J. Math. Soc. Japan
Vol. 28, No. 4, 1976

On locally symmetric Kaehler submanifolds in a complex projective space

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(Received Sept. 5, 1975)
(Revised July 6, 1976)

We denote by $M_n(c)$ an n-dimensional Kaehler manifold of constant holomorphic sectional curvature c, which is called a *complex space form*. An isometric and holomorphic immersion of a Kaehler manifold into a Kaehler manifold is said to be a *Kaehler immersion*. The study of Kaehler submanifolds immersed into a complex space form arose from a work of E. Calabi [**5**], who proved the local rigidity theorem to the effect that a Kaehler submanifold with analytic metric imbedded into $M_N(c)$ is locally rigid, and found the necessary and sufficient condition for a simply connected Kaehler manifold to be globally imbedded into a complete and simply connected complex space form as a Kaehler submanifold. Moreover, he completely classified Kaehler imbeddings of an n-dimensional complex projective space P_n into an N-dimensional complex projective space P_N.

After a while, B. Smyth [**23**] determined all complete and simply connected Einstein Kaehler hypersurfaces immersed into a complete and simply connected complex space form from the differential geometric point of view. The corresponding local theorem was proved by S. S. Chern [**8**]. As for extensions of these theorems, there are results of K. Nomizu and B. Smyth [**20**] and T. Takahashi [**24**]. With relation to these works, Kaehler submanifolds immersed in a complex space form are studied from various standpoints. In particular, K. Ogiue investigated these topics systematically, and related results are collected in [**22**]. Furthermore, concerning Einstein Kaehler submanifolds in P_N, J. Hano [**13**] obtained an interesting and suggestive result, and the first named author and K. Ogiue [**18**] studied the local version of Calabi's classification mentioned above. We note here that all examples of Einstein Kaehler submanifolds in P_N we know so far are symmetric.

Now, a complex projective space is one of the simplest examples of compact irreducible Hermitian symmetric spaces. Moreover, it is known that they

The authors would like to express their hearty thanks to the referee whose valuable suggestions led to the improvement of the original paper.

have various geometric properties. As one of them, they admit equivariant Kaehler imbeddings into P_N by virtue of theorems due to A. Borel and A. Weil [4] and G. Goto [11].

In consideration of these subjects, it seems interesting and fitting to the authors to study some properties about Kaehler imbeddings of compact Hermitian symmetric spaces into P_N. This paper has two purposes. One is to classify completely Kaehler imbeddings of such spaces into P_N. This classification is considered in a more general situation. As a result, we obtain many Einstein Kaehler submanifolds in P_N which are not symmetric (Theorem 4.1). The other is to compute various differential geometric quantities on symmetric Kaehler submanifolds in P_N. In particular, we find a close relation between a higher covariant derivative of the second fundamental form of each compact irreducible symmetric Kaehler submanifold in P_N and its rank as a symmetric space (Theorem 6.2).

The authors wish to express their hearty thanks to Professor M. Takeuchi for his many valuable suggestions.

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§ 1. Kaehler manifolds.

In this section, we recall basic formulas on Kaehler manifolds and define the linear operator Q represented by the curvature tensor. Let M' be a Kaehler manifold of complex dimension N. We choose a local field of unitary frames $\{e_1, \cdots, e_N\}$ defined in a neighborhood of M'. Its dual coframe field $\{\omega^1, \cdots, \omega^N\}$ consists of complex-valued linear differential forms of type (1, 0) on M' such that $\{\omega^1, \cdots, \omega^N, \bar{\omega}^1, \cdots, \bar{\omega}^N\}$ are linearly independent. The Kaehler metric g'

of M' can be then expressed as $g'=2\sum_A \omega^A\cdot\bar{\omega}^{A}$*). Associated with the frame $\{e_1, \cdots, e_N\}$, there exist complex-valued differential forms $\omega_B{}^A$, which are usually called connection forms on M', such that

$$d\omega^A+\sum_B \omega_B{}^A\wedge\omega^B=0\,,\qquad \omega_A{}^B+\bar{\omega}_B{}^A=0\,, \tag{1.1}$$

$$d\omega_B{}^A+\sum_C \omega_C{}^A\wedge\omega_B{}^C=\Phi_B{}^A\,,\qquad \Phi_B{}^A=\sum_{C,D} K_{\bar{A}BC\bar{D}}\omega^C\wedge\bar{\omega}^D\,, \tag{1.2}$$

where $\Phi_B{}^A$ (resp. $K_{\bar{A}BC\bar{D}}$) denotes the curvature form (resp. the curvature tensor). The second equation of (1.1) means the skew-hermitian symmetry of $\Phi_B{}^A$, which is equivalent to the symmetric conditions

$$K_{\bar{A}BC\bar{D}}=\bar{K}_{\bar{B}AD\bar{C}}\,. \tag{1.3}$$

The Bianchi identities obtained by the exterior derivative of (1.1) and (1.2) give

$$\sum_B \Phi_B{}^A\wedge\omega^B=0\,,$$

which implies the further symmetric relations

$$K_{\bar{A}BC\bar{D}}=K_{\bar{A}CB\bar{D}}=K_{\bar{D}BC\bar{A}}=K_{\bar{D}CB\bar{A}}\,. \tag{1.4}$$

Now, with respect to the frame chosen above, the Ricci tensor S' of M' can be expressed as follows:

$$S'=\sum_{C,D}(K_{C\bar{D}}\omega^C\otimes\bar{\omega}^D+K_{\bar{C}D}\bar{\omega}^C\otimes\omega^D)\,, \tag{1.5}$$

where $K_{C\bar{D}}=\sum_B K_{\bar{B}BC\bar{D}}=K_{\bar{D}C}=\bar{K}_{\bar{C}D}$. The scalar curvature K is also given by

$$K=2\sum_D K_{D\bar{D}}\,. \tag{1.6}$$

M' is said to be *Einstein*, if the Ricci tensor $K_{C\bar{D}}$ is expressed by

$$K_{C\bar{D}}=\lambda\delta_{CD}\,,\qquad \lambda=K/2N \tag{1.7}$$

for a constant λ, where λ is called the *Ricci curvature* of the Einstein manifold.

We shall here give a brief survey concerning complex space forms. We denote by $M_N(c)$ a complex N-dimensional complex space form of constant holomorphic sectional curvature c. $M_N(c)$ is said to be *elliptic*, *flat* or *hyper-*

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*) In order to avoid repetitions, the following convention on the range of indices will be used throughout this paper, unless otherwise stated:

$$A, B, \cdots=1, \cdots, n, n+1, \cdots, n+q=N\,,$$
$$i, j, \cdots=1, \cdots, n\,,$$
$$\alpha, \beta, \cdots=n+1, \cdots, N\,.$$

bolic, according as c is positive, zero or negative, respectively. The standard models of complex space forms of each type are the complex projective space P_N endowed with the Fubini-Study metric, the complex Euclidean space $\boldsymbol{C}^N$ with the flat metric and the open unit ball D_N in $\boldsymbol{C}^N$ equipped with the Bergman metric. P_N, $\boldsymbol{C}^N$ and D_N are, of course, all complete and simply connected complex space forms, which are elliptic, flat and hyperbolic, respectively. After multiplying the metric of an N-dimensional complex space form $M_N(c)$ by a suitable positive constant, $M_N(c)$ is locally holomorphically isometric to P_N, $\boldsymbol{C}^N$ or D_N, according as $M_N(c)$ is elliptic, flat or hyperbolic, respectively.

Now, the curvature tensor $K_{\bar{A}BC\bar{D}}$ on $M_N(c)$ can be given by

$$K_{\bar{A}BC\bar{D}}=\frac{c}{2}(\delta_{AB}\delta_{CD}+\delta_{AC}\delta_{BD}). \tag{1.8}$$

Then $M_N(c)$ is Einstein, and in the above notation the scalar curvature K is given by $K=N(N+1)c$ and the Ricci curvature λ by $\lambda=(N+1)c/2$.

Next, from the symmetric relation (1.4), on the $N(N+1)/2$-dimensional complex vector space Ξ consisting of symmetric tensor (ξ_{AB}) at each point on any Kaehler manifold M', we can define a linear transformation Q by

$$Q(\xi_{AB})=(\eta_{AB}), \qquad \eta_{AB}=\sum_{C,D}K_{\bar{C}AB\bar{D}}\xi_{CD}. \tag{1.9}$$

Since Q is a self-adjoint operator with respect to the metric canonically defined on Ξ, every eigenvalue of Q is a real-valued function. At each point of M', let $\mu_1, \cdots, \mu_t$ $(\mu_1<\cdots<\mu_t)$ be all distinct eigenvalues of Q and m_a the multiplicity of μ_a $(a=1, \cdots, t)$. As is easily seen, the trace of the operator Q is equal to a half of the scalar curvature.

As for some special Kaehler manifolds, these eigenvalues are known. For instance for $M_N(c)$ it follows from (1.8) that $t=1$ and $\mu_1=c$. E. Calabi and E. Vesentini [**6**] studied also the operator Q on compact irreducible Hermitian symmetric spaces M' of classical type. They proved that Q has exactly two distinct constant eigenvalues, always opposite in sign, if M' is not a complex projective space, and moreover determined m_1, m_2 and μ_1, μ_2. Successively, A. Borel [**2**] complemented their results by proving that Q has also two distinct constant eigenvalues, always opposite in sign, in the case where M' is of exceptional type, and by determining m_1, m_2 and μ_1, μ_2. By the way, M. Takeuchi obtains an *a priori* proof of these facts applying his theorem [**25**, p. 443]. Let $'M$ be a non-compact Hermitian symmetric space corresponding to a compact irreducible Hermitian symmetric space M'. It is obvious that all eigenvalues of Q on $'M$ are then opposite in sign to, and with the same multiplicities as, the ones on M'.

§ 2. Kaehler submanifolds.

In this section, we develop the general theory of Kaehler submanifolds immersed in $M_{n+q}(c)$ and prepare a useful formula and a few properties of the self-adjoint operator Q defined on the submanifold. Let M be an n-dimensional complex manifold and ι an isometric and holomorphic immersion of M into $M_{n+q}(c)$. Then, M is a Kaehler manifold endowed with the induced metric. We call such ι simply a *Kaehler immersion*. When the argument is local, M need not be distinguished from $\iota(M)$, and to simplify the discussion, we shall identify any point x in M with $\iota(x)$ in $M_{n+q}(c)$. Moreover we identify the tangent space $T_x(M)$ with $d\iota\,(T_x(M))\subset T_{\iota(x)}(M_{n+q}(c))$ by means of the differential $d\iota$ of ι. We choose a local field of unitary frames $\{e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+q}\}$ on $M_{n+q}(c)$ in such a way that, restricted to M, $e_1, \cdots, e_n$ are tangent to M. With respect to the frame field on $M_{n+q}(c)$, let $\{\omega^1, \cdots, \omega^n, \omega^{n+1}, \cdots, \omega^{n+q}\}$ be the field of dual frames. Then the Kaehler metric of $M_{n+q}(c)$ is given by $2\sum_A \omega^A\cdot\bar{\omega}^A$. We denote by $\omega_B{}^A$ the connection form on $M_{n+q}(c)$. The canonical forms ω^A and the connection forms $\omega_B{}^A$ on the ambient space satisfy the structure equations (1.1) and (1.2).

