J. Math. Soc. Japan
Vol. 13, No. 2, 1961

On some relations between the Martin boundary and the Feller boundary

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(Received Nov. 9, 1960)
(Revised Jan. 10, 1961)

1. In this paper we shall consider the integral representation of bounded harmonic functions by means of a regular Borel measure on the Feller boundary $\mathcal{M}(\mathfrak{S})$ (cf. Section 9). For this purpose we investigate mutual relations between the family of bounded harmonic functions, a function lattice on the Martin boundary and a function lattice on the Feller boundary, by use of the Martin representation theorem of harmonic functions (cf. J. L. Doob [**3**] and T. Watanabe [**12**], [**13**]). This subject is closely related to some results of D. G. Kendall [**9**] which we shall prove here by a different method.

2. Let X be a countable state space with the discrete topology. Let $X\cup\{\rho\}$ be denoted by $\tilde{X}$ in which $\{\rho\}$ is added to X as an isolated point. Let W be the totality of $\tilde{X}$-valued right-continuous functions w on the interval $T=[0,\infty]$. The value of w at time t is denoted by $w(t)$ or $x_t(w)$. Let $\mathbf{M}=\{X, W, P_x, x\in\tilde{X}\}$ be a minimal Markov process[1] where X is the state space, W is the sample space and P_x is the probability measure on the Borel field $\mathcal{F}(W)$ generated by the sets $\{w\,;x_t(w)\in A\}$(A: a Borel set on $\tilde{X}$). Define

$$\begin{aligned}\sigma_A(w)&=\inf\{t>0\,;x_t(w)\in A\} &&\text{if } x_t(w)\in A \text{ for some } t>0,\\ &=+\infty &&\text{otherwise,}\\ \tau_A(w)&=\inf\{t>0\,;x_t(w)\notin A\} &&\text{if } x_t(w)\notin A \text{ for some } t>0,\\ &=+\infty &&\text{otherwise.}^{2)}\end{aligned}$$

For $x,y\in\tilde{X}$, we set $\Pi(x,y)=P_x\{w\,;x_{\tau_x}(w)=y,\tau_x<+\infty\}$. Then $\Pi(x,\rho)=1-\sum_{y\in X}\Pi(x,y)$ and $\Pi(\rho,\rho)=1$.

In this paper, a finite real valued function $u(\cdot)$ over X will be called x_t-harmonic if it satisfies $u(x)=\sum_{y\in X}\Pi(x,y)u(y)$ (in the sense of absolute convergence) for any x in X.

1) The term 'minimal process' is used in the sense of W. Feller [**6**, pp. 535-537]. Also a precise definition of such process is seen in [**13**, Chapter 1].

2) We denote τ_x in case $A=\{x\}$.

Now, we shall define the following sequence of Markov times:[3)]

$$\tau_0(x, w) \equiv 0, \tau_1(x, w) = \tau_x(w), \cdots, \tau_n(x, w) = \tau_1(x_{\tau_{n-1}}(w), w^+{}_{\tau_{n-1}}) + \tau_{n-1}(x, w),$$

where $w_\tau^+(t) = w(t+\tau)$.

Then, τ_n is the time at which the n-th jump of a path $w(t)$ will occur. Since $\tau_n(x, w)$ is a monotone increasing function of n, we can define $\lim_{n\to\infty} \tau_n(x, w) = \tau_\omega(x, w) \leqq +\infty$.

3. We shall summarize here some results in the theory of Martin boundaries associated with Markov chains, following J. L. Doob [**3**] and T. Watanabe [**12**], [**13**].

We shall assume that there exists a center c such that $p(c, x) = P_c(\sigma_x < +\infty) > 0$ holds for any $x \in X$ and that all states are transient.[4)] Then, the Martin boundary ∂X induced by x_t is the set of all the limit functions of the family of x_t-superharmonic functions[5)] $K(c, x, y) = \dfrac{p(x, y)}{p(c, y)}$. An element of ∂X is denoted by b and we define the value of b at x by $K(c, x, b)$.

