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Periodicity and inequality

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§ 0. Introduction.

In this paper, we consider a certain kind of periodicity by Fubini's theorem. Next in such a point of view, we prove the following inequality.

$$\Gamma(n/2)/2\sqrt{\pi}\,\Gamma((n+1)/2) < \max_{I\subset\{1,2,\cdots,p\}} \|\sum_{i\in I} a_i\| \Big/ \sum_{j=1}^{p} \|a_j\| ,$$

where a_j $(j=1, 2, \cdots, p)$ is a real n-dimensional vector $(n \geqq 2)$.

For example, A. Pietsch [2] used the lemma such that for any set of complex numbers $\{a_j \in \boldsymbol{C};\ j\in J\}$, where $|\sum_{i\in I} a_i| \leqq r$ for all finite subset I of J, we have $\sum_{i\in I} |a_i| \leqq 4r$ for all I.

We can take π in place of 4, and this estimate is the best, moreover this is the generalization of Blaschke's theorem on the oval.

§ 1. Periodicity and Fubini's theorem.

THEOREM 1. *Let $H(X, Y)$ be* (1) *real valued bounded measurable function on $\boldsymbol{R}^n \times \boldsymbol{R}^n$,* (2) $H(X, Y)=H(Y, X)$, (3) *there exist M and*

$$M=\lim_{T_1, T_2, \cdots, T_n\to\infty} (2^n T_1 T_2 \cdots T_n)^{-1} \int_{-T_1}^{T_1} \cdots \int_{-T_n}^{T_n} H(X, Y)dY ,$$

and this convergence is uniform. Let $\boldsymbol{B}$ be $[0, 1]^n$, $n \geqq 1$. Let $C_{\boldsymbol{B}}^1$ be the set of all once continuously differentiable functions on $\boldsymbol{B}$. Then

$$\sup_{f\in C_{\boldsymbol{B}}^1} \inf_{Y\in \boldsymbol{R}^n} \int_{\boldsymbol{B}} H(f(Z), Y)dZ = M = \inf_{f\in C_{\boldsymbol{B}}^1} \sup_{Y\in \boldsymbol{R}^n} \int_{\boldsymbol{B}} H(f(Z), Y)dZ .$$

PROOF. For any $f\in C_{\boldsymbol{B}}^1$, any $T_1, T_2, \cdots, T_n > 0$, $H(f(Z), Y)$ is a bounded measurable function on $\boldsymbol{B}\times[-T_1, T_1]\times[-T_2, T_2]\times\cdots\times[-T_n, T_n]$. By Fubini's theorem, we have

$$(2^n T_1 T_2 \cdots T_n)^{-1} \int_{-T_1}^{T_1} \cdots \int_{-T_n}^{T_n} \int_{\boldsymbol{B}} H(f(Z), Y)dZdY$$

$$= \int_{\boldsymbol{B}} (2^n T_1 T_2 \cdots T_n)^{-1} \int_{-T_1}^{T_1} \cdots \int_{-T_n}^{T_n} H(f(Z), Y)dYdZ .$$

By Lebesgue's theorem, it holds that

$$\lim_{T_1, T_2, \cdots, T_n \to \infty} (2^n T_1 T_2 \cdots T_n)^{-1} \int_{-T_1}^{T_1} \cdots \int_{-T_n}^{T_n} \int_B H(f(Z), Y) dZ dY = M,$$

for all $f \in C_B^1$. Then we have

$$\sup_{f \in C_B^1} \inf_{Y \in R^n} \int_B H(f(Z), Y) dZ \leqq M \leqq \inf_{f \in C_B^1} \sup_{Y \in R^n} \int_B H(f(Z), Y) dZ.$$

