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On intermediate many-valued logics.

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There have been many reseaches on many-valued propositional logics. Rosser and Turquette [1], Dienes [2] and Church [3] investigated many-valued logical extensions of two-valued logic which have the analogous properties to classical logic. Łukasiewicz and Tarski [4] and Kleene [5] gave many-valued propositional logics which are not considered to be classical logic. Furthermore, the truth-tables given in [4] and [5] do not contain all formulas which are provable in intuitionistic propositional logic. In fact, $(A\supset \neg A)\supset \neg A$ which is provable intuitionistically does not always take the designated truth value in [4] and $A\supset A$ in [5] where $\supset$ and $\neg$ denote implication and negation respectively.

A treatment of many-valued propositional logics, in which every intuitionistically provable formula is true but not necessarily all classically provable formulas, viz. of intermediate many-valued logics in our terminology, was first achieved by Jaśkowski [6]. The purpose of this paper is to investigate details of intermediate many-valued logics.

A sufficient condition for a many-valued propositional logic to contain every intuitionistically provable propositional formula is given in § 1. Let $L_1,\cdots,L_n$ be arbitrary many-valued logics. We call $L_1,\cdots,L_n$ mutually independent, if for every distinct i and j there is a formula which is true in L_i and not true in L_j. In § 2, it is proved that there are at least enumerably infinite mutually independent many-valued propositional logics.

In § 3 we construct a sequence of intermediate many-valued propositional logics in which every member is a sublogic of the preceding ones. This sequence is well-ordered and the ordinal number of the sequence is called the length of the sequence. It is proved that there is a sequence of intermediate many-valued propositional logics whose length is $\omega^{\omega^{\omega}}$. In § 4, special many-valued propositional logics $\mathfrak{R}_n$ and $\mathfrak{R}_\omega$ are discussed. The many-valued logics which can be reduced to $\mathfrak{R}_n$ is studied. Every provable formula in LR_n and LP_2, special intermediate propositional logics in axiomatic stipulation (cf. Umezawa [8] and [9]), is true in $\mathfrak{R}_n$ and $\mathfrak{R}_\omega$ respectively.

In § 5 we extend the results in § 2 and § 3 to predicate calculus. Quantifiers $\forall$ and $\exists$ can be defined in the propositional logics which appear in the proof of Theorems 2 and 3 and hence these logics can be regarded as predicate logics.

§1. A sufficient condition for a many-valued propositional logic to contain all propositional formulas which are intuitionistically provable.

Let L be any many-valued propositional logic, the set S of whose elements is non-empty. We denote the logical operations in L i. e. conjunction, disjunction, implication and negation by $\wedge$, $\vee$, $\supset$ and $\neg$ respectively. For elements a, b of S, $a \equiv b$ means that a and b are in a same subclass for a classification of S. We make use of set-theoretic notations such as $\{\ \}$, $\{\ |\ \}$ and $\in$.

The following is called (J)-*condition.*

(J)-condition. There is a classification of S such that the following holds. Let a, b, c be elements of S.

1. If $a \equiv b$ and $b \equiv c$, then $a \equiv c$.
2. If $a \equiv b$, then $a \wedge c \equiv b \wedge c$ and $a \vee c \equiv b \vee c$.
3. If $a \equiv b$, then $a \wedge b \equiv a \vee b \equiv a$.
4. $a \wedge b \equiv b \wedge a$ and $a \vee b \equiv b \vee a$.
5. $a \wedge (b \wedge c) \equiv (a \wedge b) \wedge c$ and $a \vee (b \vee c) \equiv (a \vee b) \vee c$.
6. $a \wedge (a \vee b) \equiv a$ and $a \vee (a \wedge b) \equiv a$.
7. $a \wedge (b \vee c) \equiv (a \wedge b) \vee (a \wedge c)$ and $a \vee (b \wedge c) \equiv (a \vee b) \wedge (a \vee c)$.
8. There are sets T and F defined thus:
   $T = \{t \mid \text{for all } x \in S\ x \wedge t \equiv x\}$ and
   $F = \{f \mid \text{for all } x \in S\ x \wedge f \equiv f\}$
9. For a and b, there is a set which contains $a \supset b$ and whose element, say r, satisfies the condition:
   For all $x \in S$, $(a \wedge x) \wedge b \equiv a \wedge x$ is equivalent to $x \wedge r \equiv x$.
10. For any $a \in S$, there is a set which contains $\neg a$ and whose element, say r, satisfies the condition:
    For all $x \in S$, $(a \wedge x) \wedge f \equiv a \wedge x$ is equivalent to $x \wedge r \equiv x$ where f is an element of F in 8.

In virtue of 6, T and F can be also defined as follows:

$T = \{t \mid \text{for all } x \in S\ x \vee t \equiv t\}$ and

$F = \{f \mid \text{for all } x \in S\ x \vee f \equiv x\}$.

Lemma 1. *For any elements $a, b \in S$, $a \supset b \in T$ is equivalent to $a \wedge b \equiv a$.*

Proof. Let $a \supset b \in T$. By 9, we see that for all $x \in S$ $(a \wedge x) \wedge b \equiv a \wedge x$ is equivalent to $x \wedge (a \supset b) \equiv x$. From the assumption and 8, $x \wedge (a \supset b) \equiv x$ for all $x \in S$. Then for all $x \in S$ $(a \wedge x) \wedge b \equiv a \wedge x$ follows. Hence $(a \wedge a) \wedge b \equiv a \wedge a$.

Since $a\equiv a$ holds, we obtain $a\wedge b\equiv a$, using 1, 2 and 3. Conversely, assume that $a\wedge b\equiv a$. By means of 1, 2, 4 and 5, we obtain $(a\wedge x)\wedge b\equiv a\wedge x$. Consequently, the set $\{r \mid \text{for all } x\in S\ (a\wedge x)\wedge b\equiv a\wedge x \text{ is equivalent to } x\wedge r\equiv x\}$ is equal to $\{r \mid \text{for all } x\in S\ x\wedge r\equiv x\}$, i. e. to T. Hence, $a\supset b\in T$.

