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Invariant tensors under the real representation of unitary group and their applications.

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N. H. Kuiper and K. Yano [3][1)] determined all tensors of certain kinds interesting in differential geometry, which are invariant under the proper orthogonal group of the n-dimensional vector space. They also studied tensors invariant under the group of proper orthogonal transformations fixing a unit vector in the n-dimensional vector space and gave some applications of the results. The purpose of this paper is to determine all tensors of the types they studied, which are invariant under the real representation of unitary group. We obtain some theorems in differential geometry by applying the results.

1. Let C^n be the n-dimensional complex Cartesian space and R^{2n} be the $2n$-dimensional real Cartesian space. We assign to $(z^1,\cdots,z^n)$ of C^n $(x^1,\cdots, x^{2n})$ of R^{2n}, where $z^\alpha=x^\alpha+\sqrt{-1}\ x^{n+\alpha}$ [2)]. Then to every linear transformation σ of C^n corresponds a linear transformation σ' of R^{2n}. If $A_1+\sqrt{-1}\ A_2$ with real matrices A_1, A_2 of degree n is the matrix of σ, then

$$\begin{pmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{pmatrix}$$

is the matrix of σ'. A real matrix A of degree $2n$ corresponds to a complex matrix of degree n in this way if and only if it commutes with J_n, $AJ_n=J_nA$, where

$$J_n=\begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix},$$

E_n being the unit matrix of degree n. Let $(x^1,\cdots,x^{2n})$ and $(y,\cdots,y^{2n})$ of R^{2n} correspond to $(z^1,\cdots,z^n)$ and $(w^1,\cdots,w^n)$ of C^n respectively. Then we have

$$\sum_{\alpha=1}^{n}\bar{z}^\alpha w^\alpha=\sum_{i=1}^{2n}x^iy^i+\sqrt{-1}\sum_{\alpha=1}^{n}(x^\alpha y^{n+\alpha}-x^{n+\alpha}y^\alpha).$$

1) Numbers in brackets refer to the bibliography at the end of the paper.

2) Throughout the paper Greek indices $\alpha,\beta,\cdots$ run over the range $1,2,\cdots,n$ and Latin indices $h,i,j,k,\cdots$ over the range $1,\cdots,2n$.

It follows that to a unitary transformation of C^n corresponds a linear transformation of R^{2n} which leaves two bilinear forms $\sum_i x^i y^i$ and $\sum_\alpha (x^\alpha y^{n+\alpha} - x^{n+\alpha} y^\alpha)$ invariant, and vice versa. In this way the unitary group of C^n is represented as a subgroup of special orthogonal group of R^{2n}. We call this subgroup the *unitary group of* R^{2n} and its element a *unitary transformation of* R^{2n} for brevity. The bilinear forms $\sum_i x^i y^i$ and $\sum_\alpha (x^\alpha y^{n+\alpha} - x^{n+\alpha} y^\alpha)$ will be denoted by $g_{ij}x^i y^j$ and by $\phi_{ij}x^i y^j$ respectively relative to any base of R^{2n}. The linear transformation corresponding to the matrix J_n will be denoted by J and the components of the tensor J by $\phi_j{}^i$. J is a unitary transformation of R^{2n}. We have

$$\phi_k{}^j \phi_j{}^i = -\delta_k{}^i, \qquad \phi_{ih} = \phi_i{}^k g_{kh} = -\phi_{hi}.$$

The natural base of R^{2n} can be denoted as $\{\delta_1, \cdots, \delta_n, J\delta_1, \cdots, J\delta_n\}$. Let σ be a unitary transformation of R^{2n}. If we have $\sigma\delta_\alpha = e_\alpha$, then we see $\sigma J\delta_\alpha = J\sigma\delta_\alpha = Je_\alpha$. Therefore σ transforms the natural base of R^{2n} to an orthonormal base $\{e_1, \cdots, e_n, Je_1, \cdots, Je_n\}$. Conversely, if a base $\{e_1, \cdots, e_n, Je_1, \cdots, Je_n\}$ is orthonormal, then the linear transformation σ of R^{2n} such that we have $\sigma\delta_\alpha = e_\alpha$, $\sigma J\delta_\alpha = Je_\alpha$ is unitary.