Next, by the condition (3), for any $\varepsilon > 0$, there exist $T_1, T_2, \cdots, T_n > 0$, such that

$$M - \varepsilon \leqq (2^n T_1 T_2 \cdots T_n)^{-1} \int_{-T_1}^{T_1} \cdots \int_{-T_n}^{T_n} H(X, Y) dY \leqq M + \varepsilon,$$

for all $X \in \boldsymbol{R}^n$. By the change of variables such that $Z_j = (Y_j - T_j)/2T_j$, $j = 1, 2, \cdots, n$, we obtain

$$M - \varepsilon \leqq \int_0^1 \cdots \int_0^1 H(X, 2TZ + T) dZ \leqq M + \varepsilon,$$

and then

$$M - \varepsilon \leqq \int_B H(2TZ + T, Y) dZ \leqq M + \varepsilon,$$

for all $Y \in \boldsymbol{R}^n$. Hence we have

$$M - \varepsilon \leqq \inf_Y \int_B H(2TZ + T, Y) dZ \leqq \sup_Y \int_B H(2TZ + T, Y) dZ \leqq M + \varepsilon.$$

Therefore

$$\inf_{f \in C_B^1} \sup_Y \int_B H(f(Z), Y) dZ \leqq M \leqq \sup_{f \in C_B^1} \inf_Y \int_B H(f(Z), Y) dZ.$$

This completes the proof.

REMARK. If $H(X, Y)$ is continuous, then we can take the set of all measurable functions on $\boldsymbol{B}$, in place of C_B^1. If $H(X, Y)$ and f are Borel measurable, then we can prove the theorem as above and the following.

DEFINITION. $H(x)$ is said to be almost periodic, if for any $\varepsilon > 0$, there exists a number $d_\varepsilon > 0$, such that for any real number r, there exists $v \in [r, r + d_\varepsilon]$, which satisfies

$$\sup_{x \in R} |H(x + v) - H(x)| \leqq \varepsilon.$$

(See, A. S. Besicovith, [**1**].)

THEOREM 2. *Let $H(x)$ be almost periodic real valued measurable function, then*

$$\sup_{f\in C^1_{[0,1]}}\inf_{y\in R}\int_0^1 H(f(x)+y)dx=\lim_{T\to\infty}(2T)^{-1}\int_{-T}^{T}H(x)dx$$
$$=\inf_{f\in C^1_{[0,1]}}\sup_{y\in R}\int_0^1 H(f(x)+y)dx\,.$$

PROOF. We can apply Theorem 1 to $E(x, y)=H(x+y)$. It is clear that $E(x, y)$ is bounded, and $E(x, y)=E(y, x)$. We must prove that the condition (3) is satisfied. For any $\varepsilon>0$, there exist T_0, N_0, such that $4\|H\|d_\varepsilon/T_0\leqq\varepsilon$, $3\|H\|/N_0\leqq\varepsilon$. Then for any $X_1, X_2\geqq N_0T_0$, we have

$$\left|(2X_1)^{-1}\int_{-X_1}^{X_1}H(x+y)dy-(2X_2)^{-1}\int_{-X_2}^{X_2}H(x+y)dy\right|\leqq 4\varepsilon\,.$$

(See, A. S. Besicovith, [**1**].)

THEOREM 3. *Let $H(X)$ be (1) real valued bounded measurable function on $\boldsymbol{R}^n$, (2) periodic function; its period is $L=(L_1, L_2, \cdots, L_n)$. Let M be*

$$(L_1L_2\cdots L_n)^{-1}\int_0^{L_1}\cdots\int_0^{L_n}H(X)dX\,,$$

then we have

$$\sup_f\inf_Y\int_B H(f(Z)+Y)dZ=M=\inf_f\sup_Y\int_B H(f(Z)+Y)dZ\,,$$

and the following three propositions are equivalent, if $H(X)$ is continuous.

(a) $\quad\sup_Y\int_B H(f(Z)+Y)dZ=M,$

(b) $\quad\inf_Y\int_B H(f(Z)+Y)dZ=M,$

(c) $\quad\int_B H(f(Z)+Y)dZ=M,\qquad$ *for all* Y.