LEMMA 2. *Let f be an element of F. $\neg a\in T, a\supset f\in T$ and $a\in F$ are equivalent one another.*

PROOF. Let $\neg a\in T$. Then $x\wedge \neg a\equiv x$ for all $x\in S$. By 10, $(a\wedge x)\wedge f\equiv a\wedge x$ is equivalent to $x\wedge \neg a\equiv x$ for all $x\in S$. Hence, $(a\wedge x)\wedge f\equiv a\wedge x$ for all $x\in S$. Substituting a for x and using $a\wedge a\equiv a$, we obtain $a\wedge f\equiv a$. By Lemma 1, this means $a\supset f\in T$. From $a\supset f\in T$, Lemma 1 and $a\wedge f\equiv f$, we obtain $a\equiv f$. Hence, $a\in F$. Finally, let $a\in F$. By the definition of F, $x\wedge a\equiv a$ for all $x\in S$ and hence $f\wedge a\equiv a$. Consequently, for all $x\in S$ $(a\wedge x)\wedge f\equiv (f\wedge a)\wedge x\equiv a\wedge x$. In terms of the equivalence of $(a\wedge x)\wedge f\equiv a\wedge x$ to $x\wedge \neg a\equiv x$, we obtain $x\wedge \neg a\equiv x$ for all $x\in S$. Hence $\neg a\in T$.

A formula is called *true in* L or L-*true* if the formula always takes the designated element of L no matter what set of elements of L is assigned to the variables of the formula.

THEOREM 1. *Every intuitionistically provable propositional formula is true in any L which satisfies the* (J)-*condition and takes T as the set of designated elements.*

PROOF. We make use of Gentzen's LJ [7] to deduce all the intuitionistically provable formulas. Since Gentzen adopts the sequent calculus, we interprete a sequent as follows. A sequent $\Gamma\to\Delta$ with non-empty Γ, Δ is considered $\Gamma^*\supset\Delta^*$ where Γ^* and Δ^* denote $A_1\wedge\cdots\wedge A_m$ and $B_1\vee\cdots\vee B_n$ if Γ and Δ represent $A_1,\cdots, A_m$ and $B_1,\cdots, B_n$ respectively. $\Gamma\to\Delta$ with empty Γ or with empty Δ is considered Δ^* or $\neg\Gamma^*$ with the same meaning of * as the above.

As for initial sequent $A\to A$, the theorem holds by 3, because $a\supset a\in T$ is equivalent to $a\wedge a\equiv a$ by virtue of Lemma 1. Then we proceed inductively.

Thinning-in-antecedent. This inference has the shape $\dfrac{\Gamma\to H}{A, \Gamma\to H}$ because of the intuitionistic limitation. Hence, it should be proved that if $\gamma\supset h\in T$, then $(a\wedge\gamma)\supset h\in T$ where γ is an element of L representing the value of Γ^*. By assumption and Lemma 1, it follows that $\gamma\wedge h\equiv\gamma$. Using 2 and 5, we obtain $(a\wedge\gamma)\wedge h\equiv a\wedge\gamma$ and hence $(a\wedge\gamma)\supset h\in T$.

Thinning-in-succedent. It suffices to prove that if $\neg\gamma\in T$, then $\gamma\supset a\in T$ for any element a of L. Let $\neg\gamma\in T$. In virtue of Lemma 2, $\gamma\in F$ and hence $\gamma\wedge a\equiv\gamma$. By Lemma 1, we obtain $\gamma\supset a\in T$.

Cut. This inference has the shape $\dfrac{\Gamma\to A\quad A, \Delta\to H}{\Gamma, \Delta\to H}$. Hence, it should be proved that if $\gamma\supset a\in T$ and $(a\wedge\delta)\supset h\in T$, then $(\gamma\wedge\delta)\supset h\in T$. By assumption and Lemma 1, we may assume that $\gamma\wedge a\equiv\gamma$ and $(a\wedge\delta)\wedge h\equiv a\wedge\delta$. Then it follows successively that $(\gamma\wedge\delta)\wedge h\equiv(\gamma\wedge a)\wedge\delta\wedge h\equiv\gamma\wedge((a\wedge\delta)\wedge h)\equiv\gamma\wedge(a\wedge\delta)$ $\equiv(\gamma\wedge a)\wedge\delta\equiv\gamma\wedge\delta$. Hence, $(\gamma\wedge\delta)\supset h\in T$.

Since other rules of inference can be proved similarly, we omit the rest of proof.

§ 2. Many-valued propositional logics which are mutually independent.

Let $L_1,\cdots,L_n$ be arbitrary many-valued propositional logics. $L_1,\cdots,L_n$ are called *mutually independent* if for each i and j $(i\neq j\ 1\leqq i,\ j\leqq n)$ there is a formula which is true in L_i and not true in L_j.

THEOREM 2. *There are at least enumerably infinite many-valued propositional logics which are mutually independent.*

PROOF. Let S_i be the set defined as

$$S_i=\{(x,y)\,|\,x=y=0 \quad \text{or} \quad (x=1,2,\cdots,2^{n+1-i}(n+1) \ \text{ and } \ y=1,2,\cdots,i)\}$$

where $1<n$ and $2\leqq i\leqq n+1$.

Let $S_i\ni a,b$ and $a=(a_1,a_2)$ and $b=(b_1,b_2)$. Logical operations are defined thus:

$$a\wedge b=(\mathrm{Max}(a_1,b_1),\ \mathrm{Max}(a_2,b_2)).$$

$$a\vee b=(\mathrm{Min}(a_1,b_1),\ \mathrm{Min}(a_2,b_2)).$$

$$a\supset b=\begin{cases}(0,0) & \text{if } a_1\geqq b_1 \text{ and } a_2\geqq b_2,\\ (1,b_2) & \text{if } a_1\geqq b_1 \text{ and } a_2<b_2,\\ (b_1,1) & \text{if } a_1<b_1 \text{ and } a_2\geqq b_2,\\ b & \text{if } a_1<b_1 \text{ and } a_2<b_2.\end{cases}$$

$$\neg a=a\supset(2^{n+1-i}(n+1),\ i).$$

The many-valued logic whose truth values are elements of S_i and whose logical operations are the just defined ones is denoted by L_i where (0,0) is the designated element.