The Martin space $\hat{X} = X \cup \partial X$ is compact with the metric

$$\rho(\xi, \eta) = \sum_{x \in X} |K(c, x, \xi) - K(c, x, \eta)| p(c, x) m(x)$$

where, $\xi, \eta \in \hat{X}$ and $m(x)$ is a strictly positive measure on X such that $\sum_{x\in X} m(x) < +\infty$. Then ∂X is relatively compact and X is a dense set in $\hat{X}$.

$\lim_{n\to\infty} x_{\tau_n}(w)$ exists for almost all the paths in $W_f = \{w ; \tau_n(w) < +\infty$ for all $n < +\infty\}$, and for all w in W_f^c and for any $n \geqq n_0(w)$, $x_{\tau_n}(w) = +\infty$ holds. Therefore, $y(w) = \lim x_{\tau_n}^{(x)}(w)$[6)] constitutes a random variable on $\partial X \cup \{\rho\}$ for almost all w. Hence, $y(w)$ determines a measure on ∂X by restricting it on ∂X which is the so-called harmonic measure $h(x, \cdot)$.

If $u(x)$ is a bounded x_t-harmonic function on X, there exists a unique bounded measurable function $f(\cdot)$ on ∂X except for the set of $h(c, \cdot)$-measure zero such that $u(x)$ is expressed by

$$u(x) = \int_{\partial X} K(c, x, b) f(b) h(c, db) = \int_{\partial X} f(b) h(x, db) \tag{3.1}$$

where we have put $f(\rho) = 0$.

Moreover, given the bounded Borel measurable function on ∂X, the right hand side of (3.1) defines a bounded x_t-harmonic function on X.

3) For definitions of unexplained terminologies in this section, see K. Ito [8].

4) Namely, $\sum_{n=1}^{\infty} \Pi^n(x, x) < +\infty$ for any x in X.

5) For this concept, see T. Watanabe [**13**] or J. L. Doob [**3**].

6) Supersufix (x) means starting point.

4. Let $\mathfrak{H}$ be the set of all bounded harmonic functions u on $X, L^\infty(\partial X, h(c, \cdot))$ be the set of all $h(c, \cdot)$-essentially bounded Borel measurable functions f on ∂X, and $N(\partial X, h(c, \cdot))$ be the set of all functions f in $L^\infty(\partial X, h(c, \cdot))$ such that $h(c, \{b; f(b) \neq 0\}) = 0$.

Then, as it is well known, L^∞/N constitutes a σ-complete Boolean lattice with respect to

$$(f \cap g)(\cdot) = \min (f(\cdot), g(\cdot)), \qquad (f \cup g)(\cdot) = \max (f(\cdot), g(\cdot)).$$

Define the one-to-one mapping $\varphi: \mathfrak{H} \to L^\infty/N$ by the formula (3.1). Then, since φ and φ^{-1} are positivity-preserving, it follows that $\mathfrak{H}$ is a σ-complete Boolean lattice and that φ is a lattice isomorphism. The union (intersection) of u_1 and u_2 in $\mathfrak{H}$ will be denoted by $u_1 \vee u_2$ $(u_1 \wedge u_2)$.

Next, by (3.1)

$$\| u(x) \| = \sup \{ | u(x) | ; x \in X \} \geqq \int_{\partial X} f(b) h(x, db).$$

Therefore, by taking the fine limit along a path $x_t^{(c)}(w)$, we have $\| u(x) \| \geqq f(y(w))$. Since $h(c, \{b; |f(b)| > k\}) = 0$ implies $P_c(w; |f(y(w))| > k) = 0$ and *vice versa*, we can easily show that

$$\begin{aligned} &\text{ess. sup} \{ |f(b)| : b \in \partial X, \text{with respect to } h(c, \cdot)\text{-measure} \} \\ &= \text{ess. sup} \{ |f(y(w))| ; \text{with respect to } w \in W, P_c\text{-measure} \}. \end{aligned}$$

Hence, we have

$$\| u(x) \| \geqq \| f \| = \text{ess. sup} \{ |f(b)| ; b \in \partial X, \text{with respect to } h(c, \cdot)\text{-measure} \}.$$

We have also

$$| u(x) | \leqq \| f \| h(x, \partial X) \leqq \| f \|.$$

Consequently, we obtain

$$\begin{aligned} \| u(x) \| = \sup \{ | u(x) | ; x \in X \} &= \text{ess. sup} \{ | f(b) | ; b \in \partial X, \\ &\text{with respect to } h(c, \cdot)\text{-measure} \} = \| f \|. \end{aligned}$$

Thus, we have

THEOREM 1. *The one-to-one mapping*

$$\varphi : \mathfrak{H} \to L^\infty/N$$

is an isometric σ-complete lattice isomorphism.