Restricting these forms to M, we have

$$\omega^\alpha = 0\,, \tag{2.1}$$

and the induced Kaehler metric g on M is given by $g=2\sum_i \omega^i\cdot\bar{\omega}^i$. $\{e_1, \cdots, e_n\}$ is a local field of unitary frames with respect to this metric and $\{\omega^1, \cdots, \omega^n\}$ is the field of coframes dual to $\{e_1, \cdots, e_n\}$, which consists of complex-valued linear differential forms of type (1, 0) on M. $\omega^1, \cdots, \omega^n, \bar{\omega}^1, \cdots, \bar{\omega}^n$ are, of course, linearly independent, and they are canonical forms on M. It follows from (1.1) and Cartan's lemma that the exterior derivatives of (2.1) give rise to

$$\omega_i{}^\alpha = \sum_j h_{ij}^\alpha \omega^j\,, \qquad h_{ij}^\alpha = h_{ji}^\alpha\,. \tag{2.2}$$

The quadratic form $\sum_{i,j} h_{ij}^\alpha \omega^i\cdot\omega^j$ is called the *second fundamental form* of the Kaehler immersion ι on M in the direction of e_α. M is totally geodesic if and only if $h_{ij}^\alpha=0$. From the structure equations (1.1) and (1.2) of $M_{n+q}(c)$ it follows that the structure equations of M are given by

$$d\omega^i + \sum_j \omega_j{}^i\wedge\omega^j = 0\,, \qquad \omega_j{}^i + \bar{\omega}_i{}^j = 0\,, \tag{2.3}$$

$$d\omega_j{}^i + \sum_k \omega_k{}^i\wedge\omega_j{}^k = \Omega_j{}^i\,, \tag{2.4}$$

$$\Omega_j{}^i = \sum_{k,l} R_{\bar{i}jk\bar{l}}\,\omega^k\wedge\bar{\omega}^l\,,$$

where $\omega_j{}^i$ (resp. $\Omega_j{}^i$) denotes the connection (resp. the curvature) form on the submanifold. Moreover, we have the following relation

$$d\omega_\beta{}^\alpha+\sum_\gamma \omega_\gamma{}^\alpha \wedge \omega_\beta{}^\gamma=\Omega_\beta{}^\alpha , \tag{2.5}$$

$$\Omega_\beta{}^\alpha=\sum_{k,l} R_{\bar\alpha\beta k\bar l}\, \omega^k \wedge \bar\omega^l ,$$

where $\Omega_\beta{}^\alpha$ is called the normal curvature form of M. From (2.2) and (2.4) we have the equation of Gauss

$$R_{\bar i j k \bar l}=\frac{c}{2}(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl})-\sum_\alpha h_{jk}^\alpha \bar h_{il}^\alpha , \tag{2.6}$$

and from (2.2), (2.3) and (2.5) we have

$$R_{\bar\alpha\beta k\bar l}=\frac{c}{2}\delta_{\alpha\beta}\delta_{kl}+\sum_j h_{jk}^\alpha \bar h_{jl}^\beta . \tag{2.7}$$

The Ricci tensor $S_{k\bar l}$ and the scalar curvature R of M are given by

$$S_{k\bar l}=\frac{n+1}{2}c\delta_{kl}-\sum_{\alpha,j} h_{jk}^\alpha \bar h_{jl}^\alpha , \tag{2.8}$$

$$R=n(n+1)c-2\sum_{\alpha,k,l} h_{kl}^\alpha \bar h_{kl}^\alpha . \tag{2.9}$$

Thus we have

$$n(n+1)c-R\geqq 0 ,$$

where the equality is valid if and only if M is totally geodesic.

Now, we define the covariant derivatives h_{ijk}^α and $h_{ij\bar k}^\alpha$ of h_{ij}^α by

$$\sum_k h_{ijk}^\alpha \omega^k+\sum_k h_{ij\bar k}^\alpha \bar\omega^k=dh_{ij}^\alpha-\sum_k h_{kj}^\alpha \omega_i{}^k-\sum_k h_{ik}^\alpha \omega_j{}^k+\sum_\beta h_{ij}^\beta \omega_\beta{}^\alpha .$$

Then, substituting dh_{ij}^α in this equation into the exterior derivative of (2.2), we get

$$h_{ijk}^\alpha=h_{jik}^\alpha=h_{ikj}^\alpha , \qquad h_{ij\bar k}^\alpha=0 . \tag{2.10}$$

Inductively we shall define the covariant derivatives $h_{i_1\cdots i_m i_{m+1}}^\alpha$ and $h_{i_1\cdots i_m \bar i_{m+1}}^\alpha$ of $h_{i_1\cdots i_m}^\alpha$ for $m\geqq 2$. Suppose that $h_{i_1\cdots i_m}^\alpha$ are defined for $m\geqq 3$. Then $h_{i_1\cdots i_m j}^\alpha$ and $h_{i_1\cdots i_m \bar j}^\alpha$ are defined by

$$\sum_j h_{i_1\cdots i_m j}^\alpha \omega^j+\sum_j h_{i_1\cdots i_m \bar j}^\alpha \bar\omega^j \tag{2.11}$$

$$=dh_{i_1\cdots i_m}^\alpha-\sum_{r=1}^m \sum_j h_{i_1\cdots i_{r-1} j i_{r+1}\cdots i_m}^\alpha \omega_{i_r}{}^j+\sum_\beta h_{i_1\cdots i_m}^\beta \omega_\beta{}^\alpha .$$

Similarly $h_{i_1\cdots i_m \bar j k}^\alpha$, $h_{i_1\cdots i_m \bar j \bar k}^\alpha$, $(\bar h_{i_1\cdots i_m}^\alpha)_j$ and $(\bar h_{i_1\cdots i_m}^\alpha)_{\bar j}$ can be defined, where $\bar h_{i_1\cdots i_m}^\alpha$ denotes the complex conjugation of $h_{i_1\cdots i_m}^\alpha$. It is clear that $(\bar h_{i_1\cdots i_m}^\alpha)_j=\bar h_{i_1\cdots i_m \bar j}^\alpha$

and $(\bar{h}^\alpha_{i_1\cdots i_m})_{\bar{j}}=\bar{h}^\alpha_{i_1\cdots i_m j}$. By taking the exterior derivative of (2.11) and by using (2.5) and so on, the following formulas are proved [19]:

(2.12) $\quad h^\alpha_{i_1\cdots i_m}$ *is symmetric with respect to* $i_1, \cdots, i_m$

and

(2.13)
$$\begin{aligned} & h^\alpha_{i_1\cdots i_m j\bar{k}}-h^\alpha_{i_1\cdots i_m\bar{k}j} \\ &=\frac{c}{2}\{(m-1)h^\alpha_{i_1\cdots i_m}\delta_{jk}+\sum_{r=1}^{m} h^\alpha_{i_1\cdots i_{r-1}ji_{r+1}\cdots i_m}\delta_{i_r k}\} \\ &\quad -\sum_{r=1}^{m}\sum_{\beta,l} h^\alpha_{i_1\cdots i_{r-1}li_{r+1}\cdots i_m}h^\beta_{i_r j}\bar{h}^\beta_{lk}-\sum_{\beta,l} h^\alpha_{lj}h^\beta_{i_1\cdots i_m}\bar{h}^\beta_{lk}\,. \end{aligned}$$

LEMMA 2.1. *The following relation is true:*

(2.14)
$$\begin{aligned} h^\alpha_{i_1\cdots i_m\bar{j}}= & \frac{m-2}{2}c\sum_{r=1}^{m} h^\alpha_{i_1\cdots\hat{i}_r\cdots i_m}\delta_{i_r j} \\ & -\sum_{r=1}^{m-2}\frac{1}{r!(m-r)!}\sum_{\sigma,\beta,l} h^\alpha_{li_{\sigma(1)}\cdots i_{\sigma(r)}}h^\beta_{i_{\sigma(r+1)}\cdots i_{\sigma(m)}}\bar{h}^\beta_{lj} \end{aligned}$$

for $m\geqq 3$, *where the summation on* σ *is taken over all permutations of* $(1, \cdots, m)$.

PROOF. We prove (2.14) by induction on m. At first, the case where $m=2$ in (2.13) is considered. This shows that (2.14) holds for $m=3$. Next, suppose that (2.14) holds for some m. Then, using (2.10), we have

$$\begin{aligned} h^\alpha_{i_1\cdots i_m\bar{j}i_{m+1}}= & \frac{m-2}{2}c\sum_{r=1}^{m} h^\alpha_{i_1\cdots\hat{i}_r\cdots i_m i_{m+1}}\delta_{i_r j} \\ & -\sum_{r=1}^{m-2}\frac{1}{r!(m-r)!}\sum_{\sigma,\beta,l} h^\alpha_{li_{\sigma(1)}\cdots i_{\sigma(r)}i_{m+1}}h^\beta_{i_{\sigma(r+1)}\cdots i_{\sigma(m)}}\bar{h}^\beta_{lj} \\ & -\sum_{r=1}^{m-2}\frac{1}{r!(m-r)!}\sum_{\sigma,\beta,l} h^\alpha_{li_{\sigma(1)}\cdots i_{\sigma(r)}}h^\beta_{i_{\sigma(r+1)}\cdots i_{\sigma(m)}i_{m+1}}\bar{h}^\beta_{lj}\,. \end{aligned}$$

Combining this equation together with (2.13), one gets

$$\begin{aligned} h^\alpha_{i_1\cdots i_m i_{m+1}\bar{j}} &= h^\alpha_{i_1\cdots i_m\bar{j}i_{m+1}}+\frac{m-1}{2}ch^\alpha_{i_1\cdots i_m}\delta_{ji_{m+1}}+\frac{c}{2}\sum_{r=1}^{m} h^\alpha_{i_1\cdots\hat{i}_r\cdots i_m i_{m+1}}\delta_{i_r j} \\ &\quad -\sum_{r=1}^{m}\sum_{\beta,l} h^\alpha_{i_1\cdots i_{r-1}li_{r+1}\cdots i_m}h^\beta_{i_r i_{m+1}}\bar{h}^\beta_{lj}-\sum_{\beta,l} h^\alpha_{li_{m+1}}h^\beta_{i_1\cdots i_m}\bar{h}^\beta_{lj} \\ &=\frac{m-1}{2}c\sum_{r=1}^{m+1} h^\alpha_{i_1\cdots\hat{i}_r\cdots i_{m+1}}\delta_{i_r j} \\ &\quad -\sum_{r=1}^{m-1}\frac{1}{r!(m+1-r)!}\sum_{\sigma,\beta,l} h^\alpha_{li_{\sigma(1)}\cdots i_{\sigma(r)}}h^\beta_{i_{\sigma(r+1)}\cdots i_{\sigma(m+1)}}\bar{h}^\beta_{lj}\,. \end{aligned}$$

This implies that (2.14) holds for $m+1$, which completes the proof.

LEMMA 2.2. *Let M^i be an n_i-dimensional Kaehler manifolds $(i=1, 2)$. Assume that a Kaehler manifold $M=M^1\times M^2$ admits a Kaehler immersion into $M_{n_1+n_2+q}(c)$. Then c is non-negative. If $c>0$, then $q\geqq n_1n_2$.*

PROOF. We use the following convention on the range of indices in this proof: $a, b, \cdots=1, \cdots, n_1$; $r, s, \cdots=n_1+1, \cdots, n_1+n_2$. One can choose a local field of unitary frames $\{e_a, e_r, e_\alpha\}$ on $M_{n_1+n_2+q}(c)$ in such a way that, restricted to M, e_a are tangent to M^1 and e_r are tangent to M^2. We have then $R_{\bar{a}rk\bar{l}}=0$, since $\Omega_r{}^\alpha=0$ on M. By (2.6) this can be written as

$$\text{(2.15)}\qquad \begin{cases} \sum_\alpha h^\alpha_{ab}\bar{h}^\alpha_{cr}=\sum_\alpha h^\alpha_{ab}\bar{h}^\alpha_{rs}=\sum_\alpha h^\alpha_{ar}\bar{h}^\alpha_{st}=0\,, \\ \sum_\alpha h^\alpha_{br}\bar{h}^\alpha_{as}=c\delta_{ab}\delta_{rs}/2\,, \end{cases}$$

and the last equation implies that $c\geqq 0$, and q-dimensional vectors $h_{ar}=(h^\alpha_{ar})$ are linearly independent, if c is positive. Hence we have $q\geqq n_1n_2$ if $c>0$.

Q. E. D.

We shall next define three kinds of matrices A, H and H^α for any α by

$$A=(A_\beta{}^\alpha)\,,\qquad A_\beta{}^\alpha=\sum_{i,j} h^\alpha_{ij}\bar{h}^\beta_{ij}\,,$$
$$H=(h^\alpha_{(ij)})\,,$$
$$H^\alpha=(h^\alpha_{ij})\,.$$

Then it is evident that the matrix A is a positive semi-definite Hermitian one of order q and the second matrix H is a $q\times n(n+1)/2$-one and H^α is an $n\times n$ symmetric matrix. We have the following relation among them:

$$A=(\mathrm{Tr}\,(H^\alpha\bar{H}^\beta))\,.$$

We study the relations between distinct eigenvalues $\mu_1, \cdots, \mu_t$ of the linear operator Q on a submanifold immersed in $M_{n+q}(c)$ and those of the Hermitian matrix A.