We prove that $L_2,L_3,\cdots,L_{n+1}$ are mutually independent. It suffices to prove that for $2\leqq j<i\leqq n+1$ L_i and L_j are mutually independent. Let us consider a formula $\bigvee_C(A_p\supset A_q)$ where $\bigvee$ denotes the disjunction of $(A_p\supset A_q)$'s with p and q which satisfy the condition C: $1\leqq p\leqq i,\ 1\leqq q\leqq i$ and $p\neq q$. This formula is L_j-true, since, by the assumption $j<i$, there are p and q such that the truth value corresponding to $(A_p\supset A_q)\vee(A_q\supset A_p)$ is (0,0). However, this is not L_i-true if the value of A_r $(1\leqq r\leqq n)$ is $(r,\ i+1-r)$. Next we consider $\bigvee_D(A_p\supset A_q)$ where D is the condition: $p,q\in S_j$ and $p\wedge q\neq p$. Since the number of elements of S_j is greater than that of elements of S_i, there are t and s,

distinct elements of S_j, such that A_t and A_s take a same truth value in S_i. For any distinct $p, q \in S_j$, $p \wedge q \neq p$ or $p \wedge q \neq q$ and hence the value of $A_t \supset A_s$, a fortiori, of $\bigvee_D (A_p \supset A_q)$ is (0,0). However, this formula is not L_j-true, if the truth value which A_r takes is r. Since n is an arbitrary positive integer, the theorem follows.

§ 3. A sequence of intermediate many-valued logics.

First we introduce some definitions.

$$\alpha = \omega^{\omega^i t_i + \cdots + \omega t_1 + t_0} (t_i, \cdots, t_0 \geqq 0) \quad \text{and} \quad w_h = \sum_{j=h}^{i} (j+1) t_j$$

($h \leqq i$) where α naturally depends upon $t_i, \cdots, t_0$ and w_h upon $t_i, \cdots, t_h$.

$$p(k) = \frac{2^{k w_0} - 1}{2^{w_0} - 1} \quad \text{where} \quad w_0 \neq 0 .$$

We define $S(t_i, \cdots, t_0, n)$ recursively. Let A, B, A_x be arbitrary sets. $A \cup B$ denotes the sum set of A and B and $\bigcup_C A_x$ the sum set of A_x's which satisfy the condition C.

$$S(n) = \{(k, k) \mid 0 \leqq k \leqq n\} .$$

$S(t_i, \cdots, t_0, n) = \{(\alpha(2^l p(k)), \alpha(2^m p(k))) \mid 0 \leqq k \leqq n$ and $[l = 0, 1, \cdots, w_h\ m = \sum_{j=h}^{i} (j-h) t_j$ where $0 < h \leqq i$; $l = 0, 1, \cdots, w_0\ m = \sum_{j=h}^{i} t_j, w_0]\} \cup \{(\alpha(2^{w_0} p(k)) + x, \alpha(2^{w_0} p(k)) + y) \mid 0 \leqq k < n$ and $(x, y) \in \bigcup_C S(t_i, \cdots, t_{j+1}, t_j - 1, s_{j-1}, \cdots, s_0, m)$ where j is determined by the condition $t_0 = \cdots = t_{j-1} = 0$ and $t_j > 0$ and C denotes that $0 < s_{j-1} < \omega$ and $0 < m < \omega\}$.

$$S(0, t_i, \cdots, t_0, n) = S(t_i, \cdots, t_0, n) .$$

Example. $S(1, n)$ is the set, $\{(\omega(2^l p(k)), \omega(2^m p(k))) \mid 0 \leqq k \leqq n$ and $l, m = 0, 1\}$ $\cup \{(\omega(2p(k)) + x, \omega(2p(k)) + y) \mid 0 \leqq k < n$ and $(x, y) \in \bigcup_{0<m<\omega} S(m)\}$ where $p(k) = 2^k - 1$.

We express the set of $t_i, \cdots, t_0, n$ occurring in the definition of $S(t_i, \cdots, t_0, n)$ by $(t_i, \cdots, t_0, n)$. Given n, m ($n > 0$), $(t_i, \cdots, t_0, n) \succ (s_j, \cdots, s_0, m)$ ($(s_j, \cdots, s_0, m) \prec (t_i, \cdots, t_0, n)$) means that i) $i > j$ or ii) there is an x such that $i = j, t_i = s_j, \cdots, t_{x+1} = s_{x+1}, t_x > s_x$ or iii) $i = j, t_i = s_j, \cdots, t_0 = s_0, n > m$ or iv) $m = 0$. $(t_i, \cdots, t_0, n) = (s_j, \cdots, s_0, m)$ means that $i = j, t_i = s_j, \cdots, t_0 = s_0, n = m$.

$S(t_i, \cdots, t_0, n)$ contains $S(s_j, \cdots, s_0, m)$ as a proper subset if $(t_i, \cdots, t_0, n) \succ (s_j, \cdots, s_0, m)$.

We denote by $\boldsymbol{n}$ a finite sequence of $(t_i, \cdots, t_0, n)$'s such that if $(s_j, \cdots, s_0, m)$ is a preceding member of $(u_k, \cdots, u_0, l)$, then $(s_j, \cdots, s_0, m) \succ (u_k, \cdots, u_0, l)$ and $j \neq k$ or for some x $s_x \neq u_x$. Let $\boldsymbol{n}_x$ and $\boldsymbol{m}_x$ be x-th members of $\boldsymbol{n}$ and $\boldsymbol{m}$ respectively. $\boldsymbol{n} \succ \boldsymbol{m}$ ($\boldsymbol{m} \prec \boldsymbol{n}$) means that there is an x such that $\boldsymbol{n}_1 = \boldsymbol{m}_1, \cdots, \boldsymbol{n}_{x-1} = \boldsymbol{m}_{x-1}$,

$\boldsymbol{n}_x \succ \boldsymbol{m}_x$. The number of members of $\boldsymbol{n}$ is denoted by $lh(\boldsymbol{n})$.