Let us suppose that $\{e_1, \cdots, e_m, Je_1, \cdots, Je_m, e_{m+1}\}$ is an orthonormal set of vectors in R^{2n}. Then so is the set $\{e_1, \cdots, e_m, Je_1, \cdots, Je_m, e_{m+1}, Je_{m+1}\}$. In fact, denoting by $e_a{}^i$ $(a=1, \cdots, m)$ the components of the vector e_a, we have, since J is unitary,

$$g_{ij}(\phi_k{}^i e^k_{m+1})(\phi_l{}^j e^l_{m+1}) = g_{kl} e^k_{m+1} e^l_{m+1} = 1,$$
$$g_{ij} e^i_{m+1}(\phi_k{}^j e^k_{m+1}) = \phi_{ki} e^i_{m+1} e^k_{m+1} = 0,$$
$$g_{ij} e_a{}^i (\phi_h{}^j e^h_{m+1}) = g_{rs}\phi_i{}^r \phi_j{}^s e_a{}^i \phi_h{}^j e^h_{m+1} = -g_{rs}(\phi_i{}^r e_a{}^i) e^s_{m+1} = 0,$$
$$g_{rs}(\phi_i{}^r e_a{}^i)(\phi_j{}^s e^j_{m+1}) = g_{ij} e_a{}^i e^j_{m+1} = 0 \qquad (a=1, \cdots, m).$$

It follows from this that for any unit vector e of R^{2n} there exists an orthonormal base $\{e_1, \cdots, e_n, Je_1, \cdots, Je_n\}$ such that we have $e=e_1$. In other words for any unit vector e there exists a unitary transformation σ such that we have $\sigma\delta_1 = e$.

The two-dimensional subspace spanned by two vectors x and Jx is called holomorphic section determined by x [1]. Any unitary transformation leaving invariant a holomorphic section induces unitary transformations in the holomorphic section and in its orthogonal complement.

2. We now state the following

THEOREM 1. *Let the tensors* $v_i, h_{ij}, T^h{}_{ij}$ *and* $R^h{}_{ijk} = -R^h{}_{ikj}$ *be invariant under the unitary group of* R^{2n}. *Then*

$$v_i = 0, \tag{1.a}$$

(1.b) $$h_{ij}=\underset{1}{k}\, g_{ij}+\underset{2}{k}\, \phi_{ij}\,,$$

(1.c) $$T^h{}_{ij}=0\,,$$

(1.d) $$\begin{aligned} R^h{}_{ijk}=&\underset{1}{c}\,(\delta_k{}^h g_{ij}-\delta_j{}^h g_{ik})+\underset{2}{c}\,(\phi_k{}^h g_{ij}-\phi_j{}^h g_{ik}) \\ &+\underset{3}{c}\,(\delta_k{}^h \phi_{ij}-\delta_j{}^h \phi_{ik})+\underset{4}{c}\,(\phi_k{}^h \phi_{ij}-\phi_j{}^h \phi_{ik}) \\ &+\underset{5}{c}\,\delta_i{}^h \phi_{jk}+\underset{6}{c}\,\phi_i{}^h \phi_{jk} \qquad \textit{for } n>1\,, \end{aligned}$$

(1.d′) $$R^h{}_{ijk}=\underset{1}{c'}\,(\delta_k{}^h g_{ij}-\delta_j{}^h g_{ik})+\underset{2}{c'}\,(\phi_k{}^h g_{ij}-\phi_j{}^h g_{ik}) \qquad \textit{for } n=1\,,$$

where k's and c's are constants.