PROOF. By Fubini's theorem

$$(L_1L_2\cdots L_n)^{-1}\int_0^{L_1}\cdots\int_0^{L_n}\int_B H(f(Z)+Y)dZdY$$
$$=\int_B(L_1L_2\cdots L_n)^{-1}\int_0^{L_1}\cdots\int_0^{L_n}H(f(Z)+Y)dYdZ=\int_B MdZ=M\,,$$

for all $f\in C^1_B$. Then we have

$$\sup_f\inf_Y\int_B H(f(Z)+Y)dZ\leqq M\leqq\inf_f\sup_Y\int_B H(f(Z)+Y)dZ\,.$$

Here the equivalence relation is clear. On the other hand,

$$M=(L_1L_2\cdots L_n)^{-1}\int_0^{L_1}\cdots\int_0^{L_n}H(2LZ+L+Y)dY$$
$$=\int_B H(2LZ+L+Y)dZ\,,$$

for all Y, and the theorem is proved.

§ 2. Periodicity on $\boldsymbol{S}$.

We use the notations below.

$\boldsymbol{R}^n$; real n-dimensional euclidean space.
$\langle X, Y\rangle$; inner product in $\boldsymbol{R}^n$.
$\boldsymbol{S}$; $\{X\in\boldsymbol{R}^n;\ \|X\|=1\}$.
m ; surface measure on $\boldsymbol{S}$.
Q_n ; $\Gamma(n/2)/2\sqrt{\pi}\,\Gamma((n+1)/2)$.

As in the following lemma, the family of functions on $\boldsymbol{S}$, $\{|\langle X, Y\rangle|\,;\ Y\in\boldsymbol{S}\}$, has a certain kind of periodicity.

LEMMA 4. *$L(Y)=\int_S|\langle X, Y\rangle|dm$ is a constant function on $\boldsymbol{S}$. Denote this constant by number R_n, then we have $R_n\Big/\int_S dm=2Q_n$.*

PROOF. For any $Y_1, Y_2\in\boldsymbol{S}$, $\{X\in\boldsymbol{S};\ \langle X, Y_1\rangle=\cos\theta\}$ is congruent with $\{X\in\boldsymbol{S};\ \langle X, Y_2\rangle=\cos\theta\}$ by the rotation γ; $\gamma(Y_1)=Y_2$. Let $\mathrm{mes}_{(n-2)}$ be $(n-2)$-dimensional euclidean measure, then for any $Y\in\boldsymbol{S}$

$$L(Y)=\int_S|\langle X, Y\rangle|dm=\int_0^{\pi}|\cos\theta|\,\mathrm{mes}_{(n-2)}\{X\in\boldsymbol{S};\ \langle X, Y\rangle=\cos\theta\}d\theta\,.$$

So $L(Y)$ is independent of Y. If we set $Y_0=(0, 0, \cdots, 0, 1)$, then by the n-dimensional polar co-ordinate method, we have

$$R_n=\int_S|x_n|dm=\int_0^{\pi}\cdots\int_0^{\pi}\int_0^{2\pi}|\cos\theta_{n-1}|\sin\theta_2\sin^2\theta_3\cdots\sin^{n-2}\theta_{n-1}d\theta_1d\theta_2\cdots d\theta_{n-1}\,.$$

On the other hand

$$\int_S dm=\int_0^{\pi}\int_0^{\pi}\int_0^{2\pi}\sin\theta_2\sin^2\theta_3\cdots\sin^{n-2}\theta_{n-1}d\theta_1d\theta_2\cdots d\theta_{n-1}\,.$$

Therefore

$$R_n\Big/\int_S dm=\int_0^{\pi}\cos\theta_{n-1}\sin^{n-2}\theta_{n-1}d\theta_{n-1}\Big/\int_0^{\pi}\sin^{n-2}\theta_{n-1}\,d\theta_{n-1}$$
$$=\int_0^{\pi/2}\cos\theta\sin^{n-2}\theta\,d\theta\Big/\int_0^{\pi/2}\sin^{n-2}\theta\,d\theta$$
$$=2\Gamma(n/2)/(n-1)\sqrt{\pi}\,\Gamma((n-1)/2)=2Q_n\,.$$

This completes the proof.