Let $\beta(\boldsymbol{n}_x) = \alpha(2^{w_0}p(n))$ where α, w_0 and $p(n)$ are defined for $t_i, \cdots, t_0, n$ in $\boldsymbol{n}_x$. $S(t_i, \cdots, t_0, n)$ is also denoted by $S(\boldsymbol{n}_x)$ if $\boldsymbol{n}_x$ is $(t_i, \cdots, t_0, n)$. Now we define $S_1(\boldsymbol{n}_r)$ and $T(\boldsymbol{n})$.

$$S_1(\boldsymbol{n}_r) = \left\{ (\sum_{x=1}^{r-1} \beta(\boldsymbol{n}_x) + y,\ \sum_{x=1}^{r-1} \beta(\boldsymbol{n}_x) + z) \mid (y, z) \in S(\boldsymbol{n}_r) \right\}$$

where $\sum_{x=1}^{r-1} \beta(\boldsymbol{n}_x) = \beta(\boldsymbol{n}_1) + \beta(\boldsymbol{n}_2) + \cdots + \beta(\boldsymbol{n}_{r-1})$.

$$T(\boldsymbol{n}) = \bigcup_{1 \leqq r \leqq lh(\boldsymbol{n})} S_1(\boldsymbol{n}_r)\,.$$

Logical operations are defined in $T(\boldsymbol{n})$ in what follows. Let a and b are elements of $T(\boldsymbol{n})$ and $a = (a_1, a_2)$ and $b = (b_1, b_2)$.

$$a \wedge b = (\mathrm{Max}(a_1, b_1),\ \mathrm{Max}(a_2, b_2))\,.$$

$$a \vee b = (\mathrm{Min}(a_1, b_1),\ \mathrm{Min}(a_2, b_2))\,.$$

Let c be the least ordinal number of x's occurring in $(x, b_2) \in T(\boldsymbol{n})$ and d the least ordinal number of y's occurring in $(b_1, y) \in T(\boldsymbol{n})$.

$$a \supset b = \begin{cases} (0, 0) & \text{if } a_1 \geqq b_1 \text{ and } a_2 \geqq b_2\,, \\ (c, b_2) & \text{if } a_1 \geqq b_1 \text{ and } a_2 < b_2\,, \\ (b_1, d) & \text{if } a_1 < b_1 \text{ and } a_2 \geqq b_2\,, \\ (b_1, b_2) & \text{if } a_1 < b_1 \text{ and } a_2 < b_2\,. \end{cases}$$

Let γ be the greatest ordinal number of all x's occurring in $(x, y) \in T(\boldsymbol{n})$.

$$\urcorner a = a \supset (\gamma, \gamma)\,.$$

$T(\boldsymbol{n})$ is closed with regard to $\wedge$, $\vee$, $\supset$, $\urcorner$.

We denote by $L(\boldsymbol{n})$ the many-valued propositional logic as defined above where the designated element is (0,0).

Lemma 3. *$L(\boldsymbol{n})$ is an intemediate many-valued propositional logic.*

Proof. In virtue of Theorem 1, it suffices to prove that $L(\boldsymbol{n})$ satisfies (J)-condition. We take a trivial classification where every subclass consists of only one element. Then $\equiv$ becomes $=$ between elements of $L(\boldsymbol{n})$. 1 and 2 are evident. 3–7 can be easily proved. As to 8, T and F are taken to be $\{(0,0)\}$ and $\{(\gamma, \gamma)\}$ where γ is the greatest ordinal number of all x's occurring in $(x, y) \in L(\boldsymbol{n})$.

Concerning 9 and 10, we take $\{a \supset b\}$ and $\{\urcorner a\}$ as sets required in 9 and 10 respectively. We prove that for all $x \in L(\boldsymbol{n})$, $(a \wedge x) \wedge b = a \wedge x$ is equivalent to $x \wedge (a \supset b) = x$. Let $x = (x_1, x_2)$, $a = (a_1, a_2)$ and $b = (b_1, b_2)$. In case both $a_1 \geqq b_1$ and $a_2 \geqq b_2$, it is evident. Assume that $a_1 \geqq b_1$ and $a_2 < b_2$. It suffices to show that $\mathrm{Max}(b_2, x_2) = \mathrm{Max}(a_2, x_2)$ is equivalent to both $\mathrm{Max}(x_1, c) = x$ and $\mathrm{Max}(x_2, b_2)$

$=x_2$ where c denotes the least ordinal number of x's occurring in $(x, b_2)\in L(\boldsymbol{n})$. For $x_2<b_2$, it clearly holds. For $x_2=b_2$, it also holds, since if $x_2=b_2$, then $x_1\geqq c$, and for $x_2>b_2$, $x_1\geqq c$ is also valid. Hence, what is to prove holds for any $(x_1, x_2)\in L(\boldsymbol{n})$. Furthermore, it can be proved that $\{a\supset b\}$ is the only set which satisfies 9. Other cases can be treated similarly.

We say that $L(\boldsymbol{n})$ is a *sublogic* of $L(\boldsymbol{m})$, if every $L(\boldsymbol{n})$-true formula is $L(\boldsymbol{m})$-true and the converse is not the case.