LEMMA 2.3. *Let M be an n-dimensional Kaehler submanifold immersed in $M_{n+q}(c)$. Then the following assertions are valid at each point on M:*

(1) *For $a=1, \cdots, t$, $c-\mu_a\geqq 0$. If $c\neq\mu_a$, then $c-\mu_a$ is an eigenvalue of the matrix A.*

(2) *If $q<n(n+1)/2$, then the maximal eigenvalue μ_t is equal to c.*

(3) *If $c\neq\mu_t$, then the rank of the matrix A is equal to $n(n+1)/2$, and the eigenvalues of A are $c-\mu_a$ $(a=1, \cdots, t)$ and possibly 0.*

(4) *If $c=\mu_t$, then the rank of A is equal to $n(n+1)/2-m_t$, and the eigenvalues of A are $c-\mu_a$ $(a=1, \cdots, t-1)$ and possibly 0.*

PROOF. We consider Q at an arbitrary but fixed point of M. Let V_a be the eigenspace of Q corresponding to an eigenvalue μ_a $(a=1, \cdots, t)$. Then a direct decomposition

$$\Xi = V_1 + V_2 + \cdots + V_t$$

is obtained. If $\xi = (\xi_{ij}) \in V_a$, then (1.9) and (2.6) imply

$$\sum_{\beta,k,l} h^{\beta}_{ij} \bar{h}^{\beta}_{kl} \xi_{kl} = (c - \mu_a) \xi_{ij} , \tag{2.16}$$

and hence

$$\sum_{\beta,i,j,k,l} \bar{h}^{\alpha}_{ij} h^{\beta}_{ij} \bar{h}^{\beta}_{kl} \xi_{kl} = (c - \mu_a) \sum_{i,j} \bar{h}^{\alpha}_{ij} \xi_{ij} . \tag{2.17}$$

For any vector $\eta \in \Xi$, we define v_η by $v_\eta = (\langle H^\beta, \eta \rangle)$, which can be regarded as a q-dimensional vector in $\boldsymbol{C}^q$, where $\langle\ ,\ \rangle$ denotes the inner product on Ξ. For the inner product $(\ ,\ )$ on $\boldsymbol{C}^q$, we have from (2.16) and (2.17)

$$A v_\xi = (c - \mu_a) v_\xi \qquad \text{for} \quad \xi \in V_a , \tag{2.18}$$

$$(v_\xi, v_\eta) = (c - \mu_a) \langle \xi, \eta \rangle \qquad \text{for} \quad \xi \in V_a \text{ and } \eta \in \Xi . \tag{2.19}$$

Suppose that $\mu_a \neq c$ for each a. Then (2.19) implies that $v_\xi \neq 0$ for $0 \neq \xi \in V_a$, and (2.18) shows that $c - \mu_a$ is an eigenvalue of the Hermitian matrix A with eigenvector v_ξ. Since A is positive semi-definite, we see $c \geqq \mu_a$, and therefore $c > \mu_a$. Thus the first assertion is proved.

Suppose $\mu_t \neq c$. Then (2.19) implies that the linear subspace $\{v_\xi ; \xi \in V_a\}$ is of dimension m_a for each a. Hence the multiplicity of the eigenvalue $c - \mu_a$ of the matrix A is greater than or equal to $m_a = \dim V_a$ for each a. Summing up these inequalities over a, we get

$$\sum_{a=1}^{t} m_a \leqq \text{rank of } A .$$

Remark that

$$\sum_{a=1}^{t} m_a = \sum_{a=1}^{t} \dim V_a = \dim \Xi = \frac{n(n+1)}{2} ,$$

and the rank of $A \leqq q$. This proves (2).

Moreover, since the trace of the linear transformation Q is equal to $R/2$, we obtain

$$\operatorname{Tr} A \geqq \sum_{a=1}^{t} m_a (c - \mu_a) = \frac{n(n+1)}{2} c - \operatorname{Tr} Q = \frac{n(n+1)c - R}{2} ,$$

and hence by (2.9)

$$\operatorname{Tr} A = \sum_{a=1}^{t} m_a (c - \mu_a) .$$

This implies that the eigenvalues of A are $c - \mu_a$ and possibly 0, and the multiplicity of $c - \mu_a$ is equal to m_a. Thus (3) is proved.

By a discussion similar to the above, we can prove the last property.

Q. E. D.

LAMMA 2.4. *Let M be an n-dimensional Kaehler submanifold immersed in $M_{n+q}(c)$. If $\mu_t=c$, then one gets*

$$\sum_{\beta,\kappa,l} h^{\alpha}_{kl}\bar{h}^{\beta}_{\kappa l}h^{\beta}_{ij}=(c-\mu_1)h^{\alpha}_{ij}$$

at the point where $t=2$.

PROOF. It follows from Lemma 2.3 and (2.19) that we have

$$(v_{\xi}, v_{\xi})=0 \qquad \text{for all} \quad \xi=(\xi_{ij}) \text{ in } V_2 .$$

This implies that the tensor (h^{α}_{ij}), which is symmetric with respect to i and j, is orthogonal to V_2 and therefore belongs to V_1 for each α. The formula follows from (2.16). Q. E. D.

§ 3. Locally symmetric Kaehler submanifolds in $M_N(c)$.

In this section, we investigate the manifold structure of locally symmetric Kaehler submanifolds immersed in $M_N(c)$, in the case where the ambient space is flat or hyperbolic.

Let M be an n-dimensional locally symmetric Kaehler manifold and ι the Kaehler immersion of M into $M_N(c)$. For any point x in M and some positive integer m, let $N^m_x(M)$ be a subspace spanned by the vector $\sum_{\alpha} h^{\alpha}_{i_1\cdots i_m}e_{\alpha}$ in the normal space $N_x(M)$ of M. Since M is locally symmetric, we have

$$\sum_{\alpha} h^{\alpha}_{i_1i_2i_3}\bar{h}^{\alpha}_{j_1j_2}=0\,, \tag{3.1}$$

by virtue of (2.6) and (2.10). This means that two spaces $N^2_x(M)$ and $N^3_x(M)$ are mutually orthogonal. Taking the covariant derivatives of (3.1) successively, we get

$$\sum_{\alpha} h^{\alpha}_{i_1\cdots i_m}\bar{h}^{\alpha}_{j_1j_2}=0 \qquad \text{for} \quad m\geqq 3\,. \tag{3.2}$$

Now

$$\sum_{\alpha} h^{\alpha}_{i_1\cdots i_m}\bar{h}^{\alpha}_{j_1j_2j_3}=(\sum_{\alpha} h^{\alpha}_{i_1\cdots i_m}\bar{h}^{\alpha}_{j_1j_2})_{\bar{j}_3}-\sum_{\alpha} h^{\alpha}_{i_1\cdots i_m\bar{j}_3}\bar{h}^{\alpha}_{j_1j_2}\,.$$

Applying Lemma 2.1 and (3.2) to this expression, we obtain

$$\sum_{\alpha} h^{\alpha}_{i_1\cdots i_m}\bar{h}^{\alpha}_{j_1j_2j_3}=0 \qquad \text{for} \quad m\geqq 4\,.$$

Inductively we can show

$$\sum_{\alpha} h^{\alpha}_{i_1\cdots i_m}\bar{h}^{\alpha}_{j_1\cdots j_r}=0 \qquad \text{for} \quad m>r\geqq 2\,. \tag{3.3}$$

Now, we denote by A_m the square of the length of $h^{\alpha}_{i_1\cdots i_m}$, in other words, we put

$$A_m=\sum_{\alpha,i_1,\cdots,i_m} h^{\alpha}_{i_1\cdots i_m}\bar{h}^{\alpha}_{i_1\cdots i_m} \quad \text{for} \quad m\geqq 2. \tag{3.4}$$

If $m=2$, then (2.9) implies

$$A_2=\frac{n(n+1)c-R}{2}. \tag{3.5}$$

Of course, since the scalar curvature R of M is constant, A_2 must be constant. In general, one can show by Lemma 2.1 and (3.3) that A_m is constant.

PROPOSITION 3.1. *Let M be an n-dimensional locally symmetric Kaehler submanifold immersed in $M_N(c)$. Then there exists a positive integer m_0 in such a way that*

$$A_{m_0}\neq 0, \qquad A_{m_0+1}=0.$$

PROOF. Suppose that A_m is positive for each positive integer m. Then $(h^{\alpha}_{i_1\cdots i_m})$ for some fixed $i_1, \cdots, i_m$ is a non-zero q-dimensional vector, where the indices $i_1, \cdots, i_m$ depend on m. This property and (3.3) imply that there exist an infinite number of linearly independent q-vectors in the normal space, which is a contradiction. Q. E. D.

We call such m_0 the *degree* of $\iota: M\to M_N(c)$. In particular, when the emphasis is laid on the immersion, the degree is denoted by $d(M, \iota)$. To say that the degree is 1 means that h^{α}_{ij} vanishes identically on each neighborhood in M, and hence M is totally geodesic. Similarly, in view of the second equation of (2.10), that the degree is 2 means that the second fundamental form is parallel but does not vanish.

THEOREM 3.2. *Let M be an n-dimensional locally symmmetric Kaehler submanifold immersed in $M_{n+q}(c)$. If the ambient space is flat or hyperbolic, then M is totally geodesic.*

PROOF. The proof is divided into three parts.

(1) The case where M is a complex space form. This is precisely a theorem of the first named author and K. Ogiue [**19**].

(2) The case where M is a piece of an irreducible Hermitian symmetric space different from a complex space form. Then, as is already stated in the first section, the linear operator Q on M has exactly two distinct constant eigenvalues, say μ_1 and μ_2 $(\mu_1<\mu_2)$. On the other hand, by means of the first assertion of Lemma 2.3, $c-\mu_1$ and $c-\mu_2$ are non-negative, which contradicts the fact that μ_2 is positive. Thus this case is excluded.

(3) The case where M is reducible. Then, M is locally a product $U^1\times\cdots\times U^k$ of pieces of irreducible Hermitian symmetric spaces. Since each U^s $(s=1, \cdots, k)$ can be considered as a Kaehler submanifold in $M_{n+q}(c)$, c must be

zero by Lemma 2.2. Hence each U^s is a complex space form according to the case (2), and moreover it is flat according to the case (1). Consequently, the proof is reduced to the case (1). Q.E.D.

REMARK. For a Kaehler submanifold immersed in $M_N(c)$ with parallel second fundamental form, M. Kon [16] proved that if $c \leqq 0$, then M is totally geodesic. Theorem 3.2 is a slight generalization of his theorem.

§4. Examples of Einstein Kaehler submanifolds.

In this section we describe various examples of Einstein Kaehler submanifolds immersed in P_N. They are given as a class Θ of irreducible C-spaces M in the sense of H.C. Wang [28] such that $\dim H^2(M; \boldsymbol{R})=1$. Here a C-space stands for a compact simply connected complex homogeneous manifold, which was completely classified by himself. We know that Θ contains all compact irreducible Hermitian symmetric spaces. On the other hand, M. Goto [11] and A. Borel and A. Weil [4] proved, in different ways, that Kaehler C-spaces are algebraic. For later use, we begin with the construction of C-spaces in Θ and their holomorphic imbeddings into P_N after the fashion of [4]. For more details about the results mentioned without proofs from the theory of Lie algebras, see e.g. [14].