THEOREM 5. *Let C; $C=C(t)$ be normal curve in $\boldsymbol{R}^n$, namely, $\dfrac{dC(t)}{dt}$ is continuous function and never vanishes, $n\geqq 2$, and its length be 1. Let C_L be the length of the locus of C on a line L, (see the proof), then there exist lines L_1, L_2, such that $C_{L_2}\leqq 2Q_n\leqq C_{L_1}$, and the following three propositions are equivalent, for the normal curve C.*

(a) $\sup_L C_L=2Q_n$. (b) $\inf_L C_L=2Q_n$. (c) $C_L=2Q_n$, *for all line L.*

PROOF. We can assume that $t=\int_0^t\left\|\dfrac{dC(t)}{dt}\right\|dt$. The line L can be considered to be oriented. Let Y_L be L's oriented unit vector, then

$$C_L=C_{Y_L}=C_Y=\int_0^1\left|\left\langle\frac{dC(t)}{dt},\ Y\right\rangle\right|dt\,.$$

By Fubini's theorem

$$\int_S C_Y dm=\int_S\int_0^1\left|\left\langle\frac{dC(t)}{dt},\ Y\right\rangle\right|dt\,dm=\int_0^1\int_S\left|\left\langle\frac{dC(t)}{dt},\ Y\right\rangle\right|dm\,dt=\int_0^1 R_n dt\,.$$

Therefore

$$\inf_{Y\in S} C_Y\int_S dm\leqq R_n\leqq\sup_{Y\in S} C_Y\int_S dm\,,$$

and hence by Lemma 4

$$\inf_{Y\in S} C_Y\leqq 2Q_n\leqq\sup_{Y\in S} C_Y\,.$$

By the continuity on the compact set $\boldsymbol{S}$, there exist Y_1, $Y_2\in\boldsymbol{S}$, such that $C_{Y_2}\leqq 2Q_n\leqq C_{Y_1}$. The others are clear.

DEFINITION. Let $a_1, a_2, \cdots, a_p$ be n-dimensional vectors. By the notation $[a_1, a_2, \cdots, a_p]_L$, we mean the sum of the length of the *shaddow* of a_j $(j=1, 2, \cdots, p)$ on the line L.

THEOREM 6. *For any integer $n\geqq 2$, and any positive number $\varepsilon>0$, there exist finite number of n-dimensional vectors $a_1, a_2, \cdots, a_p$, such that*

$$\left|[a_1, a_2, \cdots, a_p]_L\Big/\sum_{j=1}^p\|a_j\|-2Q_n\right|<\varepsilon\,,$$

for all line L.

PROOF. $\{|\langle X, Y\rangle|\,;\,Y\in\boldsymbol{S}\}$ is uniformly compact in $C_R(\boldsymbol{S})$. m is a positive measure on $\boldsymbol{S}$. Then for any $\varepsilon>0$, there exists w, positive point measure on $\boldsymbol{S}$, such that

$$\left|\int_S|\langle X, Y\rangle|dw-\int_S|\langle X, Y\rangle|dm\right|<(\varepsilon/4)\int_S dm\,,$$

for all $Y\in\boldsymbol{S}$, and

$$\left|\int_S dw-\int_S dm\right|<\min\left\{2^{-1}\int_S dm,\ (\varepsilon/2R_n)\left(\int_S dm\right)^2\right\}.$$

We assume that $\int_{A_j} dw=p_j>0$, $j=1, 2, \cdots, p$, and $\|w\|=\sum_{j=1}^{p} p_j$. Let a_j be $p_j\overrightarrow{OA_j}$, $j=1, 2, \cdots, p$, then for any line L, we have

$$[a_1, a_2, \cdots, a_p]_L=\sum_{j=1}^{p}|\langle a_j, Y_L\rangle|=\sum_{j=1}^{p}p_j|<\overrightarrow{OA_j}, Y_L\rangle|=\int_S|\langle X, Y_L\rangle|dw\,.$$

Therefore

$$\begin{aligned}&\left|[a_1, a_2, \cdots, a_p]_L\Big/\sum_{j=1}^{p}\|a_j\|-2Q_n\right|\\&=\left|\int_S|\langle X, Y_L\rangle|dw\Big/\int_S dw-\int_S|\langle X, Y_L\rangle|dm\Big/\int_S dm\right|\\&=\left|\int_S\langle X, Y_L\rangle|dw-\int_S|\langle X, Y_L\rangle|dm\right|\Big/\int_S dw+R^n\left|\int_S dw-\int_S dm\right|\Big/\left|\int_S dw\int_S dm\right|\\&<\varepsilon/2+\varepsilon/2=\varepsilon\,.\end{aligned}$$

This completes the proof.