5. By a sojourn solution we mean a bounded positive harmonic function $s(x)$ on X with $\| s \| = 1$ which satisfies $s(x) \wedge (h(x, \partial X) - s(x)) = 0$. It follows from Theorem 4.3 in [13] that $h(\cdot, \partial X)$ is nothing but the function s_E of [5]. Therefore, according to Theorem 9.1 in [5], our definition of a sojourn solution coincides with Feller's one.[7] Therefore, by the above isomorphism in Theorem 1, we have the following expression

$$\varphi(s(x) \wedge (h(x, \partial X) - s(x)) = f_s \cap (1 - f_s) = 0\,,$$

where we put $\varphi(s) = f_s$.

Hence, $f_s = 1$ or 0 on ∂X except on the set of $h(c, \cdot)$-measure zero. Since f_s is Borel measurable, $\{b\,; f_s(b) = 1\} = A$ belongs to the family $\mathscr{B}(\partial X)$ of all Borel measurable subsets of ∂X. Conversely, for any given $A \in \mathscr{B}(\partial X)$, $h(x, A)$ is a sojourn solution.

Now, we shall set $\mathfrak{S} = \{s\,; s$ is a sojourn solution or $s \equiv 0\}$, and $\mathfrak{N} = \{A\,; h(c, A) = 0\}$. Then, we have

THEOREM 2. *The mapping*

$$\psi : \mathfrak{S} \to \mathscr{B}(\partial X)/\mathfrak{N} \qquad \text{(onto)}$$

is a σ-complete lattice isomorphism, where $\mathscr{B}(\partial X)/\mathfrak{N}$ is a σ-complete lattice in the ordinary sense.

REMARK. From this Theorem 2 and Section 6 below, we can see that in case the Feller boundary consists of finite points, the Martin boundary is not smaller than the former.

6. In this section, we shall prove some results of D. G. Kendall [**9**] by different approach.

We recall several definitions here. A set α of sojourn solutions is called a lattice ideal in $\mathfrak{S}$ if $u \in \alpha$ and $v \in \alpha$ imply $u \vee v \in \alpha$ and $w \leqq u, u \in \alpha$ and $w \in \mathfrak{S}$ imply $w \in \alpha$. An ideal is called maximal if $\mathfrak{S}$ is the only ideal containing α as a proper subset (cf. W. Feller [**5**]). We can define similarly an ideal of $\mathscr{B}(\partial X)/\mathfrak{N}$.

Denote the space of maximal lattice ideals of $\mathfrak{S}$ and $\mathscr{B}(\partial X)/\mathfrak{N}$ by $\mathscr{M}(\mathfrak{S})$ and $\mathscr{M}(\mathscr{B}(\partial X)/\mathfrak{N})$ respectively. Following M. H. Stone [**11**], put $J(A) = \{\alpha\,; \alpha \in \mathscr{M}(\mathscr{B}(\partial X)/\mathfrak{N})(\mathscr{M}(\mathfrak{S})), \alpha \not\ni A(h(x, A))\}$ for $A \in \mathscr{B}(\partial X)/\mathfrak{N}$ and we take the set $\Sigma = \{J(A)\,; A \in \mathscr{B}(\partial X)/\mathfrak{N}\}$ as the basis of open sets, then the space $\mathscr{M}(\mathscr{B}(\partial X)/\mathfrak{N})$ $(\mathscr{M}(\mathfrak{S}))$ is a totally disconnected compact Hausdorff space.

The mapping $J : \mathscr{B}(\partial X)/\mathfrak{N} \to \Sigma$ induces a lattice isomorphism and Σ contains every closed and open set (cf. M. H. Stone [**11**] or N. Dunford and J. T. Schwarz, [**4** pp. 41-43]).