Let $\mathfrak{g}$ be a complex simple Lie algebra. We choose a fundamental root system α_i $(i=1, \cdots, l)$ of $\mathfrak{g}$, where l is the rank of $\mathfrak{g}$. Then Θ can be constructed from possible pairs $(\mathfrak{g}, \alpha_i)$ as follows. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ (so $l=\dim_{\boldsymbol{C}} \mathfrak{h}$). The dual space of the complex vector space $\mathfrak{h}$ is denoted by $\mathfrak{h}^*$. An element α of $\mathfrak{h}^*$ is called a root of $(\mathfrak{g}, \mathfrak{h})$ if there exists a non-zero vector E_α in $\mathfrak{g}$ such that

$$[H, E_\alpha]=\alpha(H)E_\alpha \qquad \text{for all } H \in \mathfrak{h}.$$

We denote by Δ the set of all non-zero roots of $(\mathfrak{g}, \mathfrak{h})$ and put $\mathfrak{g}_\alpha = \boldsymbol{C}E_\alpha$. Then we have a direct sum decomposition:

$$\mathfrak{g}=\mathfrak{h}+\sum_{\alpha \in \Delta} \mathfrak{g}_\alpha .$$

Since the Killing form K of $\mathfrak{g}$ is non-degenerate on $\mathfrak{h}\times\mathfrak{h}$, for each $\xi \in \mathfrak{h}^*$ we can define $H_\xi \in \mathfrak{h}$ by

$$K(H, H_\xi)=\xi(H) \qquad \text{for all } \ H \in \mathfrak{h}.$$

Put $\mathfrak{h}_0=\sum_{\alpha\in\Delta} \boldsymbol{R}H_\alpha$. Then $\dim_{\boldsymbol{R}} \mathfrak{h}_0 = l$, and so the dual space $\mathfrak{h}_0^*$ of $\mathfrak{h}_0$ can be considered as a real subspace of $\mathfrak{h}^*$. We define an inner product on $\mathfrak{h}_0^*$ by

$$(\xi, \eta)=K(H_\xi, H_\eta) \qquad \text{for all } \ \xi, \eta \in \mathfrak{h}_0^* .$$

The fundamental root system $\alpha_1, \cdots, \alpha_l$ of $\mathfrak{g}$ already chosen can be assumed to be the set of simple roots with respect to a linear ordering in $\mathfrak{h}_0^*$. Let $\Lambda_1, \cdots, \Lambda_l$ be the fundamental weight system of $\mathfrak{g}$ associated with $\alpha_1, \cdots, \alpha_l$, that is,

$$2(\Lambda_i, \alpha_j)=(\alpha_j, \alpha_j)\delta_{ij} \qquad (i, j=1, \cdots, l).$$

For each $\alpha\in\Delta$ we select a base E_α of $\mathfrak{g}_\alpha$ so that $\{H_{\alpha_j}\ (j=1, \cdots, l),\ E_\alpha\ (\alpha\in\Delta)\}$ forms Weyl's canonical base of $\mathfrak{g}$. Then the following $\mathfrak{g}_u$ is a compact real form of $\mathfrak{g}$:

$$\mathfrak{g}_u=\sum_{\alpha\in\Delta}\boldsymbol{R}\sqrt{-1}\,H_\alpha+\sum_{\alpha\in\Delta}\boldsymbol{R}(E_\alpha+E_{-\alpha})+\sum_{\alpha\in\Delta}\boldsymbol{R}\sqrt{-1}\,(E_\alpha-E_{-\alpha}).$$

We fix a simple root α_i $(i=1, \cdots, l)$. We define a subset Δ_i of Δ and a complex subalgebra $\mathfrak{l}_i$ of $\mathfrak{g}$ by

$$\Delta_i=\{n_1\alpha_1+\cdots+n_l\alpha_l\in\Delta\,;\ n_1, \cdots, n_l: \text{integers},\ n_i<0\}\,, \tag{4.1}$$

$$\mathfrak{l}_i=\mathfrak{h}+\sum_{\alpha\in\Delta-\Delta_i}\mathfrak{g}_\alpha\,. \tag{4.2}$$

If we put $\mathfrak{h}_{u,i}=\mathfrak{g}_u\cap\mathfrak{l}_i$, then it is a subalgebra of $\mathfrak{g}_u$ expressed as

$$\mathfrak{h}_{u,i}=\sum_{\alpha\in\Delta}\boldsymbol{R}\sqrt{-1}\,H_\alpha+\sum_{\alpha\in\Delta^--\Delta_i}\boldsymbol{R}(E_\alpha+E_{-\alpha})+\sum_{\alpha\in\Delta^--\Delta_i}\boldsymbol{R}\sqrt{-1}\,(E_\alpha-E_{-\alpha}),$$

where $\Delta^-=\{\alpha\in\Delta\,;\ \alpha<0\}$.

Let G be the simply connected complex Lie group with the Lie algebra $\mathfrak{g}$. Let L_i be the connected complex Lie subgroup of G with the Lie algebra $\mathfrak{l}_i$ and G_u, $H_{u,i}$ be the connected Lie subgroups of G with the Lie algebras $\mathfrak{g}_u$, $\mathfrak{h}_{u,i}$ respectively. Then we obtain a compact homogeneous manifold $M_i=G_u/H_{u,i}$. The injection of G_u into G induces a homeomorphism of M_i onto a simply connected complex homogeneous manifold G/L_i, and furthermore under this homeomorphism M_i becomes a complex manifold on which G_u (and also G) acts transitively as a group of holomorphic transformations (cf. [4], [15] and [26]).

It is known in [3] that

$$H^2(M_i\,;\boldsymbol{R})\cong H^2(H_{u,i}\,;\boldsymbol{R})\cong \text{the center of }\mathfrak{h}_{u,i}\cong\boldsymbol{R}H_{\Lambda_i}\,.$$

Thus we have obtained an irreducible C-space M_i with $\dim H^2(M_i\,;\boldsymbol{R})=1$ from each complex simple Lie algebra $\mathfrak{g}$ and each simple roots α_i $(i=1, \cdots, l)$ of $\mathfrak{g}$. Conversely every irreducible C-space M with $\dim H^2(M\,;\boldsymbol{R})=1$ can be obtained in the way just described ([28]).

Next, we construct holomorphic imbeddings of M_i into a complex projective space. We fix a positive integer p. By a well known theorem of E. Cartan, there exists an irreducible representation $(f_i{}^p, \boldsymbol{C}^{N(p)+1})$ (resp. $(\hat{\rho}_i{}^p$,

$\boldsymbol{C}^{N(p)+1}$) of $\mathfrak{g}$ (resp. G), unique up to an equivalence, whose highest weight Λ is equal to $p\Lambda_i$. They are related by $f_i^p=d\hat{\rho}_i^p$, where $d\hat{\rho}_i^p$ denotes the differentiation of $\hat{\rho}_i^p$. Let V be the eigenspace of $(f_i^p, \boldsymbol{C}^{N(p)+1})$ belonging to the weight Λ. Then $\dim_C V=1$. $f_i^p(E_\alpha)$ $(\alpha\in\Delta)$ leaves V invariant if and only if $(\Lambda, \alpha)\geqq 0$ (cf. [**3**], [**4**] and [**26**]). We see easily

$$\Delta_i=\{\alpha\in\Delta\,;\,(\Lambda_i, \alpha)<0\}\,, \tag{4.3}$$

$$\mathfrak{l}_i=\{X\in\mathfrak{g}\,;\,f_i^p(X)V\subset V\}\,. \tag{4.4}$$

Put $\tilde{L}_i=\{g\in G\,;\,\hat{\rho}_i^p(g)V\subset V\}$. Then $\tilde{L}_i$ is a closed subgroup of G and its Lie algebra coincides with $\mathfrak{l}_i$. Hence the identity component of $\tilde{L}_i$ is equal to L_i, and it is contained in the normalizer of L_i. It is known (cf. J. A. Wolf and A. Korányi [**30**, p. 905]) that the normalizer of L_i is equal to L_i itself. Therefore $\tilde{L}_i=L_i$. Then a mapping: $g\mapsto\hat{\rho}_i^p(g)V$ of G into $P_{N(p)}$ induces an injection ρ_i^p of $M_i=G/L_i$ into $P_{N(p)}$. It is clear from the construction that ρ_i^p is holomorphic. On the other hand, since G_u is compact, we can choose a suitable unitary frame $\{e_0, \cdots, e_{N(p)}\}$ on $\boldsymbol{C}^{N(p)+1}$ such that $e_0\in V$ and $\hat{\rho}_i^p(G_u)\subset SU(N(p)+1)$. Then we can identify $P_{N(p)}$ with $SU(N(p)+1)/S(U(N(p))\times U(1))$, where

$$S(U(N(p))\times U(1))=\{A\in SU(N(p)+1)\,;\,AV\subset V\}\,.$$

Thus we have obtained countably many holomorphic imbeddings $\{\rho_i^p\}$ of M_i into $P_{N(p)}$. We shall call such ρ_i^p a *p-canonical imbedding* of M_i into $P_{N(p)}$. In particular, the 1-canonical imbedding ρ_i^1 is simply said to be *canonical.*

We assert that the Kaehler metric g_i^p on M_i induced from the metric on $P_{N(p)}$ under ρ_i^p is Einstein. In fact, the group G_u acts on M_i transitively as a group of isometries, since $\hat{\rho}_i^p(G_u)$ is a subgroup of $SU(N(p)+1)$. In particular, the scalar curvature of g_i^p is constant. Then it is well-known that the so-called Ricci form of g_i^p is harmonic (see, e.g., [**31**], p. 72]). It follows from $\dim H^2(M_i\,;\boldsymbol{R})=1$ that it is proportional to the fundamental 2-form of g_i^p, which proves our assertion. This implies that M_i is an Einstein Kaehler submanifold imbedded in $P_N(c)$.

The above argument can be summed up as

THEOREM 4.1. *Let $\mathfrak{g}$ be an arbitrary complex simple Lie algebra and $\{\alpha_1, \cdots, \alpha_l\}$ a fundamental root system of $\mathfrak{g}$. Then a compact simply connected complex homogeneous manifold $M_i=G_u/H_{u,i}$ constructed from $\mathfrak{g}$ and each i in the above way admits countably many holomorphic imbeddings $\{\rho_i^p\}$ $(p=1, 2, \cdots)$ into a complex projective space $P_{N(p)}$ for some $N(p)$, and the Kaehler metric g_i^p on M_i induced from the Fubini-Study metric on $P_{N(p)}$ under ρ_i^p is Einstein. In other words, (M_i, ρ_i^p) is an Einstein Kaehler submanifold imbedded in $P_{N(p)}$.*

REMARK 4.1. We have another expression of g_i^p as follows. Let θ^α, $\theta^{-\alpha}$

be the dual forms of E_α, $E_{-\alpha}$. Then $\bar\theta^\alpha=\theta^{-\alpha}$, and a theorem of A. Borel [**1**] says that every G_u-invariant Kaehler metric (and in particular, $g_i{}^p$) on M_i is proportional to $-\sum_{\alpha\in\Delta_i}(\Lambda_i, \alpha)\theta^\alpha\cdot\bar\theta^\alpha$.

Now, we compute the complex dimension n of M_i. Let $w(i)$ be the number of simple roots α_j of $\mathfrak{g}$ such that $(\alpha_i, \alpha_j)\neq 0$. Then we know $w(i)=1$, 2 or 3. If we take away α_i from the Dynkin diagram D of $\mathfrak{g}$, then there arise the Dynkin diagrams of $w(i)$ complex simple Lie algebras, say $\mathfrak{g}_1, \cdots, \mathfrak{g}_{w(i)}$. Then the following formula on dimensions is due to J. Tits [**26**, p. 130].

LEMMA 4.2.