THEOREM 3. *If $\boldsymbol{n}\succ\boldsymbol{m}$, then $L(\boldsymbol{n})$ is a sublogic of $L(\boldsymbol{m})$.*

PROOF. It follows from the assumption $\boldsymbol{n}\succ\boldsymbol{m}$ that there is an x such that $\boldsymbol{n}_1=\boldsymbol{m}_1,\cdots,\boldsymbol{n}_{x+1}=\boldsymbol{m}_{x+1}, \boldsymbol{n}_x\succ\boldsymbol{m}_x$. Let r be the x as required. For every y such that $r\leqq y\leqq lh(\boldsymbol{m})$, $\boldsymbol{n}_r\succ\boldsymbol{m}_y$ and it is seen from the definition of $S(t_i,\cdots, t_0, n)$ that $(\sum_{x=r}^{y-1}\beta(\boldsymbol{m}_x)+z_1, \sum_{x=r}^{y-1}\beta(\boldsymbol{m}_x)+z_2)\in S(\boldsymbol{n}_r)$ where $(z_1, z_2)\in S(\boldsymbol{m}_y)$. Consequently, $S_1(\boldsymbol{m}_y)$ $(r\leqq y\leqq lh(\boldsymbol{m}))$ is a subset of $S_1(\boldsymbol{n}_r)$ and hence $\bigcup_{r\leqq y\leqq lh(\boldsymbol{m})} S_1(\boldsymbol{m}_y)\subset S_1(\boldsymbol{n}_r)$. Therefore $T(\boldsymbol{m})$ is a subset of $T(\boldsymbol{n})$ and we obtain that if a formula is $L(\boldsymbol{n})$-true, then it is $L(\boldsymbol{m})$-true.

Then, for the proof, it suffices to show a formula which is $L(\boldsymbol{m})$-true but not $L(\boldsymbol{n})$-true. Let us define

$$S'(\boldsymbol{n}_r)=\Big\{(\sum_{x=1}^{r-1}\beta(\boldsymbol{n}_x)+\alpha(2^l p(k)),\ \sum_{x=1}^{r-1}\beta(\boldsymbol{n}_x)+\alpha(2^m p(k)))\,|\,0\leqq k\leqq n\ [l=0,\cdots, w_h\ m=\sum_{j=h}^{i}(j-h)t_j:\ l=0,\cdots, w_0\ m=\sum_{j=0}^{i}jt_j, w_0]\Big\}$$

where $\boldsymbol{n}_r=(t_i,\cdots, t_0, n)$ and $\alpha, p(k), w_h$ are defined for the $t_i,\cdots, t_0$.

We consider a formula F: $\bigvee_C(A_x\supset A_y)$ where $\bigvee$ denotes the disjunction of $(A_x\supset A_y)$'s with x, y which satisfy the condition C: $x, y\in \bigcup_{1\leqq z\leqq r} S'(\boldsymbol{n}_z)$ and for $x=(x_1, x_2)$, $y=(y_1, y_2)$, $\mathrm{Max}(x_1, y_1)\neq x_1$ or $\mathrm{Max}(x_2, y_2)\neq x_2$. Since $S'(\boldsymbol{n}_z)$ and r are finite, F is a formula in propositional calculus. F is not $L(\boldsymbol{n})$-true, because if the value of A_x is x, every $A_x\supset A_y$ in F does not take (0,0) as its value, as is seen from the definition of $\supset$.

Next we consider F in $L(\boldsymbol{m})$. Any $A_x\supset A_y$ in F where $x, y\in \bigcup_{1\leqq x<r} S'(\boldsymbol{n}_x)$ can take a value different from (0,0) in the same way as the above. Let $\boldsymbol{n}_r$ be $(t_i,\cdots, t_0, n)$ and $\boldsymbol{m}_r$ $(s_j,\cdots, s_0, m)$. Since $\boldsymbol{n}_r\succ\boldsymbol{m}_r$, i) $i>j$ or ii) there is an x such that $i=j, t_i=s_j,\cdots, t_{x+1}=s_{x+1}, t_x>s_x$ or iii) $i=j, t_i=s_j,\cdots, t_0=s_0, n>m$ or iv) $m=0$. Let $i>j$. $S'(\boldsymbol{m}_y)$ $(r\leqq y\leqq lh(\boldsymbol{m}))$ does not contain $i+1$ elements such that $a\supset b\neq(0,0)$ and $b\supset a\neq(0,0)$. A_x's in F': $\bigvee(A_x\supset A_y)$ in F where $x, y\in S'(\boldsymbol{n}_r)$ must take values from $\bigcup_{r\leqq y\leqq lh(\boldsymbol{m})} S'(\boldsymbol{m}_y)$ in order that the value of F be not (0,0). Since F' contains a subformula of form $\bigvee_{x\neq y\ x,y=1,\cdots,i+1}(B_x\supset B_y)$, then F' takes (0,0) as its value in $L(\boldsymbol{m})$. Hence F is $L(\boldsymbol{m})$-true. Also in other cases, not all $A_x\supset A_y$ in F' can take values different from (0,0) in $\bigcup_{r\leqq y\leqq lh(\boldsymbol{m})} S'(\boldsymbol{m}_y)$. F' and hence F take (0,0) as their values. Therefore F is $L(\boldsymbol{m})$-true.

We consider a sequence of $L(\boldsymbol{n})$'s in which every member is a sublogic of the preceding ones. This sequence is well-ordered. The ordinal number of such a sequence of $L(\boldsymbol{n})$ is called the *length* of the sequence.

Tehorem 4. *There is a sequence of intermediate many-valued propositional logics whose length is* ω^{ω^ω}.

Proof. For convenience, we write "the length for $\boldsymbol{n}$" instead of "the length of a sequence of all $L(\boldsymbol{m})$'s where $\boldsymbol{m} \prec \boldsymbol{n}$ and every mumber of the sequence is a sublogic of preceding ones".

We take $L(\boldsymbol{n})$ where $\boldsymbol{n}$ consists of only $\boldsymbol{n}_1 = (1,1)$. Since for any $\boldsymbol{m}$ consisting of only (m), $\boldsymbol{m} \prec \boldsymbol{n}$, the length for the $\boldsymbol{n}$ is ω. Assume that the length for $\boldsymbol{n}$ consisting of only $\boldsymbol{n}_1 = (1, 0, \cdots, 0, 1)$ with i zeros is ω^{ω^i}. Then the length for $\boldsymbol{n}$ consisting of only $(1, 0, \cdots, 0, p)$ with i zeros is $\omega^{\omega^i p}$ and hence the one for $\boldsymbol{n}$ consisting of only $\boldsymbol{n}_1 = (1, 0, \cdots, 0, 1, 1)$ with $i-1$ zeros is ω^{ω^i+1}. It can be proved that the length for $\boldsymbol{n}$ consisting of only $\boldsymbol{n}_1 = (1, 0, \cdots, 0, t_0, 1)$ with $i-1$ zeros is $\omega^{\omega^i+t_0}$ and hence for $\boldsymbol{n}$ consisting of only $\boldsymbol{n}_1 = (1, 0, \cdots, 0, 1, 0, 1)$ with $i-1$ zeros it is $\omega^{\omega^i+\omega}$. Similarly, it is proved that the length for $\boldsymbol{n}$ consisting of only $\boldsymbol{n}_1 = (1, 0, \cdots, 0, 1)$ with $i+1$ zeros is ω^{ω^i+1}. Since we can take any integer for i, the theorem follows.