W. Feller [**5**] topologized $X \cup \mathscr{M}(\mathfrak{S})$. It follows from definition that Feller's relative topology on $\mathscr{M}(\mathfrak{S})$ coincides with the above Stone topology. Therefore, we can say that the Feller boundary is the space $\mathscr{M}(\mathfrak{S})$ or $\mathscr{M}(\mathscr{B}(\partial X)/\mathfrak{N})$ with Stone topology.

Let $\{J(B_\alpha), \alpha \in \Gamma\}$ be any system of disjoint subsets of Σ. Since $h(c, \cdot)$ is

7) See W. Feller [**5**, p. 33].

a totally finite measure, we see that $J(B_\alpha)$ is non-empty for countably many α's.

We have also

$$\overline{\bigcup_{\alpha=1}^{\infty} J(A_\alpha)} = J(\bigcup_{\alpha=1}^{\infty} A_\alpha) .$$

For, if it were not so, there exists a $C \in \mathcal{B}(\partial X)/\mathfrak{N}$ such that $J(\bigcup_{\alpha=1}^{\infty} A_\alpha) \supseteqq \bigcup_{\alpha=1}^{\infty} J(A_\alpha) \cup J(C)$, $A_\alpha \cap C = \phi$ for all $\alpha, C \neq \phi$, and $\bigcup_{\alpha=1}^{\infty} A_\alpha \supseteqq C$ which is impossible.

Now let G be any open subset of $\mathcal{M}(\mathcal{B}(\partial X)/\mathfrak{N}) : G = \bigcup_{\beta \in \Lambda} J(A_\beta)$, where Λ is an arbitrary index set. Then we can select a maximal sequence $\{J(B_r), r = 1, 2, \cdots\}$ such that $G \supseteqq \bigcup_{r=1}^{\infty} J(B_r)$ and $B_r (r = 1, 2, \cdots)$ are pairwise disjoint. By Zorn's lemma, such a sequence always exists. Since $\overline{\bigcup_{r=1}^{\infty} J(B_r)} = J(\bigcup_{r=1}^{\infty} B)$ and $\{J(B_r)\}$ is a maximal sequence, we have

$$\bigcup_{r=1}^{\infty} J(B_r) \subseteqq G \subseteqq \bar{G} = \overline{\bigcup_{r=1}^{\infty} J(B_r)} = J(\bigcup_{r=1}^{\infty} B_r) . \tag{6.1}$$

Hence, any open set in $\mathcal{M}(\mathcal{B}(\partial X)/\mathfrak{N})$ has a closed and open closure. Thus, $\mathcal{M}(\mathcal{B}(\partial X)/\mathfrak{N})$ is a so-colled Stonian space.

Now we shall define a measure on $\mathcal{M}(\mathcal{B}(\partial X)/\mathfrak{N})$ by means of harmonic measure $h(x, \cdot)$. At first, we define a measure $\bar{h}_0$ on the ring Σ by putting $\bar{h}_0(x, J(A)) = h(x, A)$ for $A \in \mathcal{B}(\partial X)/\mathfrak{N})$.

Let S_0 be the σ-ring generated by Σ. Then, S_0 coincides with the class of Baire[8] sets. Since $\bar{h}_0$ is countably additive on the ring Σ, $\bar{h}_0$ has a unique extention over S_0. Furthermore, there exists a unique Borel measure $\bar{h}$ such that $h(A) = \bar{h}_0(A)$ for every A in S_0.

Then, we have for any disjoint sequence $\{J(B_r), r = 1, 2, \cdots\}$

$$\begin{aligned} h(x, \bigcup_{r=1}^{\infty} J(B_r)) &= \lim_{n \to \infty} \bar{h}_0(x, \bigcup_{r=1}^{n} J(B_r)) = \sum_{r=1}^{\infty} h(x, B_r) \\ &= h(x, \bigcup_{r=1}^{\infty} B_r) = \bar{h}_0(x, J(\bigcup_{r=1}^{\infty} B_r)) \\ &= \bar{h}(x, J(\bigcup_{r=1}^{\infty} B_r)) . \end{aligned}$$

Therefore, by (6.1), we have

$$\bar{h}(G) = \bar{h}(\bar{G}) .$$

From this formula we can easily prove that for any closed set F in $\mathcal{M}(\mathcal{B}$

8) For the terminology of this section, see P. R. Halmos [7].