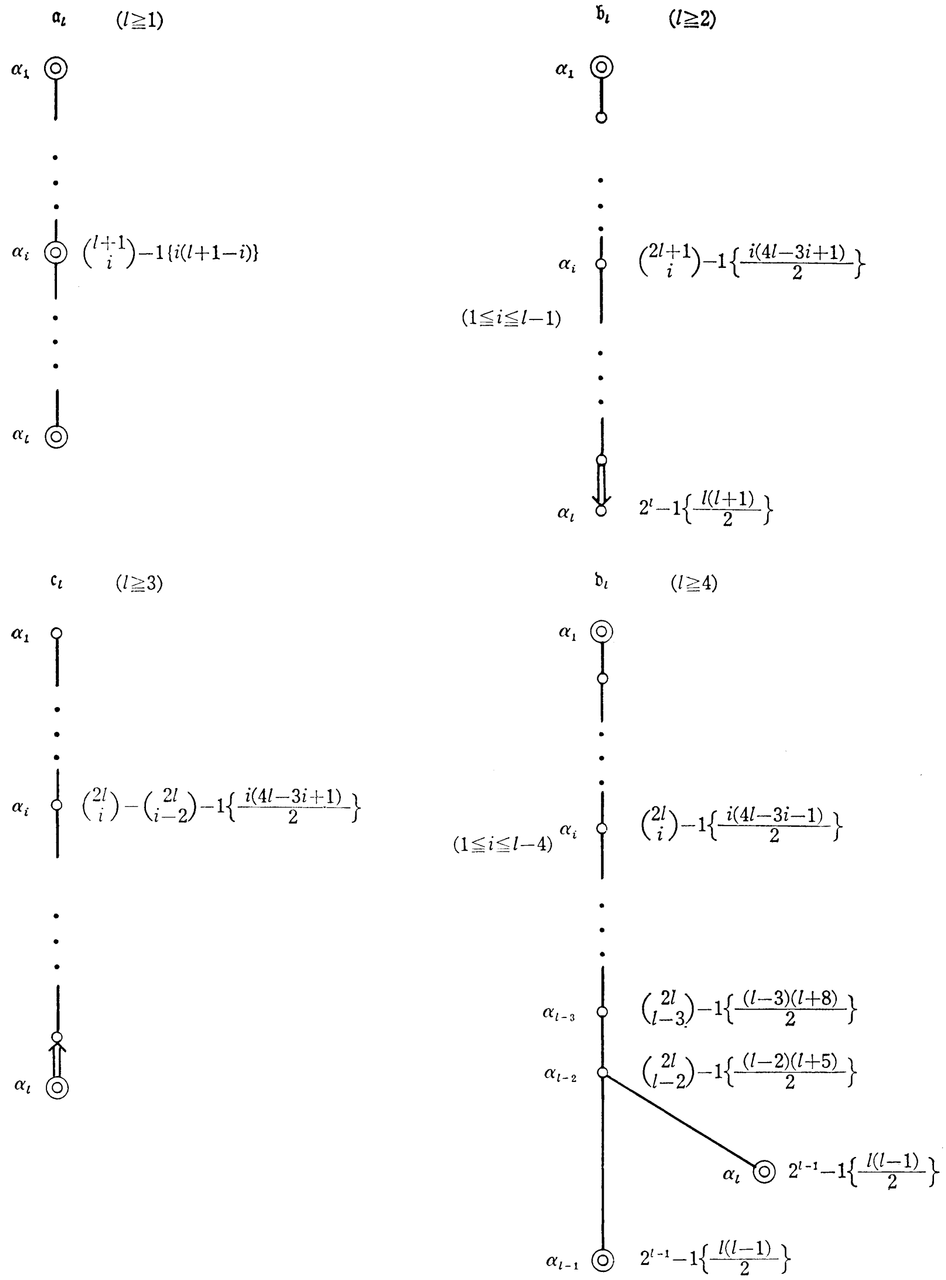
$$n=\frac{1}{2}(\dim_C \mathfrak{g}-\dim_C \mathfrak{g}_1-\cdots-\dim_C \mathfrak{g}_{w(i)}-1).$$

Next, we shall be concerned with the dimension $N(p)$ of the ambient space $P_{N(p)}$. Every imbedding $\rho_i{}^p$ of M_i into $P_{N(p)}$ is full, since $\hat\rho_i{}^p$ is irreducible. The dimension $N(p)$ is given by Weyl's formula

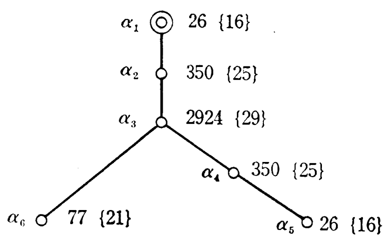
$$N(p)+1=\prod_{\alpha\in\Delta^+}(\alpha, \delta+p\Lambda_i)/\prod_{\alpha\in\Delta^+}(\alpha, \delta), \tag{4.5}$$

where $\delta=(\sum_{\alpha\in\Delta^+}\alpha)/2$. Therefore, for a fixed $(\mathfrak{g}, \alpha_i)$, $N(p)$ is a strictly monotone increasing function of p, in particular, the canonical imbedding $\rho_i{}^1$ of M_i into P_N has the smallest codimension among $\{\rho_i{}^p\}$, where $N=N(1)$. Now, E. Cartan [**7**] calculated the dimension N for all $f_i{}^1$ except for $\mathfrak{g}=\mathfrak{e}_8$, and indicated a principle of computation of $(f_i{}^1, \mathfrak{e}_8)$. On the other hand, E. B. Dynkin [**9**, Table 30] computed N for $(f_i{}^1, \mathfrak{e}_8)$ using the formula (4.5). For the sake of completeness we quote their tables and attach the dimension n of M_i to the table. Thus, with respect to the canonical imbedding, we have Table 1 on dimensions n and N. In this table the notation α_i ◎ means that the C-space M_i corresponding to α_i is Hermitian symmetric (cf. J. A. Wolf [**29**]), and the notation α_i◎$N\{n\}$ or $\alpha_i\circ N\{n\}$ means that $\dim_C M_i=n$, and $\rho_i{}^1$ is a full imbedding of M_i into P_N.

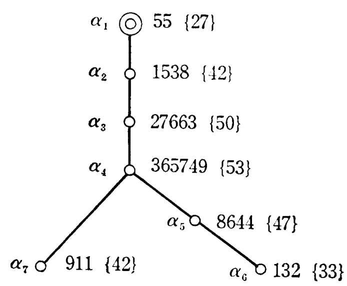
Table 1.


$\mathfrak{a}_l$ $(l\geqq 1)$
α_1
α_i $\binom{l+1}{i}-1\{i(l+1-i)\}$
α_l
$\mathfrak{b}_l$ $(l\geqq 2)$
α_1
α_i $\binom{2l+1}{i}-1\left\{\frac{i(4l-3i+1)}{2}\right\}$
$(1\leqq i\leqq l-1)$
α_l $2^l-1\left\{\frac{l(l+1)}{2}\right\}$
$\mathfrak{c}_l$ $(l\geqq 3)$
α_1
α_i $\binom{2l}{i}-\binom{2l}{i-2}-1\left\{\frac{i(4l-3i+1)}{2}\right\}$
α_l
$\mathfrak{d}_l$ $(l\geqq 4)$
α_1
$(1\leqq i\leqq l-4)$ α_i $\binom{2l}{i}-1\left\{\frac{i(4l-3i-1)}{2}\right\}$
α_{l-3} $\binom{2l}{l-3}-1\left\{\frac{(l-3)(l+8)}{2}\right\}$
α_{l-2} $\binom{2l}{l-2}-1\left\{\frac{(l-2)(l+5)}{2}\right\}$
α_l $2^{l-1}-1\left\{\frac{l(l-1)}{2}\right\}$
α_{l-1} $2^{l-1}-1\left\{\frac{l(l-1)}{2}\right\}$

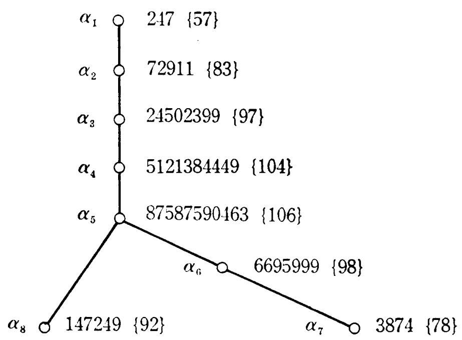
$\mathfrak{e}_6$



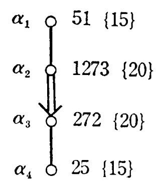
$\mathfrak{e}_7$



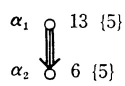
$\mathfrak{e}_8$



$\mathfrak{f}_4$



$\mathfrak{g}_2$



REMARK 4.2. We give another example of Einstein Kaehler submanifold of P_N. Define a mapping f of $P_{n_1}\times\cdots\times P_{n_r}$ into P_N by

$$
\begin{aligned}
&(z_0^1, \cdots, z_{n_1}^1, \cdots, z_0^r, \cdots, z_{n_r}^r) \\
&\qquad\longrightarrow (z_0^1\cdots z_0^r, \cdots, z_{i_1}^1\cdots z_{i_r}^r, \cdots, z_{n_1}^1\cdots z_{n_r}^r) \\
&\qquad\qquad i_\alpha=0, 1, \cdots, n_\alpha, \qquad \alpha=1, \cdots, r,
\end{aligned}
$$

where $N=(n_1+1)\cdots(n_r+1)-1$ and $(z_0^\alpha, \cdots, z_{n_\alpha}^\alpha)$ are homogeneous coordinates of P_{n_α}. It is easy to see that f induces a Kaehler imbedding of a Kaehler manifold $P_{n_1}(c_1)\times\cdots\times P_{n_r}(c_r)$ into $P_N(c)$ if and only if $c_1=\cdots=c_r=c$, and that $P_{n_1}(c)\times\cdots\times P_{n_r}(c)$ is Einstein if and only if $n_1=\cdots=n_r$. Thus we obtain an Einstein Kaehler submanifold $(\underbrace{P_n(c)\times\cdots\times P_n(c)}_{r\text{-times}}, f)$ of $P_N(c)$, where $N=(n+1)^r-1$.

It is obvious that f is a full Kaehler imbedding.

We have so far constructed the irreducible C-spaces with $\dim H^2(M; \boldsymbol{R})=1$ and their imbeddings into the complex projective space P_N. In the same way, that is, by making use of the representation theory of semi-simple Lie groups, we can construct other algebraic C-spaces and their imbeddings into P_N (cf. [**4**]). The following theorem, essentially due to E. Calabi [**5**], however, asserts that all imbeddings of every algebraic C-space into P_N are obtained in this way.

THEOREM 4.3. *Let $M=G/L=G_0/G_0 \cap L$ be a connected C-space, where G is a simply connected complex semi-simple Lie group and L is a complex subgroup of G and G_0 is a maximal compact subgroup of G. Let g be a G_0-invariant Kaehler metric on M. If (M, g) admits a full Kaehler imbedding κ into $P_N(c)$, then κ is equivariant, that is, there exists a complex homomorphism ρ of G into $GL(N+1, \boldsymbol{C})$ which induces the holomorphic imbedding κ in a canonical way.*

PROOF. For every element ϕ of G_0, we have another Kaehler imbedding $\kappa\circ\phi$ of M into $P_N(c)$. Then the rigidity theorem of E. Calabi [**5**] says that there exists an element $\tilde{\phi}$ in $PU(N+1)$ such that $\tilde{\phi}\circ\kappa=\kappa\circ\phi$. We shall show that $\tilde{\phi}$ is uniquely determined.*)

For this, it suffices to show that if an element $\tilde{\phi}$ in $PU(N+1)$ satisfies $\tilde{\phi}(\kappa(m))=\kappa(m)$ for all $m\in M$, then $\tilde{\phi}$ is the identity. Let $\tilde{\phi}$ be induced from $\Phi\in SU(N+1)$. Since Φ is conjugate to a diagonal matrix in $SU(N+1)$, we may assume

$$\Phi=\begin{pmatrix} \alpha_1 & & & & & & & & 0 \\ & \ddots & & & & & & & \\ & & \alpha_1 & & & & & & \\ & & & \alpha_2 & & & & & \\ & & & & \ddots & & & & \\ & & & & & \alpha_2 & & & \\ & & & & & & \ddots & & \\ 0 & & & & & & & & \alpha_r \end{pmatrix}, \qquad \alpha_i\neq\alpha_j \ (i\neq j).$$

Then the set of fixed points of $\tilde{\phi}$ can be expressed as the disjoint union $S_1\cup S_2\cup\cdots\cup S_r$, where S_i $(i=1,\cdots,r)$ denotes the linear subvariety of $P_N(c)$ corresponding to the eigenvalue α_i. Since $\tilde{\phi}$ is identical on $\kappa(M)$ and M is connected, M is contained in some S_i. Moreover, since κ is full, we see $r=1$, which implies that $\tilde{\phi}$ is the identity.