§ 4. Special many-valued propositional logics.

In this section we treat special many-valued logics. $L(\boldsymbol{n})$ with $\boldsymbol{n}$ consisting of only $\boldsymbol{n}_1 = (n)$ in the preceding section is denoted by $\mathfrak{R}_n$. We represent $\mathfrak{R}_n$ in terms of truth-tables. Let $0, 1, 2, \cdots, n$ be truth values of $\mathfrak{R}_n$ and 0 the designated element. Logical operation $\wedge$, $\vee$, $\supset$ and $\neg$ are defined in what follows:

$\wedge$	0 1 2 $\cdots$ n
0	0 1 2 $\cdots$ n
1	1 1 2 $\cdots$ n
2	2 2 2 $\cdots$ n
.	....
n	n n n $\cdots$ n

$\vee$	0 1 2 $\cdots$ n
0	0 0 0 $\cdots$ 0
1	0 1 1 $\cdots$ 1
2	0 1 2 $\cdots$ 2
.	....
n	0 1 2 $\cdots$ n

$\supset$	0 1 2 3 $\cdots$ n	$\neg$
0	0 1 2 3 $\cdots$ n	n
1	0 0 2 3 $\cdots$ n	n
2	0 0 0 3 $\cdots$ n	n
.	....	.
$n-1$	0 0 0 0 $\cdots$ n	n
n	0 0 0 0 $\cdots$ 0	0

In virtue of Theorem 3, it follows that if $n > m$, then $\mathfrak{R}_n$ is a sublogic of $\mathfrak{R}_m$. $\mathfrak{R}_0$ is the contradictory logic and $\mathfrak{R}_1$ is the usual two-valued logic.

Two many-valued logics are called *equivalent* if the sets of true formulas are the same. We now give two many-valued logics which are equivalent to $\mathfrak{R}_n$.

4.1. Let $S_0, S_1, \cdots, S_n$ be arbitrary non-empty sets, each two of which is disjoint. The elements of $S_0, S_1, \cdots, S_n$ are taken as truth values of L_1. Logical operations are defined in the following.

Let $a_i \in S_i$ and $b_j \in S_j$.

i) $a_i \wedge b_j \in S_{\max(i,j)}$.

ii) $a_i \vee b_j \in S_{\min(i,j)}$.

iii) $a_i \supset b_j \in S_0$ if $i \geqq j$ and $a_i \supset b_j \in S_j$ if $i < j$.

iv) $\neg a_n \in S_0$ and $\neg a_i \in S_n$ if $i < n$.

The designated elements of L_1 are elements of S_0. Then L_1 is equivalent to $\mathfrak{R}_n$, because if we classify elements of L_1 into $S_0, S_1, \cdots, S_n$, then the resulting logic is isomorphic to $\mathfrak{R}_n$.

4.2. Let S be any non-empty set. f_i $(0 \leqq i \leqq n)$ is defined to be a function such that for all $x \in S, f_i(x) = i$. Let F be the set of all functions of one variable x with S as the range of x and with $\{0,1\}$ as the domain. f_i $(1 < i \leqq n)$ and elements of F are truth values of L_2. f_0 and f_1 are elements of F. Let $f, g \in L_2$.

$f \wedge g = h_1$ where $h_1(x) = \mathrm{Max}(f(x), g(x))$.

$f \vee g = h_2$ where $h_2(x) = \mathrm{Min}(f(x), g(x))$.

$f \supset g = h_3$ where $h_3(x) = 0$ if $f(x) \geqq g(x)$ and $h_3(x) = g(x)$ if $f(x) < g(x)$.

$\neg f = f \supset f_n$.

f_0 is the only designated element of L_2. Then we prove that L_2 is equivalent to $\mathfrak{R}_n$.

$f_0, f_1, \cdots, f_n$ form a subtable isomorphic to $\mathfrak{R}_n$. Hence, if a formula is L_2-true, then it is also $\mathfrak{R}_n$-true.

Let $S \ni a$ and $T_a = \{f \mid f(a) = 0 \text{ and } f \in F\}$. We denote the relative complement of T_a with regard to F by $F - T_a$. For $g, h \in T_a$ and for $k, l \in F - T_a$, the following hold:

i) $g \wedge h \in T_a$ and $g \wedge k$, $k \wedge g$, $k \wedge l \in F - T_a$. For $2 \leqq j \leqq i \leqq n$, $g \wedge f_i = f_i \wedge g = k \wedge f_i = f_i \wedge k = f_j \wedge f_i = f_i \wedge f_j = f_i$.

ii) For $2 \leqq i \leqq n$, $g \vee h, g \vee k, k \vee g, g \vee f_i, f_i \vee g \in T_a$ and $k \vee l, k \vee f_i, f_i \vee k \in F - T_a$. For $2 \leqq j \leqq i \leqq n$, $f_j \vee f_i = f_i \vee f_j = f_j$.

iii) For $2 \leqq j \leqq i \leqq n$, $g \supset h$, $k \supset g$, $k \supset l$, $f_i \supset g$, $f_i \supset k$, $f_i \supset f_j \in T_a$ and for $2 \leqq j < i \leqq n$, $g \supset f_i = k \supset f_i = f_j \supset f_i = f_i$.

iv) For $i < n$, $\neg g = \neg k = \neg f_i = f_n$ and $\neg f_n \in T_a$.