$(\partial X)/\mathfrak{N})$

(6.2) $$\bar{h}(F)=\bar{h}(F_0)\,,$$

where F_0 is the interior of F.

Then, it can be observed that the regular Borel measure $\bar{h}$ is normal.[9] For, assuming F to be any closed set with empty interior, we have by (6.2)

$$\bar{h}(F)=\bar{h}(F_0)=h(x,\phi)=0\,.$$

Hence, if A is any Borel set with closure $\bar{A}$ and interior A_0, we have $\bar{h}(A_0)=\bar{h}(A)=\bar{h}(\bar{A})$.

Let N_0 be the class of all Borel sets of the first category. Then, $N\ni A$ if and only if $\bar{h}(A)=0$.

Since $\mathcal{M}(\mathcal{B}(\partial X)/\mathfrak{N})$ is a Baire space we see that the support of h is a dense subset of $\mathcal{M}$ (cf. N. Bourbaki [**1**, p. 76, Theorem 1]).

Accordingly we have the following

THEOREM 3 (D. G. Kendall). *The Stonian space* $\mathcal{M}(\mathcal{B}(\partial X)/\mathfrak{N})(\mathcal{M}(\mathfrak{S}))$ *is hyperstonian.*

7. We shall prove that the space L^∞/N is isometrically isomorphic to the space $C(\mathcal{M}(\mathcal{B}(\partial X)/\mathfrak{N}))$ of all continuous functions on $\mathcal{M}(\mathcal{B}(\partial X)/\mathfrak{N})$. For this purpose, we map $a\cdot\chi_A(\cdot)\in L^\infty/N$ to $a\cdot\chi_{J(A)}(\cdot)\in C(\mathcal{M}(\mathcal{B}(\partial X)/\mathfrak{N}))$, where $A\in\mathcal{B}(\partial X)/\mathfrak{N}$, $\chi_A(\cdot)$ is its indicator function and a is a real number.

But, the uniform closure of the family $\mathcal{A}$ of functions of the form $\sum_{i=1}^{\infty}a_i\cdot\chi_{A_i}(\cdot)$ is L^∞/N. (a_i real, $A_i\in\mathcal{B}(\partial X)/\mathfrak{N}$ and $A_i\cap A_j=\phi$, $i\neq j$).

On the other hand, the family of continuous functions $\mathcal{L}=\{\sum_{i=1}^{n}a_i\cdot\chi_{J(A_i)}(\cdot)$, $J(A_i)\in\Sigma, a_i$ real$\}$ separates any two points of $\mathcal{M}(\mathcal{B}(\partial X)/\mathfrak{N})$. For, for any points $\alpha,\beta\in\mathcal{M}(\mathcal{B}(\partial X)/\mathfrak{N}), \alpha\neq\beta$, there exists at least an $A\in\mathcal{B}(\partial X)/\mathfrak{N}$ such that $\alpha\not\ni A$ and $\beta\ni A$. Therefore, we can see that $J(A)\ni\alpha$ and $J(A)\not\ni\beta$. Thus the function $\chi_{J(A)}(\cdot)$ separates α and β. Hence, by Stone-Weierstrass' theorem, the uniform closure of $\mathcal{L}$ coincides with $C(\mathcal{M}(\mathcal{B}(\partial X)/\mathfrak{N}))$.

We can see easily that the mapping from $\mathcal{A}$ to $\mathcal{L}$:

$$\xi(\sum_{i=1}^{n}a_i\cdot\chi_{A_i}(\cdot))=\sum_{i=1}^{n}a_i\cdot\chi_{J(A_i)}(\cdot)\,,$$

where A_i are pairwise disjoint, is linear, and bounded. Since

$$\|\sum_{i=1}^{n}a_i\cdot\chi_{A_i}(\cdot)\|=\max_{1\leq i\leq n}|a_i|=\|\sum_{i=1}^{n}a_i\cdot\chi_{J(A_i)}(\cdot)\|\,,$$

the mapping ξ is bounded, linear and isometric. Therefore, we can see that

9) A positive totally finite regular Borel measure λ on a Stonian space is called normal if each rare set can be covered by a Borel set of λ-measure zero.

there exists an isometric isomorphism ξ from L^∞/N to $C(\mathcal{M}(\mathcal{B}(\partial X)/\mathfrak{N}))$ as Banach algebras.