If moreover $R^h{}_{[ijk]}=0$, *then we have in* (1.d)

(1.e) $$2\underset{3}{c}+\underset{5}{c}=0\,, \qquad 2\underset{4}{c}+\underset{6}{c}=0\,.$$

If $g_{ai}R^a{}_{hjk}+g_{ha}R^a{}_{ijk}=0$, *then we have in* (1.d)

(1.f) $$\underset{2}{c}+\underset{3}{c}=0\,, \qquad \underset{5}{c}=0 \qquad \textit{for } n>1\,,$$

and in (1.d′)

(1.f′) $$\underset{2}{c'}=0 \qquad \textit{for } n=1\,.$$

If $\phi_i{}^a R^h{}_{ajk}-\phi_a{}^h R^a{}_{ijk}=0$, *then we have in* (1.d)

(1.g) $$\underset{2}{c}-\underset{3}{c}=0\,, \qquad \underset{1}{c}+\underset{4}{c}=0\,.$$

PROOF. (1.a) No non-zero vector is invariant under the unitary group of R^{2n}.

(1.b) If h_{ij} is invariant under a group of linear transformations, then so are the symmetric part $h_{(ij)}$ and the anti-symmetric part $h_{[ij]}$ of h_{ij}. We study two cases.

In case h_{ij} is symmetric, we consider a constant $\underset{1}{k}$ such that, for one particular unit vector e_0,

$$h_{ij}e_0{}^i e_0{}^j=\underset{1}{k}\,.$$

The tensor $h_{ij}-\underset{1}{k}g_{ij}$ is invariant under the unitary group. For any unit vector e, those unitary transformations which send e_0 to e transform $(h_{ij}-\underset{1}{k}g_{ij})e_0{}^i e_0{}^j$ which is equal to zero to $(h_{ij}-\underset{1}{k}g_{ij})e^i e^j$. Hence $(h_{ij}-\underset{1}{k}g_{ij})e^i e^j=0$ for any unit vector e. This implies

$$h_{ij}=\underset{1}{k}g_{ij}\,.$$

In case h_{ij} is anti-symmetric, we consider the tensor $f_{ij}=\phi_i{}^k h_{jk}$. The tensor f_{ij} is invariant under the unitary group and symmetric. In fact, since J is a unitary transformation, we have

$$f_{ji}=\phi_j{}^{j'}\phi_i{}^{i'}f_{j'i'}=\phi_j{}^{j'}\phi_i{}^{i'}\phi_{j'}{}^k h_{i'k}=-\phi_i{}^{i'}h_{i'j}=\phi_i{}^k h_{jk}=f_{ij}\,.$$

Hence there exists a constant k such that we have $f_{ij}=\underset{2}{k}g_{ij}$. From this we get

$$h_{ij}=\phi_i{}^k f_{kj}=\underset{2}{k}\phi_i{}^k g_{kj}=\underset{2}{k}\phi_{ij}\,.$$

Since the tensor h_{ij} is the sum of its symmetric part and antisymmetric part, (1.b) is proved.

(1.c) Let $T^h{}_{ij}$ be invariant under the unitary group. Since the linear transformation $-\delta_j{}^i$ is a unitary transformation, we have

$$T^h{}_{ij}=(-\delta^h_{h'})(-\delta^{i'}_i)(-\delta^{j'}_j)T^{h'}{}_{i'j'}=-T^h{}_{ij}\,,$$

and hence

$$T^h{}_{ij}=0\,.$$

3. To prove (1.d) and (1.d′) we prove two lemmas, and this section is devoted to the proof of the lemmas. Let $R^h{}_{ijk}=-R^h{}_{ikj}$ be invariant under the unitary group. The tensor $R^h{}_{ijk}$ is determined by the bilinear mapping

$$(v^i, f^{jk}) \rightarrow R^h{}_{ijk}v^i f^{jk}\,,$$

in short

$$(v, f) \rightarrow R(v, f)$$

of pairs of a vector and a bivector, to vectors. Since the tensor $R^h{}_{ijk}$ is invariant under the unitary group, the vector $R(v,f)$ is invariant under unitary transformations leaving invariant the vector v and the bivector f.