It is seen from i)—iv) that T_a, $F - T_a$, $f_2, \cdots, f_n$ form a subtable isomorphic to $\mathfrak{R}_n$ where T_a corresponds to the designated element of $\mathfrak{R}_n$. Therefore, if a formula is $\mathfrak{R}_n$-true, then it takes an element of T_a as its value. Since $a \in S$ is an arbitrary element, the value of $\mathfrak{R}_n$-true formula is $\bigcap_{a \in S} T_a = f_0$ and hence

L_2-true.

4.3. We here show some relations between many-valued logics and logics by axiomatic stipulation. Concerning our axiomatic stipulation, we refer to Gentzen [**7**] and Umezawa [8], [**9**].

LR_n is defined in [8, § 4] or in [**9**, § 4] to be the intermediate logic resulting from LJ′ (cf. [**9**, § 1]) by adding the following as a new schema of initial sequents

$$R_n: \quad \rightarrow A_1, A_1 \supset A_2, A_2 \supset A_3, \cdots, A_{n-1} \supset A_n, \neg A_n .$$

For any sequent Z, the formula which we obtain from Z in the same way as in the proof of Theorem 1 is denoted by Z^*.

R_n^* is $\mathfrak{R}_n$-true, as is seen from the truth-tables of $\mathfrak{R}_n$. Since $\mathfrak{R}_n$ satisfies the (J)-condition, we obtain that every R_n-provable propositional formula is $\mathfrak{R}_n$-true. R_i^* $(i<n)$ is not $\mathfrak{R}_n$-true, since if the truth value of A_j is j, then R_i^* takes 1 as its truth value.

LP_2 is defined in [8, § 2] to be the logic resulting from LJ′ by adding P_2 as a new schema of initial sequents

$$P_2: \quad \rightarrow A_1 \supset A_2, \; A_2 \supset A_1 .$$

We denote by $\mathfrak{R}_\omega$ the truth-tables similar to $\mathfrak{R}_n$ except we take $0, 1, 2, \cdots, \omega$ as truth values instead of $0, 1, \cdots, n$.

P_2^* is $\mathfrak{R}_\omega$-true, because for any a and b of $\mathfrak{R}_\omega$, $a \geqq b$ or $b \geqq a$. Since $\mathfrak{R}_\omega$ also satisfies the (J)-condition, it follows that every P_2-provable propositional formula is $\mathfrak{R}_\omega$-true. However, any R_n^* is not $\mathfrak{R}_\omega$-true, as is seen from the truth-tables of $\mathfrak{R}_\omega$. It remains open whether the converse holds or not.

§ 5. Extension of propositional to predicate calculus.

Now quantifiers $\forall$ and $\exists$ are adjoined to the set of logical operations in propositional calculus. We consider quantifiers to be defined for any subset of our basic set of truth values. Let S be the basic set of truth values. For any subset M of S, $\forall x Mx$ and $\exists x Mx$ take some elements of S as their values.

(J)-condition with the following is called the *extended* (J)-*condition*.

11. For any subset M of S, there is a set which contains $\forall x Mx$ and whose element, say r, satisfies the condition:

For all $y \in S$, that for all $z \in M$ $y \wedge z \equiv y$ is equivalent to $y \wedge r \equiv y$.

12. For any subset M of S, there is a set which contains $\exists x Mx$ and whose element, say r, satisfies the condition:

For all $y \in S$, that for all $z \in M$ $y \wedge z \equiv z$ is equivalent to $y \wedge r \equiv r$.

13. Let $M \wedge b$ be the set, $\{x \wedge b \mid x \in M\}$. For any subset M of S and for

any $b\in S$, $\exists x(M\wedge b)x\equiv\exists xMx\wedge b$.

A formula in predicate calculus is called *true in L* or *L-true* if the formula always takes the designated element of L under the interpretation that a predicate variable $A(x)$ represents an element Mx of a subset M of S and $\forall xA(x)$, $\exists xA(x)$ represent $\forall xMx$, $\exists xMx$ respectively.

THEOREM 5. *Every intuitionistically provable formula in predicate calculus (of the first order) is true in any logic which satisfies the extended* (J)*-condition and takes T as the set of designated elements.*

PROOF. We use the same method as in the proof of Theorem 1. In virtue of Theorem 1, it suffices to treat the rules of inference for predicate calculus.

$\forall$-in-antecedent has the shape: $\dfrac{A(a), \Gamma\to H}{\forall xA(x), \Gamma\to H}$. We prove that for any subset M of S, if $(Ma\wedge\gamma)\supset h\in T$, then $(\forall xMx\wedge\gamma)\supset h\in T$ where Ma is an element of M and γ and h are elements of S. From 11, we obtain for all $y\in S$, that for all $z\in M$ $y\wedge z\equiv y$ is equivalent to $y\wedge\forall xMx\equiv y$. Taking $\forall xMx$ as y, it follows that for all $z\in M$ $\forall xMx\wedge z\equiv\forall xMx$. Hence $\forall xMx\wedge Ma\equiv\forall xMx$. In virtue of Lemma 1 and the assumption, $(Ma\wedge\gamma)\wedge h\equiv Ma\wedge\gamma$. Therefore, $\forall xMx\wedge(Ma\wedge\gamma\wedge h)\equiv\forall xMx\wedge(Ma\wedge\gamma)$. By the above fact, we obtain $(\forall xMx\wedge\gamma)\wedge h\equiv\forall xMx\wedge\gamma$ and hence, in virtue of Lemma 1, $(\forall xMx\wedge\gamma)\supset h\in T$.

For $\exists$-in-antecedent: $\dfrac{A(a), \Gamma\to H}{\exists xA(x), \Gamma\to H}$ with the restriction on variable that a shall not occur in the lower sequent, it suffices to prove that for any subset M of S, if $(Ma\wedge\gamma)\supset h\in T$ where a is an arbitrary element of M, then $(\exists xMx\wedge\gamma)\supset h\in T$. By Lemma 1 and the assumption, $(Ma\wedge\gamma)\wedge h\equiv Ma\wedge\gamma$. Since a is an arbitrary element of M, it follows that for all $z\in M\wedge\gamma$ $z\wedge h\equiv z$. Hence, we obtain, using 12, that $h\wedge\exists x(M\wedge\gamma)x\equiv\exists x(M\wedge\gamma)x$. In virtue of 13, $(\exists xMx\wedge\gamma)\wedge h\equiv\exists xMx\wedge\gamma$ follows and hence $(\exists xMx\wedge\gamma)\supset h\in T$.