8. L^∞/N is a completetly symmetric ring in the sense of M. A. Naimark [**10**] and $\|f^2\|=\|f\|^2$ for $f\in L^\infty/N$. Let $M(L^\infty/N)$ be the space of all maximal ideals of the ring L^∞/N and let its topology be defined by weak topology. Then this representation is onto, isometric and isomorphic (M. A. Naimark [**10**, p. 218, Corollary 2]). We note that the space $M(L^\infty/N)$ is hyperstonian (see J. Dixmier [**2**]).

Combining with the results in section 7 we see that $C(M(L^\infty/N))$ is isometric and isomorphic to $C(\mathcal{M}(\mathcal{B}(\partial X)/\mathfrak{N}))$ as a Banach algebra. Hence, by a well-known theorem (see N. Dunford and J. T. Schwarz [**4**, p. 279]), $M(L^\infty/N)$ and $\mathcal{M}(\mathcal{B}(\partial X)/\mathfrak{N})$ are homeomorphic. Thus, we have

THEOREM 4. *The Feller boundary is homeomorphic to* $M(L^\infty/N)$.

9. By summarizing the results of the preceding sections we can deduce the following conclusion.

THEOREM 5. *The Boolean lattices* $\mathfrak{H}$, $C(\mathcal{M}(\mathcal{B}(\partial X)/\mathfrak{N}))$, $L^\infty(\partial X, h(c,\cdot))/N$ *and* $C(M(L^\infty/N))$ *are isometrically isomorphic to one another.*

Let $\mathfrak{l}$ be the isometric isomorphism from $C(\mathcal{M}(\mathcal{B}(\partial X)/\mathfrak{N}))$ to $\mathfrak{H}$. Then, $\mathfrak{l}$ is positive linear, isometric, and hence bounded.

Accordingly, we have the representation of $u=\mathfrak{l}(\bar f)\in\mathfrak{H}$ ($\bar f\in C(\mathcal{M}(\mathcal{B}(\partial X)/\mathfrak{N}))$) by a theorem of Riesz:

$$u(x)=\int_{\mathcal{M}(\mathcal{B}(\partial X)/\mathfrak{N})} f d\mu_x\,,$$

where, μ_x is a regular Borel measure on $\mathcal{M}(\mathcal{B}(\partial X)/\mathfrak{N})$.

However, if $\bar f=\chi_{J(A)}(\cdot)$, $u(x)=h(x,A)=\mu_x(J(A))$ holds. Hence, for every element in Σ, we have $\mu_x(J(A))=\bar h(x,J(A))$. Thus, by the uniqueness of the extension of Borel measure, we can see $\bar h=\mu_x$ for every Borel set in $\mathcal{M}(\mathcal{B}(\partial X)/\mathfrak{N})$. Consequently, we have

$$u(x)=\int_{\partial X} f(b)h(x,db)=\int_{\mathcal{M}(\mathcal{B}(\partial X)/\mathfrak{N})}\bar f(\bar b)\bar h(x,d\bar b)\,.$$

Accordingly, we have proved the following conclusion for the P. W. B. problem on Feller boundaries.

THEOREM 6. *If* $u(x)$ *is a bounded* x_t*-harmonic function on* X, *there exists a bounded measurable function* $\bar f(\cdot)$ *on* $\mathcal{M}(\mathfrak{S})$ *and a Borel measure* $\bar h(x,\cdot)$ *on* $\mathcal{M}(\mathfrak{S})$ *such that* $u(x)$ *is expressed by*

$$u(x)=\int_{\mathcal{M}(\mathfrak{S})}\bar f(\bar b)h(x,d\bar b)\,. \tag{9.1}$$

Conversely, for any given bounded measurable function $\bar f(\cdot)$ *on* $\mathcal{M}(\mathfrak{S})$ *the right hand side of* (9.1) *defines a bounded* x_t*-harmonic function on* X.

REMARK. A bounded measurable function $f(\cdot)$ on $\mathcal{M}(\mathfrak{S})$ coincides with a continuous function on $\mathcal{M}(\mathfrak{S})$ except on the set of $\bar{h}(c, \cdot)$-measure zero.

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