COROLLARY 7. *For any curve C with length 1 in $\boldsymbol{R}^n$, $n\geqq 2$, we have* $\inf_L C_L\leqq 2Q_n\leqq\sup_L C_L$, *and this estimate is the best. Moreover the following three propositions are equivalent.*

(a) $\sup_L C_L=2Q_n$. (b) $\inf_L C_L=2Q_n$. (c) $C_L=2Q_n$, *for all line L.*

PROOF. For any curve C with length l, its length can be approximated by the zigzag line method. Conversely, the length of any zigsag line can be approximated by the normal curve. Then the corollary is proved by Theorems 5 and 6.

So far, we have considered the family of functions $\{|\langle X, Y\rangle|\,;\ Y\in\boldsymbol{S}\}$, and proved Lemma 4, Theorems 5, 6, and Corollary 7. On the family of functions on $\boldsymbol{S}$, $\{\langle X, Y\rangle^+=(|\langle X, Y\rangle|+\langle X, Y\rangle)/2\,;\ Y\in\boldsymbol{S}\}$, we can prove the same lemma, theorems and corollary. In this case, we have $\int_S\langle X, Y\rangle^+dm=R_n/2$.

Particularly, it is clear that for any n-dimensional vectors $a_1, a_2, \cdots, a_p$, there exists $Y_0\in\boldsymbol{S}$ such that

$$Q_n\leqq\sum_{j=1}^{p}\langle a_j, Y_0\rangle^+\Big/\sum_{j=1}^{p}\|a_j\|\,.$$

On the other hand, we can assume that Y_0 satisfies the following relation.

$$\sum_{j=1}^{p}\langle a_j, Y_0\rangle^+=\max_{Y\in S}\sum_{j=1}^{p}\langle a_j, Y\rangle^+=\max_{I\subset\{1,2,\cdots,p\}}\|\sum_{i\in I}a_i\|\,.$$

Therefore

$$Q_n \leqq \max_I \|\sum_{i\in I} a_i\| \Big/ \sum_{j=1}^{p} \|a_j\| .$$

In the same way as the proof of Theorem 6, we can prove that this estimate is the best. And by Theorem 5, we can prove that this inequality is strict. So the following theorem and its corollary are obtained.

THEOREM 8. *For any n-dimensional vectors,* $a_1, a_2, \cdots, a_p, n \geqq 2$, *we have*

$$Q_n < \max_I \|\sum_{i\in I} a_i\| \Big/ \sum_{j=1}^{p} \|a_j\| ,$$

and this estimate is the best.

COROLLARY 9. *For any complex numbers,* $a_1, a_2, \cdots, a_p$, *we have*

$$1/\pi < \max_I |\sum_{i\in I} a_i| \Big/ \sum_{j=1}^{p} |a_j| ,$$

and this estimate is the best.

REMARK. If $n=1$, then $Q_1 \equiv \Gamma(1/2)/2\sqrt{\pi}\,\Gamma(1) = 1/2$. Here, it is clear that for any 1-dimensional vectors, $a_1, a_2, \cdots, a_p$, we have

$$1/2 \leqq \max_I \|\sum_{i\in I} a_i\| \Big/ \sum_{j=1}^{p} \|a_j\| .$$

I am grateful to referee.

References

[1] A.S. Besicovith, Almost periodic functions, Cambridge, 1932.
[2] A. Pietsch, Nukleare Lokalkonvexe Räume, Akademie-Verlag, Berlin, 1969.

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