Now, we have obtained a homomorphism $\phi\mapsto\tilde{\phi}$ of G_0 into $PU(N+1)$, which is denoted by $\hat{\rho}$. Then we have on M

$$\hat{\rho}(\phi)\circ\kappa=\kappa\circ\phi \qquad \text{for all} \quad \phi\in G.$$

Since G_0 is also simply connected, we may assume that a homomorphism

*) The proof of uniqueness of $\tilde{\phi}$ is due to Professor H. Ozeki.

$\hat{\rho}$ of G_0 into $SU(N+1)$ induces $\hat{\rho}$*). Let $\mathfrak{g}$ and $\mathfrak{g}_0$ be the Lie algebras of G and G_0, respectively. Since $\mathfrak{g}_0$ is a compact form of $\mathfrak{g}$, we can uniquely extend $\hat{\rho}$ to a holomorphic representation: $G \to SL(N+1, \boldsymbol{C})$, which is denoted by ρ. We put

$$F = \{\phi \in G\,;\ \rho(\phi)\circ\kappa = \kappa\circ\phi\}\,.$$

Clearly F is a closed subgroup of G. Let $\mathfrak{f}$ be the Lie algebra of F. For $X \in \mathfrak{g}$, we denote by X^* (resp. $d\rho(X)^*$) the vector field on M (resp. $P_N(c)$) induced by the 1-parameter transformation group $\exp tX$ (resp. $\rho(\exp tX)$) ($t \in \boldsymbol{R}$). Then we have

$$\begin{aligned} X \in \mathfrak{f} &\Leftrightarrow (\exp t d\rho(X))(\kappa(m)) = \rho(\exp tX)(\kappa(m)) \\ &\qquad = \kappa((\exp tX)(m)) \qquad \text{for all } m \in M \text{ and } t \in \boldsymbol{R} \\ &\Leftrightarrow d\rho(X)^*_{\kappa(m)} = d\kappa_m(X^*_m) \qquad \text{for all } m \in M\,. \end{aligned}$$

Taking account of this relation and the fact that the action of G on M, the representation ρ and the imbedding κ are all holomorphic, we see that $\mathfrak{f}$ is a complex Lie subalgebra of $\mathfrak{g}$. Thus $\mathfrak{f} = \mathfrak{g}$ since $\mathfrak{f} \supset \mathfrak{g}_0$. Hence $F = G$, in other words,

$$\rho(\phi)\circ\kappa = \kappa\circ\phi \qquad \text{for all } \phi \in G\,.$$

We put $V = \kappa(G_0 \cap L)$. Then we have

$$L = \{\phi \in G\,;\ \rho(\phi)V = V\}\,.$$

Moreover κ coincides with the holomorphic imbedding induced canonically from the mapping: $\phi \to \rho(\phi)V$. Q. E. D.

The corresponding local theorem is due to E. Calabi [**5**], which can be stated as

THEOREM 4.4. *Let M be a simply connected Kaehler manifold with analytic metric. If an open Kaehler submamifold U in M admits a full Kaehler imbedding κ_0 into P_N, then κ_0 can be extended to a full Kaehler imbedding κ of M into P_N.*

We can say that these theorems classify all C-spaces imbedded into P_N even locally.

REMARK 4.3. Let $\mathfrak{g}$ be a simple Lie algebra of type A_n and consider the case where $i=1$ or $i=n$. Then $M_i = P_n$ and so we have a full Kaehler imbedding $\rho_i{}^p$ of P_n into $P_{N(p)}(\tilde{c})$. Applying (4.5) to $f_i{}^p$, we find $N(p) = \binom{n+p}{p} - 1$ by a simple calculation. On the other hand, E. Calabi [**5**] gave a full Kaehler immedding ι^p of $P_n(c)$ into $P_{N(p)}(pc)$ by

*) The following proof is due to the referee. The original proof was incomplete.

$$(z_0, \cdots, z_n) \longrightarrow \left(z_0^p, \cdots, \sqrt{\frac{p!}{p_0! \cdots p_n!}}\, z_0^{p_0} \cdots z_n^{p_n}, \cdots, z_n^p\right),$$

where $(z_0, \cdots, z_n)$ are homogeneous coordinates of $P_n(c)$ and $p_0, \cdots, p_n$ range over all non-negative integers with $p_0 + \cdots + p_n = p$. Moreover he proved that if there exists a full Kaehler imbedding of $P_n(c)$ into $P_{N'}(c')$, then $c' = pc$ for some positive integer p and $N' = N(p)$. Now, according to the local rigidity theorem of E. Calabi, we may conclude that $\tilde{c} = pc$ and $\rho_i{}^p$ coincides with ι^p for each $p = 1, 2, \cdots$.

REMARK 4.4. Let $\mathfrak{g}$ be a simple Lie algebra of type G. In the case of $i = 2$, the main theorem of B. Smyth [23] and Table 1 imply that M_i must be Q_5.

§ 5. Scalar curvatures of Hermitian symmetric spaces imbedded in $P_N(c)$.

We keep the notation in § 4. Let $M_i = G_u/H_{u,i}$ be a compact irreducible Hermitian symmetric space, that is, a C-space corresponding to α_i ◎ in Table 1. The purpose of this section is to compute the scalar curvature of the G_u-invariant Kaehler metric $g_i{}^p$ on M_i induced from the Fubini-Study metric g_0 in $P_{N(p)}(c)$ under a p-canonical imbedding $\rho_i{}^p$ of M_i into $P_{N(p)}(c)$. We denote by Ad the adjoint representation of G_u. Then the group $\mathrm{Ad}(H_{u,i})$ acts on the tangent space $\mathfrak{m}$ of M_i at the origin o irreducibly, and it leaves $K|_{\mathfrak{m}\times\mathfrak{m}}$ invariant as well as $g_i{}^p(o)$. From Schur's lemma it follows $K = kg_i{}^p$ on $\mathfrak{m}\times\mathfrak{m}$ for a constant k. Then k is given by

LEMMA 5.1.

$$k = -c/p(\alpha_i, \alpha_i).$$

To give a proof we need some preparations. In this section, we denote an isomorphism $f_i{}^p$ of $\mathfrak{g}$ into $\mathfrak{sl}(N(p)+1)$ simply by f. Note that $f(\mathfrak{g}_u)$ is a subalgebra of $\mathfrak{su}(N(p)+1)$. We define a subalgebra $\mathfrak{k}$ $(=\mathfrak{sl}(\mathfrak{u}(1)\times\mathfrak{u}(N(p)))$ and a subspace $\mathfrak{p}$ of $\mathfrak{su}(N(p)+1)$ by

$$\mathfrak{k} = \left\{ \left(\begin{array}{c|c} \sqrt{-1}\,a & 0 \\ \hline 0 & X \end{array}\right);\ a \in \boldsymbol{R},\ X \in \mathfrak{u}(N(p)),\ \sqrt{-1}\,a + \mathrm{Tr}\,X = 0 \right\}$$

$$\mathfrak{p} = \left\{ [x] = \left(\begin{array}{c|c} 0 & x \\ \hline -{}^t\bar{x} & 0 \end{array}\right);\ x \in \boldsymbol{C}^{N(p)} \right\}.$$

Then we have a direct sum decomposition

$$\mathfrak{su}(N(p)+1)=\mathfrak{k}+\mathfrak{p}$$

and we may identify $\mathfrak{p}$ with the tangent space of $P_{N(p)}$ at $\rho_i^p(o)$. For an element X of $\mathfrak{su}(N(p)+1)$ we denote by $X_{\mathfrak{p}}$ the $\mathfrak{p}$-component of X relative to this decomposition. Then g_0 and g_i^p are by definition expressed as

$$g_0(X, X)=\frac{4}{c}|x|^2 \qquad \text{for} \quad X=[x]\in\mathfrak{p}, \tag{5.1}$$

$$g_i^p(X, X)=g_0(f(X)_{\mathfrak{p}}, f(X)_{\mathfrak{p}}) \qquad \text{for} \quad X\in\mathfrak{m}, \tag{5.2}$$

where $|\ |$ denotes the norm with respect to the canonical inner product $\langle\ ,\ \rangle$ on $\boldsymbol{C}^{N(p)}$. Then from (5.1) we have

$$g_0(f(X)_{\mathfrak{p}}, f(X)_{\mathfrak{p}})=\frac{4}{c}|f(X)_{\mathfrak{p}}e_0|^2 \qquad \text{for} \quad X\in\mathfrak{m}, \tag{5.3}$$

because of $e_0=(1, 0, \cdots, 0)\in\boldsymbol{C}^{N(p)+1}$.

PROOF OF LEMMA 5.1. We put $F_\alpha=E_\alpha+E_{-\alpha}$ and $G_\alpha=\sqrt{-1}(E_\alpha-E_{-\alpha})$ for $\alpha\in\Delta$. Since elements $f(F_\alpha)$ and $f(G_\alpha)$ of $\mathfrak{su}(N(p)+1)$ are both skew Hermitian, we have easily

$$\langle f(E_\alpha)x, y\rangle+\langle x, f(E_{-\alpha})y\rangle=0 \qquad \text{for} \quad x, y\in\boldsymbol{C}^{N(p)+1}. \tag{5.4}$$

Since e_0 is a highest weight vector, we see $f(E_\alpha)e_0=0$ for $\alpha\in\Delta^+$. This and (5.4) imply

$$\langle f(F_\alpha)e_0, e_0\rangle=\langle f(E_{-\alpha})e_0, e_0\rangle=\langle e_0, f(E_\alpha)e_0\rangle=0$$

for $\alpha\in\Delta$. It follows that

$$|f(F_\alpha)e_0|=|f(F_\alpha)_{\mathfrak{p}}e_0| \qquad \text{for} \quad \alpha\in\Delta. \tag{5.5}$$

Similarly we obtain, for $\alpha\in\Delta^+$,

$$\begin{aligned} f(F_\alpha)^2e_0 &= f(E_\alpha)f(E_{-\alpha})e_0+f(E_{-\alpha})^2e_0 \\ &= -(\Lambda, \alpha)e_0+f(E_{-\alpha})^2e_0, \end{aligned} \tag{5.6}$$

because $[E_\alpha, E_{-\alpha}]=-H_\alpha$ and therefore $f(E_\alpha)f(E_{-\alpha})e_0=-f(H_\alpha)e_0$. However, by (5.4), we see

$$\langle f(E_{-\alpha})^2e_0, e_0\rangle=-\langle f(E_{-\alpha})e_0, f(E_\alpha)e_0\rangle=0 \qquad \text{for} \quad \alpha\in\Delta.$$

Thus (5.6) gives

$$\langle f(F_\alpha)^2e_0, e_0\rangle=-(\Lambda, \alpha) \qquad \text{for} \quad \alpha\in\Delta^+. \tag{5.7}$$

On the other hand, by virtue of (5.3), (5.4) and (5.5), we have

$$\langle f(F_\alpha)^2e_0, e_0\rangle=-\frac{c}{4}g_0(f(F_\alpha)_{\mathfrak{p}}, f(F_\alpha)_{\mathfrak{p}}) \qquad \text{for} \quad -\alpha\in\Delta_i.$$

Then, by the definition of the constant k and E_α, we have

$$\langle f(F_\alpha)^2 e_0, e_0\rangle = -\frac{c}{4k} K(F_\alpha, F_\alpha) = \frac{c}{2k} \qquad \text{for} \quad -\alpha \in \Delta_i .$$

Combining this with (5.7), we find

$$k = -\frac{c}{2(\Lambda, \alpha)} \qquad \text{for} \quad -\alpha \in \Delta_i .$$

Here we take α_i as α. Then we have

$$2(\Lambda, \alpha) = 2(\Lambda, \alpha_i) = p(\alpha_i, \alpha_i) .$$

Q. E. D.

We denote by $\nu(\mathfrak{g}, i)$ the number of roots $\alpha \in \Delta_i$ such that $\alpha \neq \alpha_i$ and $\alpha + \alpha_i \in \Delta$. The following proposition is due to A. Borel [2].