We take an orthonormal base $\{e_1,\cdots,e_n, Je_1,\cdots,Je_n\}$ and consider in the proof of the lemmas and of (1.d) and (1.d′) components of tensors with respect to this base. We denote $\alpha+n$ by $\bar{\alpha}$ and Je_α by $e_{\bar{\alpha}}$. We say that α and $\bar{\alpha}$ belong to the same class. The numbers from 1 to $2n$ separate into n classes.

LEMMA 1. *The component $R^h{}_{ijk}$ vanishes unless h, i, j, k belong to the same class or two of them belong to a class and others do to another one.*

PROOF. We distinguish two cases A and B.

Case A. j and k are in the same class.

Subcase 1. i, j and k are in the same class. The vector $R(e_i, e_j\wedge e_k) =R^h{}_{ijk}e_h$ is the sum of two orthogonal component vectors in the holomorphic section determined by e_i and in its orthogonal complement. $R(e_i, e_j\wedge e_k)$ is invariant under those unitary transformations which induce the identity transformation in the holomorphic section. Therefore the vector $R(e_i, e_j\wedge e_k)$ coincides with its component vector in the holomorphic section. It follows from this that, if the class of h is different from that of i, we have $R^h{}_{ijk}=0$.

Subcase 2. The class of i is different from that of j. Then the vector $R(e_i, e_j\wedge e_k)$ is invariant under those unitary transformations which fix the vector e_i and leave invariant the holomorphic section determined by e_j. Thus we have $R^h{}_{ijk}=0$ unless the class of h is that of i.

Case B. j and k are in different classes.

Subcase 1. The class of i is either that of j or that of k. Unitary transformations leaving fixed e_j and e_k induce unitary transformations in the orthogonal complement to the subspace spanned by the set of vectors $\{e_j, Je_j, e_k, Je_k\}$. From this we see $R^h{}_{ijk}=0$ if the class of h is different from those of j and k. That is, the vector $R(e_i, e_j\wedge e_k)$ is the sum of two component vectors in the holomorphic section determined by e_j and in the holomorphic section determined by e_k. Without loss of generality we assume the class of i to be that of j. Apply to $R(e_i, e_j\wedge e_k)$ those unitary transformations which send e_j to e_j and e_k to $-e_k$. Then the component vector of $R(e_i, e_j\wedge e_k)$ in the holomorphic section determined by e_j remains fixed, but the component vector in the holomorphic section determined by e_k reverses its direction. On the other hand $R(e_i, e_j\wedge e_k)$ transforms into $-R(e_i, e_j\wedge e_k)$. It follows from this that the component vector in the holomorphic section determined by e_j is zero, i.e. $R^h{}_{ijk}=0$ if the class of h is that of i.

Subcase 2. The class of i is neither that of j nor that of k. The vector $R(e_i, e_j\wedge e_k)$ is invariant under unitary transformations leaving invariant e_i and $e_j\wedge e_k$. We see that $R(e_i, e_j\wedge e_k)$ is in the holomorphic section determined by e_i. We next apply to $R(e_i, e_j\wedge e_k)$ any unitary transformation sending e_i to $-e_i, e_j$ to $-e_j$ and e_k to e_k. Then the component vector of $R(e_i, e_j\wedge e_k)$ in the holomorphic section determined by e_i reverses its direction, while the vector $R(e_i, e_j\wedge e_k)$ remains fixed. Hence we have $R(e_i, e_j\wedge e_k)=0$. This completes the proof of Lemma 1.

