1 and G_2 by application of Combination Theorem I and if $H=G_1\cap G_2$ is elliptic cyclic or trivial, then $PH^1(G, \Pi_2)=\beta^*(B_2(\Omega(G), G))$.*

PROOF. We assume that $\Omega(G_i)/G_i$ has the signature $(0;\nu_{i1},\nu_{i2},\nu_{i3})$ for $i=1, 2$ and that $H=G_1\cap G_2$ is elliptic cyclic. By Theorem 1 we have

$$\dim PH^1(G, \Pi_2)=\dim PZ^1(G_1, \Pi_2)+\dim PZ^1(G_2, \Pi_2)-\dim PZ^1(H, \Pi_2)-3.$$

By Lemma 9, we have $\dim PZ^1(G_i, \Pi_2)=3$ and $\dim PZ^1(H, \Pi_2)=2$, which yields $\dim PH^1(G, \Pi_2)=1$. On the other hand, $\dim B_2(\Omega(G), G)=1$ by Lemma 7. Hence $\dim PH^1(G, \Pi_2)=\dim B_2(\Omega(G), G)$, that is $PH^1(G, \Pi_2)=\beta^*(B_2(\Omega(G), G))$. We can also prove all other cases in the same way as above.

LEMMA 11. *If G is a non-elementary Kleinian group which is generated by its elementary subgroup G_1 and an element f by application of Combination Theorem II and if H_1 (or H_2) is elliptic cyclic or parabolic cyclic or trivial, then*

$PH^1(G, \Pi_2)=\beta^*(B_2(\Omega(G), G))$.

PROOF. Assume that $\Omega(G_1)/G_1$ has the signature $(0; \nu_1, \nu_2, \nu_3)$ and that H_1 is an elliptic cyclic group. Using Theorem 2, Lemma 8 and Lemma 9, we have $\dim PH^1(G, \Pi_2)=\dim B_2(\Omega(G), G)$, that is, $PH^1(G, \Pi_2)=\beta^*(B_2(\Omega(G), G))$. All other cases can be proved in the same manner as in the above case.

LEMMA 12. *Let G be a Kleinian group which is generated by its finitely generated subgroups G_1 and G_2 by application of Combination Theorem I and let $H=G_1\cap G_2$ be elliptic cyclic or parabolic cyclic or trivial, where G_1 is non-elementary and G_2 is elementary. Then $PH^1(G, \Pi_2)=\beta^*(B_2(\Omega(G), G))$ if and only if $PH^1(G_1, \Pi_2)=\beta^*(B_2(\Omega(G_1), G_1))$.*

PROOF. Using Theorem 1, Lemma 7 and Lemma 9, we can easily verify this Lemma.

6. Let G be a non-elementary finitely generated Kleinian group with an invariant component. Then, as stated in preliminaries, we can construct G from basic groups in a finite number of steps by using Combination Theorems I and II, where, in each step, the amalgamated subgroups and the conjugated subgroups are trivial or elliptic cyclic or parabolic cyclic.

Now we can prove the Theorem stated in preliminaries which is restated as follows.

THEOREM 5. *Let G be a non-elementary finitely generated Kleinian group with an invariant component and let G be constructed from basic groups $G_1, \cdots, G_s$, by using Combination Theorems I and II. Then G_i is an elementary group or a quasi-Fuchsian group for $i=1, \cdots, s$ if and only if $PH^1(G, \Pi_2)=\beta^*(B_2(\Omega(G), G))$.*

PROOF. First we assume that G_i is an elementary group or a quasi-Fuchsian group for $i=1, \cdots, s$. For a quasi-Fuchsian group G_i, we have $\dim PH^1(G_i, \Pi_2)=\dim B_2(\Omega(G_i), G_i)$, so $PH^1(G_i, \Pi_2)=\beta^*(B_2(\Omega(G_i), G_i))$. As mentioned already, in each step of using Combination Theorems I and II, the amalgamated subgroups and the conjugated subgroups are trivial or elliptic cyclic or parabolic cyclic. Therefore, by Theorem 3, Theorem 4, Lemma 10, Lemma 11 and Lemma 12, we have $PH^1(G, \Pi_2)=\beta^*(B_2(\Omega(G), G))$.

Next assume that for some i, G_i is a degenerate basic group. Then G_i has no accidental parabolic transformation, so we have $\dim PH^1(G_i, \Pi_2)=2\dim B_2(\Omega(G_i), G_i)$ (see [2]). We have $\dim B_2(\Omega(G_i), G_i)\neq 0$, since G_i is a degenerate group. Therefore we have $PH^1(G_i, \Pi_2)\supsetneqq\beta^*(B_2(\Omega(G_i), G_i))$. Hence by Theorem 3, Theorem 4 and Lemma 12, we have $PH^1(G, \Pi_2)\supsetneqq\beta^*(B_2(\Omega(G), G))$.

This completes the proof of Theorem 5.

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Masami NAKADA
Department of Mathematics
Yamagata University
Koshirakawa-cho
Yamagata, Japan