PROPOSITION 5.2. *The scalar curvature of G_u-invariant Kaehler metric on M_i given by $-K|_{\mathfrak{m}\times\mathfrak{m}}$ at o is equal to $n(\nu(\mathfrak{g}, i)+2)(\alpha_i, \alpha_i)$ everywhere. Moreover, the maximal eigenvalue μ_2 of Q is equal to (α_i, α_i) everywhere.*

Combining Lemma 5.1 with Proposition 5.2, we find

LEMMA 5.3. *The scalar curvature of the G_u-invariant Kaehler metric $g_i{}^p$ on M_i is equal to $n(\nu(\mathfrak{g}, i)+2)c/p$ everywhere. Moreover, the maximal eigenvalue μ_2 of Q is equal to c/p everywhere.*

REMARK. We denote by q_{ad} the number of roots $\alpha \in -\Delta_i$ such that $(\alpha, \alpha_i) > 0$. Then it is easy to see $\nu(\mathfrak{g}, i) = q_{\mathrm{ad}} - 1$. S. Murakami [**17**, p. 113] shows that $q_{\mathrm{ad}} = \dfrac{1}{(\alpha_i, \alpha_i)} - 1$. The scalar curvature in Proposition 5.2 is therefore equal to n, which can also be proved directly (cf. [**17**, p. 94]).

THEOREM 5.4. *Let U be a connected open set of an n-dimensional irreducible Hermitian symmetric space M_i. Let ι be a full Kaehler imbedding of U into $P_N(c)$, and R be the scalar curvature of U. Then $n(\nu(\mathfrak{g}, i)+2)\, c/R$ is a positive integer, say p, and ι is the restriction of the p-canonical imbedding $\rho_i{}^p$ of M_i into $P_{N(p)}(c)$ (so $N=N(p)$) to U, that is, there exists an isometry σ of $P_N(c)$ such that $\sigma \circ \iota = \rho_i{}^p | U$.*

PROOF. We express M_i as G/L_i using the notation in § 4. By Theorem 4.4, ι can be extended to a full Kaehler imbedding κ of M_i into $P_N(c)$. By Theorem 4.3, we have a holomorphic representation ρ of G over $\boldsymbol{C}^{N+1}$ which induces canonically κ. The highest weight of ρ must be of the form $p\Lambda_i$ for a positive integer p. Then κ is the p-canonical imbedding, and R is equal to $n(\nu(\mathfrak{g}, i)+2)c/p$ by Lemma 5.3. Q. E. D.

From this theorem, we have another interpretation of the p-canonical imbedding $\rho_i{}^p$ of M_i into $P_{N(p)}(c)$ as follows. Let ι^p denote the p-canonical imbedding of $P_n(c/p)$ into $P_N(c)$, where $N = \binom{n+p}{p} - 1$ (cf. Remark 4.3). Then

ρ_i^p is nothing but the composition $\iota^p \circ \rho_i^1$.

§6. Symmetric Kaehler submanifolds in $P_N(c)$.

In this section, we investigate the second fundamental form of the p-canonical Kaehler imbedding ρ^p of a compact irreducible Hermitian symmetric space M into $P_N(c)$, which is closely related to the scalar curvature R of M and the eigenvalues μ_1 and μ_2 of the operator Q associated with M. Under this situation, we can make use of many equalities obtained in the preceding sections. First, we recall that $\mu_2=c/p$ by Lemma 5.3. Next, we compute the constant A_m for $m\geqq 2$. If $m=2$, then, since g is Einstein, we get by (2.9)

$$A_2=n\lambda=\frac{n(n+1)c-R}{2} \tag{6.1}$$

with the Ricci curvature λ. We define $f_{ma}(p)$ by

$$f_{ma}(p)=n(n+m)c-mR-nm(m-1)\mu_a$$

for $a=1, 2$. Then we find

LEMMA 6.1.

$$A_{m+1}=f_{m1}(1)A_m/2n \qquad \text{for} \quad m\geqq 2,$$

and

$$A_3=\{f_{21}(p)B_2+f_{22}(p)C_2\}/2n\,,$$

where B_2 and C_2 are both non-negative functions. Furthermore, in type A III$_1$,

$$A_{m+1}=f_{m2}(p)A_m/2n\,.$$

PROOF. From (3.3) we have

$$\begin{aligned}A_{m+1}&=\Big(\sum_{\alpha,i_1,\cdots,i_{m+1}} h^\alpha_{i_1\cdots i_{m+1}}\bar{h}^\alpha_{i_1\cdots i_m}\Big)_{\bar{i}_{m+1}}\\&\quad-\sum_{\alpha,i_1,\cdots,i_{m+1}} h^\alpha_{i_1\cdots i_{m+1}\bar{i}_{m+1}}\bar{h}^\alpha_{i_1\cdots i_m}\\&=-\sum_{\alpha,i_1,\cdots,i_{m+1}} h^\alpha_{i_1\cdots i_{m+1}\bar{i}_{m+1}}\bar{h}^\alpha_{i_1\cdots i_m}.\end{aligned}$$

Lemma 2.1 and (3.2), however, imply

$$\begin{aligned}&\sum_{i_{m+1}} h^\alpha_{i_1\cdots i_{m+1}\bar{i}_{m+1}}\\&=\frac{m-1}{2}c\sum_{i_{m+1}}\sum_{r=1}^{m+1} h^\alpha_{i_1\cdots\hat{i}_r\cdots i_{m+1}}\delta_{i_r i_{m+1}}\\&\quad-\sum_{r=1}^{m-1}\frac{1}{r!(m+1-r)!}\sum_{\sigma,\beta,l,i_{m+1}} h^\alpha_{l i_{\sigma(1)}\cdots i_{\sigma(r)}} h^\beta_{i_{\sigma(r+1)}\cdots i_{\sigma(m+1)}}\bar{h}^\beta_{l i_{m+1}}\end{aligned}$$

$$
\begin{aligned}
&= \frac{(m-1)(n+m)}{2} c h^{\alpha}_{i_1 \cdots i_m} \\
&\quad - \frac{1}{2!(m-1)!} \sum_{\sigma,\beta,l,i_{m+1}} h^{\alpha}_{li\sigma(1)\cdots i\sigma(m-1)} h^{\beta}_{i\sigma(m) i\sigma(m+1)} \bar{h}^{\beta}_{l i_{m+1}} \\
&= \frac{(m-1)(n+m)}{2} c h^{\alpha}_{i_1 \cdots i_m} \\
&\quad - \frac{1}{(m-1)!} \sum_{\sigma,\beta,l,i_{m+1}} h^{\alpha}_{li\sigma(1)\cdots i\sigma(m-1)} h^{\beta}_{i\sigma(m) i_{m+1}} \bar{h}^{\beta}_{l i_{m+1}} \\
&\quad - \frac{1}{2!(m-2)!} \sum_{\sigma,\beta,l,i_{m+1}} h^{\alpha}_{l i_{m+1} i\sigma(1)\cdots i\sigma(m-2)} h^{\beta}_{i\sigma(m-1) i\sigma(m)} \bar{h}^{\beta}_{l i_{m+1}},
\end{aligned}
$$

where the last two summations on σ are taken over all permutations of $(1, \cdots, m)$.

Suppose that $p=1$. Since g is Einstein, by (2.8), (2.9), (3.3), (6.1) and Lemma 2.4, the right hand side of the above equation is equal to

$$
-\left\{-\frac{(m-1)(n+m)}{2} c + m \frac{n(n+1)c-R}{2n} + \frac{m(m-1)}{2}(c-\mu_1)\right\} h^{\alpha}_{i_1 \cdots i_m}.
$$

This completes the proof in the case $p=1$.

Suppose that $p \geqq 2$. We consider here the Hermitian matrix A. Then $\mu_1 = c/p \neq c$, and by Lemma 2.3, $c-\mu_1$ is an eigenvalue of A and so also is $c-\mu_2$. Consequently, in view of Lemma 2.3, distinct eigenvalues of A are $c-\mu_a$ $(a=1, 2)$ and possibly 0. We diagonalize the matrix A by a suitable unitary matrix. By a suitable choice of $e_{n+1}, \cdots, e_N$, A may be represented in the form

$$
A = \begin{pmatrix}
c-\mu_1 & & & & & & & & \mathbf{0} \\
& \ddots & & & & & & & \\
& & c-\mu_1 & & & & & & \\
& & & c-\mu_2 & & & & & \\
& & & & \ddots & & & & \\
& & & & & c-\mu_2 & & & \\
& & & & & & 0 & & \\
& & & & & & & \ddots & \\
\mathbf{0} & & & & & & & & 0
\end{pmatrix}.
$$

Accordingly, we have

$$
\sum_{\beta} A^{\gamma_a}_{\beta} H^{\beta} = (c-\mu_a) H^{\gamma_a}
$$

for $n+1 \leqq \gamma_1 \leqq n+m_1$, $n+m_1+1 \leqq \gamma_2 \leqq n+m_1+m_2$, where m_a $(a=1, 2)$ is the multiplicity of μ_a. This implies that the constant A_3 satisfies the given property, where

$$
B_2 = \sum_{\gamma_1, i, j} h^{\gamma_1}_{ij} \bar{h}^{\gamma_1}_{ij}, \qquad C_2 = \sum_{\gamma_2, i, j} h^{\gamma_2}_{ij} \bar{h}^{\gamma_2}_{ij}.
$$

For type A III_1, the assertion is trivial. Q. E. D.

Since we know the values μ_1 and μ_2 and the scalar curvature R, we can calculate all $f_{ma}(p)$ on each M and so all A_m on M for the canonical imbedding (cf. Table 2). As a result of the computation, we find the following remarkable theorem.

THEOREM 6.2. *Let M be an n-dimensional compact irreducible Hermitian symmetric space with the Kaehler metric induced under the canonical imbedding ρ into $P_N(c)$. Then the degree $d(M, \rho)$ of the imbedding coincides with the rank of M as a symmetric space.*

So far we computed some geometrical quantities on a Kaehler submanifold immersed in a complex projective space. Here we shall sum up them as Table 2 in the next page in the case where $M=M_i$ is an n-dimensional compact irreducible Hermitian symmetric space with the Kaehler metric induced under the p-canonical imbedding into $P_N(c)$. In this table, the value μ_1 and the multiplicities of μ_1 and μ_2 for A $\mathrm{III}_2\sim D$ III are quoted from Table 2 in [**6**], and those of type E III, E VII and $\nu(\mathfrak{e}_6, 1)$, $\nu(\mathfrak{e}_7, 1)$ from [**2**].

Pick out spaces with $m_0=2$ from Table 2 for the canonical imbedding ρ, where m_0 is the degree of ρ, that is, the positive integer such that $A_{m_0}\neq 0$ and $A_m=0$ for $m>m_0$. Then the following compact irreducible Hermitian symmetric spaces admit Kaehler imbeddings into a complex projective space with parallel second fundamental form:

$$P_n\ (=SU(n+1)/S(U(n)\times U(1)))\,,$$

$$Q_n\ (=SO(n+2)/SO(n)\times SO(2))\,,$$

$$SU(r+2)/S(U(r)\times U(2))\,,\qquad r\geqq 3\,,$$

$$SO(10)/U(5)\,,$$

$$E_6/\mathrm{Spin}\,(10)\times T\,.$$

REMARK 6.1. Both of spaces $M_1=SU(5)/S(U(3)\times U(2))$ and $M_2=SO(10)/U(5)$ satisfy the condition $q=N-n=n/2$, which shows that the estimate of the codimension in the first assertion on the main theorem of the first named author [**18**] is best possible.

Now, we give another example of Einstein Kaehler submanifold immersed in $P_N(c)$ with parallel second fundamental form. Consider a Kaehler imbedding ρ_i^p of $P_n(c/p)$ into $P_{N(p)}(c)$, where $N(p)=\binom{n+p}{p}-1$ and $p=2, 3, \cdots$ (cf. Remark 4.3). For the p-canonical imbedding of type A III_1 in Table 2, one gets

$$A_{m+1}=\frac{(n+m)(p-m)}{2p}cA_m \qquad \text{for}\quad m\geqq 2\,, \tag{6.2}$$

where $A_2=(p-1)n(n+1)c/2$. This was already obtained in the first named

Table 2.