Proofs for $\forall$-in-succedent and for $\exists$-in-succedent are similar.

In case the set of truth values is finite, 11, 12 and 13 are satisfied by defining $\forall xMx$ by $M_1\wedge\cdots\wedge M_n$, $\exists xMx$ by $M_1\vee\cdots\vee M_n$ where $M_1,\cdots, M_n$ are elements of M. This is proved as follows. The condition which the elements of the requied set of 11 satisfy can be expressed thus: For all $y\in S$, that for all i $(1\leqq i\leqq n)$ $y\wedge M_i\equiv y$ is equivalent to $y\wedge r\equiv y$. Thence, we obtain that $M_1\wedge\cdots\wedge M_n\wedge r\equiv M_1\wedge\cdots\wedge M_n$ and for all i $(1\leqq i\leqq n)$ $r\wedge M_i\equiv r$. Hence $r\equiv M_1\wedge\cdots\wedge M_n$. Since $M_1\wedge\cdots\wedge M_n$ naturally exists for any M, 11 can be written thus: $\forall xMx\in\{r\in S\mid r\equiv M_1\wedge\cdots\wedge M_n\}$ where $M=\{M_1,\cdots, M_n\}$. This is clearly satisfied if $\exists xMx$ is defined as $M_1\wedge\cdots\wedge M_n$. Similarly for 12. 13 is obvious for this definition of $\exists xMx$.

Hence, if we define $\forall$ and $\exists$ for S_i appeared in the proof of Theorem 2 in the above way, then S_i is a predicate logic and the proof of Theorem 2 is valid. Then we obtain

THEOREM 6. *There is a set of intermediate many-valued predicate logics which are mutually independent with any desired number of elements.*

Next we introduce $\forall$ and $\exists$ into $T(\boldsymbol{n})$ in § 3 by defining them thus. Let M be a subset of $T(\boldsymbol{n})$. $\forall xMx$ is the element r of $T(\boldsymbol{n})$ such that for all $y \in T(\boldsymbol{n})$, that for all $z \in M$ $y \wedge z = y$ is equivalent to $y \wedge r = y$. $\exists xMx$ is the element r of $T(\boldsymbol{n})$ such that for all $y \in T(\boldsymbol{n})$, that for all $z \in M$ $y \wedge z = z$ is equivalent to $y \wedge a = a$. Uniqueness of such elements can be easily proved. Existence of $\exists xMx$ for any subset M is seen from the following: For any subset M of $T(\boldsymbol{n})$ there are $d_1, \cdots, d_n$ in M such that for $i \neq j$ $d_i \wedge d_j \neq d_i$ and $d_i \wedge d_j \neq d_j$ and for every $x \in M$ there is a d_i $(1 \leqq i \leqq n)$ which satisfies $x \wedge d_i = x$. It can be proved that $\exists xMx = d_1 \vee \cdots \vee d_n$. Hence, the existence of $\exists xMx$ is clear. Existence of $\forall xMx$ can be proved from the existence of $\exists xMx$.

Therefore 11 and 12 follow. We prove that 13 holds. Let $x \wedge b \in M \wedge b$. Since $x \in M$, there is a d_i such that $x \wedge d_i = x$ and hence there is a $d_i \wedge b$ such that $(x \wedge b) \wedge (d_i \wedge b) = x \wedge b$. From 12, it is seen that for all $y \in T(\boldsymbol{n})$, that for all $z \in M \wedge b$ $y \wedge z = z$ is equivalent to $y \wedge \exists x(M \wedge b)x = \exists x(M \wedge b)x$. Taking $(d_1 \vee \cdots \vee d_n) \wedge b$ as y, it follows that $((d_1 \vee \cdots \vee d_n) \wedge b) \wedge z = z$ for all $z \in M \wedge b$ is equivalent to $((d_1 \vee \cdots \vee d_n) \wedge b) \wedge \exists x(M \wedge b)x = \exists x(M \wedge b)x$. In virtue of the above fact, for all $z \in M \wedge b$ where $z = x \wedge b$ for an $x \in M$, $(d_1 \vee \cdots \vee d_n) \wedge b \wedge z = (d_1 \vee \cdots \vee d_n) \wedge b \wedge (x \wedge b) = (d_1 \wedge \cdots \vee d_n) \wedge b \wedge (x \wedge b) \wedge (d_i \wedge b) = d_i \wedge b \wedge x \wedge b = x \wedge b = z$. Hence $(d_1 \vee \cdots \vee d_n) \wedge b \wedge \exists x(M \wedge b)x = \exists x(M \wedge b)x$. On the other hand, we obtain, by taking $\exists x(M \wedge b)x$ as y, that for all $z \in M \wedge b$ $\exists x(M \wedge b)x \wedge z = z$. Therefore for all i $(1 \leqq i \leqq n)$ $\exists x(M \wedge b)x \wedge d_i \wedge b = d_i \wedge b$ and hence $\exists x(M \wedge b)x \wedge (d_1 \vee \cdots \vee d_n) \wedge b = (d_1 \vee \cdots \vee d_n) \wedge b$. Combining the above equations, we see that $\exists x(M \wedge b)x = (d_1 \vee \cdots \vee d_n) \wedge b$. Since $\exists xMx = d_1 \vee \cdots \vee d_n$, thence 13 follows.

Hence, $L(\boldsymbol{n})$ in which $\forall$ and $\exists$ are defined in the above way is an intermediate many-valued predicate logic. The proof of Theorem 3 goes validly. Consequently, in virtue of Theorem 4, we obtain

THEOREM 7. *There is a sequence of intermediate many-valued predicate logics whose length is $\omega^{\omega^{\omega}}$.*

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