Type	M_i	$\mathfrak{g}$	i	$\nu(\mathfrak{g}, i)$	n
A III$_1$	$SU(n+1)/S(U(n)\times U(1))$ $(n\geqq 1)$	$\mathfrak{a}_n$	1 or n	$n-1$	n
A III$_2$	$SU(r+s)/S(U(r)\times U(s))$ $(r\geqq s\geqq 2)$	$\mathfrak{a}_{r+s-1}$	r or s	$r+s-2$	rs
BD I	$SO(n+2)/SO(n)\times SO(2)$ $(n\geqq 3)$	$\mathfrak{b}_{(n+1)/2}$ or $\mathfrak{d}_{(n+2)/2}$	1	$n-2$	n
C I	$Sp(r)/U(r)$ $(r\geqq 3)$	$\mathfrak{c}_r$	r	$r-1$	$\frac{r(r+1)}{2}$
D III	$SO(2r)/U(r)$ $(r\geqq 5)$	$\mathfrak{d}_r$	$r-1$ or r	$2r-4$	$\frac{r(r-1)}{2}$
E III	E_6/Spin $(10)\times T$	$\mathfrak{e}_6$	1	10	16
E VII	$E_7/E_6\times T$	$\mathfrak{e}_7$	1	16	27

Type	$N(p)+1$	N	μ_1	μ_2	m_1
A III$_1$	$\binom{n+p}{p}$	n		c/p	0
A III$_2$	$\prod_{i=1}^{r}\prod_{j=r+1}^{r+s}\frac{p+j-i}{j-i}$	$\binom{r+s}{r}-1$	$-c/p$	c/p	$\binom{r}{2}\binom{s}{2}$
BD I	$\binom{n+p}{p}+\binom{n+p-1}{p-1}$	$n+1$	$(2-n)c/2p$	c/p	1
C I	$\binom{n+p}{p}\prod_{s=1}^{r-1}\binom{2p+r+s}{2p}\Big/\binom{2p+2s}{2p}$	$\binom{2r}{r}-\binom{2r}{r-2}-1$	$-c/2p$	c/p	$\frac{r^2(r^2-1)}{12}$
D III	$\binom{n+p-1}{p}\prod_{s=1}^{r-2}\binom{p+2r-2s-1}{p}\Big/\binom{p+r+s}{p}$	$2^{r-1}-1$	$-2c/p$	c/p	$\binom{r}{4}$
E III	$\binom{p+11}{p}\binom{p+8}{p}\Big/\binom{p+3}{p}$	26	$-3c/p$	c/p	10
E VII	$\frac{p+9}{9}\binom{p+17}{p}\binom{p+13}{p}\Big/\binom{p+4}{p}$	55	$-4c/p$	c/p	27

Type	m_2	R	$f_{m_1}(1)$	m_0	$f_{m_2}(p)$
A III$_1$	$\frac{n(n+1)}{2}$	$n(n+1)c/p$			$n(n+m)(p-m)c/p$
A III$_2$	$\binom{r+1}{2}\binom{s+1}{2}$	$rs(r+s)c/p$	$rs(m-r)(m-s)c$	s	$rs\{(n+m)p-m(r+s+m-1)\}c/p$
BD I	$\frac{(n-1)(n+2)}{2}$	n^2c/p	$\frac{1}{2}n(n-2)(m-2)\left(m-\frac{2}{n-2}\right)c$	2	$n\{(n+m)p-m(n+m-1)\}c/p$
C I	$\binom{r+3}{4}$	$r(r+1)^2c/2p$	$r(r+1)(m-r)(m-r-1)c/4$	r	$n\{(n+m)p-m(r+m)\}c/p$
D III	$\frac{r^2(r^2-1)}{12}$	$r(r-1)^2c/p$	$r(r-1)(2m-r)(2m-r+1)c/16$	$\frac{r}{2}$ or $\frac{r-1}{2}$	$n\{(n+m)p-m(2r+m-3)\}c/p$
E III	126	$192c/p$	$16(m-2)(3m-8)c$	2	$16\{(16+m)p-m(m+11)\}c/p$
E VII	351	$486c/p$	$27(m-3)(4m-9)c$	3	$27\{(27+m)p-m(m+17)\}c/p$

author and K. Ogiue [**19**]. It follows that the second fundamental form of $\rho_i{}^p(P_n(c/p))$ is parallel only when $p=2$.

REMARK 6.2. We know from (6.2) that the degree of the p-canonical imbedding of type A III_1 in Table 2 is equal to p. We could also prove that the degree of the p-canonical imbedding of type A III_2 (resp. BD I) is equal to sp (resp. $2p$). From these facts, together with Theorem 6.2, we conjectured that if M is an n-dimensional compact irreducible Hermitian symmetric space of rank r, then the degree of its p-canonical imbedding is equal to rp. Recently the second named author and M. Takeuchi have solved this conjecture affirmatively, whose details will be published in the forthcoming paper.

REMARK 6.3. After a simple calculation, it is easily seen that if $p\geqq 2$, then $f_{22}(p)$ is positive except for type A III_1 and $p=2$. This yields that A_3 is a positive constant, because $f_{22}(p)<f_{21}(p)$ and M is not totally geodesic. Thus the second fundamental form of the p-canonical imbedding of M_i is not parallel, if M_i is not a complex projective space. Accordingly, we find that there exist only six kinds of compact irreducible Hermitian symmetric spaces under the p-canonical imbedding into $P_{N(p)}(c)$ with respect to which the second fundamental form are parallel, which are mentioned above.

REMARK 6.4. K. Ogiue [**22**] gave the following problem "Let $M_r(c)$ be an r-dimensional complex space form of constant holomorphic sectional curvature c. Let M be an n-dimensional Kaehler submanifold immersed in $M_{n+q}(c)$, $c>0$. If M is irreducible (or Einstein) and the second fundamental form is parallel, is M one of the following spaces? $M_n(c)$, $M_n(c/2)$ or locally Q_n." Examples stated above give a negative answer to this problem.

§7. Kaehler submanifolds with parallel second fundamental form.

In this section, we determine all Kaehler submanifolds M immersed into $P_{n+q}(c)$ with parallel second fundamental form, which are locally symmetric by (2.6). On such a manifold, $h^\alpha_{ijk}=0$ and applying (2.14), we get

$$(7.1)\qquad \left\{\begin{aligned} &\frac{c}{2}(h^\alpha_{jk}\delta_{il}+h^\alpha_{ik}\delta_{jl}+h^\alpha_{ij}\delta_{kl})\\ &-\sum_{\beta,m}(h^\alpha_{mi}h^\beta_{jk}+h^\alpha_{mj}h^\beta_{ik}+h^\alpha_{mk}h^\beta_{ij})\bar{h}^\beta_{ml}=0\,. \end{aligned}\right.$$

LEMMA 7.1. *Let M^i be an n_i-dimensional Kaehler manifolds ($i=1, 2$). If a Kaehler manifold $M^1\times M^2$ admits a Kaehler immersion into $P_{n_1+n_2+q}(c)$ with parallel second fundamental form, then M^i is locally $P_{n_i}(c)$ ($i=1, 2$).*

PROOF. Let the indices used here be as in Lemma 2.2. Put $i=a$, $j=b$ and $k=l=r$ in (7.1). Then we have

$$\frac{c}{2}h^{\alpha}_{ab}-\sum_{\beta,m}(h^{\alpha}_{ma}h^{\beta}_{br}+h^{\alpha}_{mb}h^{\beta}_{ar}+h^{\alpha}_{mr}h^{\beta}_{ab})\bar{h}^{\beta}_{mr}=0\,.$$

Applying (2.15) to this relation, we get $h^{\alpha}_{ab}=0$. Similarly, putting $i=r$, $j=s$ and $k=l=a$ in (7.1), we have $h^{\alpha}_{rs}=0$. Hence, making use of these equations and the equation (2.6) of Gauss, we get easily

$$R_{\bar{a}bc\bar{d}}=\frac{c}{2}(\delta_{ab}\delta_{cd}+\delta_{ac}\delta_{bd})\,,$$

$$R_{\bar{r}st\bar{u}}=\frac{c}{2}(\delta_{rs}\delta_{tu}+\delta_{rt}\delta_{su})\,,$$

which imply that M^i is of constant holomorphic sectional curvature c.

LEMMA 7.2. *The Kaehler imbedding f of $P_{n_1}(c)\times P_{n_2}(c)$ into $P_{n_1+n_2+n_1n_2}(c)$ defined in Remark 4.2 is equivariant and has parallel second fundamental form.*

PROOF. By the proof of Lemma 2.2, h^{α}_{ij} for the imbedding f satisfy (2.15). On the other hand, we have $h^{\alpha}_{ab}=h^{\alpha}_{rs}=0$, because of (2.6). Thus it is easily seen that h^{α}_{ij} satisfy (7.1), which means $h^{\alpha}_{ijk\bar{l}}=0$. Since $P_{n_1}(c)\times P_{n_2}(c)$ is locally symmetric, we have $\sum_{\alpha}h^{\alpha}_{ijk}\bar{h}^{\alpha}_{ml}=0$. Hence we obtain

$$0=\sum_{\alpha}h^{\alpha}_{ijk\bar{k}}\bar{h}^{\alpha}_{ij}+\sum_{\alpha}h^{\alpha}_{ijk}\bar{h}^{\alpha}_{ijk}=\sum_{\alpha}h^{\alpha}_{ijk}\bar{h}^{\alpha}_{ijk}\,,$$

which proves the second fundamental form is parallel. The equivariance of f is evident. Q. E. D.

By virtue of the above lemmas, we can classify the given submanifold in the reducible case. Finally we prepare the following

LEMMA 7.3. *Let M be an n-dimensional Kaehler submanifold different from $M_n(c)$ immersed into $P_{n+q}(c)$ with parallel second fundamental form. If M is irreducible as a locally symmetric space, then $\mu_2=c$ and M is of compact type.*

PROOF. Since M is Einstein, we have

$$\sum_{\alpha,k}h^{\alpha}_{ik}\bar{h}^{\alpha}_{kj}=\lambda\delta_{ij}\,,\qquad \lambda=\frac{n(n+1)c-R}{2n}\,.$$

Putting $k=l$ in (7.1) and summing up over $k=1,\cdots,n$, we have

$$\sum_{\beta,k,l}h^{\alpha}_{kl}\bar{h}^{\beta}_{kl}h^{\beta}_{ij}=\left(\frac{n+2}{2}c-2\lambda\right)h^{\alpha}_{ij}\,,$$

from which it follows that

$$A^2=\left(\frac{n+2}{2}c-2\lambda\right)A\,.$$

It follows that the eigenvalues of the $q\times q$ Hermitian matrix $A=(A_{\beta}{}^{\alpha})$ are 0 or $\frac{n+2}{2}c-2\lambda$ $(\geqq 0)$.

On the other hand, since M is different from $M_n(c)$, we already know that the value $c-\mu_1$ is a positive eigenvalue of A. Suppose that $c\neq\mu_2$. Then $c-\mu_2$ is also a positive eigenvalue of A by Lemma 2.3 and moreover it is different from $c-\mu_1$, which contradicts the fact that A has at most one non-zero eigenvalue. Thus we obtain $c-\mu_2=0$ and $c-\mu_1=\frac{n+2}{2}c-2\lambda$, and so

$$R=n\{(n+2)c-2\mu_1\}/2.$$

Suppose that M is of non-compact type. Then, since R is negative, we have $\mu_1>(n+2)c/2>0$, which contradicts the fact $\mu_1\mu_2<0$. Q.E.D.

Combining Lemmas 7.1, 7.2 and 7.3 together with Theorems 4.3 and 4.4, we have the following classification theorem.

THEOREM 7.4. *Let M be a complete Kaehler submanifold imbedded into $P_N(c)$ with parallel second fundamental form. If M is irreducible then M is congruent to one of six kinds of Kaehler submanifolds imbedded into $P_N(c)$ with parallel second fundamental form given in the last paragraph of* § 6. *If M is reducible, then M is congruent to $(P_{n_1}(c)\times P_{n_2}(c), f)$ given in Remark* 4.2 *for some n_1 and n_2 with* $\dim M=n_1+n_2$. *The corresponding local version is also true.*

Thus we can say that if M is a Kaehler submanifold immersed into $P_N(c)$ with parallel second fundamental form and not of constant holomorphic sectional curvature, then M is of rank two as a locally symmetric space.

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