LEMMA 2. *We have (repeated indices being not summed)*

$$R^h{}_{iih}=R^{h'}{}_{i'i'h'}\,,\qquad R^{\bar\alpha}{}_{ii\alpha}=-R^{\alpha}{}_{ii\bar\alpha}=R^{\bar\alpha'}{}_{i'i'\alpha'}\,,$$

$$R^h{}_{\alpha\bar\alpha h}=-R^h{}_{\bar\alpha\alpha h}=R^{h'}{}_{\alpha'\bar\alpha'h'}\,,\qquad R^{\bar\beta}{}_{\alpha\bar\alpha\beta}=-R^{\bar\beta}{}_{\bar\alpha\alpha\beta}=-R^{\beta}{}_{\alpha\bar\alpha\bar\beta}$$

$$=R^{\beta}{}_{\bar\alpha\alpha\bar\beta}=R^{\bar\beta'}{}_{\alpha'\bar\alpha'\beta'}\,,$$

$$R^h{}_{h\alpha\bar\alpha}=R^{h'}{}_{h'\alpha'\bar\alpha'}\,,\qquad R^{\bar\alpha}{}_{\alpha\beta\bar\beta}=-R^{\alpha}{}_{\bar\alpha\beta\bar\beta}=R^{\bar\alpha'}{}_{\alpha'\beta'\bar\beta'}\,,$$

$$R^{\alpha}{}_{\alpha\alpha\bar\alpha}=-R^{\bar\alpha}{}_{\bar\alpha\bar\alpha\alpha}=R^{\alpha'}{}_{\alpha'\alpha'\bar\alpha'}\,,\qquad R^{\bar\alpha}{}_{\alpha\alpha\bar\alpha}=R^{\alpha}{}_{\bar\alpha\bar\alpha\alpha}=R^{\bar\alpha'}{}_{\alpha'\alpha'\bar\alpha'}\qquad \textit{for any } n,$$

and

$$R^{\alpha}{}_{\alpha\alpha\bar\alpha}=-R^{\bar\beta}{}_{\alpha\alpha\beta}-R^{\beta}{}_{\alpha\bar\alpha\beta}+R^{\alpha}{}_{\alpha\beta\bar\beta}\,,$$

$$R^{\bar\alpha}{}_{\alpha\alpha\bar\alpha}=R^{\beta}{}_{\alpha\alpha\beta}-R^{\bar\beta}{}_{\alpha\bar\alpha\beta}+R^{\bar\alpha}{}_{\alpha\beta\bar\beta}\qquad \textit{for } n>1,$$

where different letters in the indices of components of the tensor belong to different classes.

PROOF. To prove the equations of the type $R^h{}_{ijk}=R^{h'}{}_{i'j'k'}$, we consider the vector $R(e_i, e_j\wedge e_k)=R^h{}_{ijk}e_h+R^{\bar h}{}_{ijk}Je_h$, where $R^{\bar h}{}_{ijk}$ denotes $-R^{\alpha}{}_{ijk}$ if $h=\bar\alpha$. Unitary transformation which sends e_i to $e_{i'}, e_j$ to $e_{j'}, e_k$ to $e_{k'}$ and e_h to $e_{h'}$

transforms the vector $R(e_i, e_j \wedge e_k)$ into $R(e_{i'}, e_{j'} \wedge e_{k'}) = R^{h'}{}_{i'j'k'} e_{h'} + R^{\bar{h}'}{}_{i'j'k'} J e_{h'}$ and $R^h{}_{ijk} e_h + R^{\bar{h}}{}_{ijk} J e_h$ into $R^h{}_{ijk} e_{h'} + R^{\bar{h}}{}_{ijk} J e_{h'}$. Hence we get $R^h{}_{ijk} = R^{h'}{}_{i'j'k'}$.

We consider the relation $R(e_i, e_i \wedge e_\alpha) = R^\alpha{}_{ii\alpha} e_\alpha + R^{\bar\alpha}{}_{ii\alpha} e_{\bar\alpha}$. Those unitary transformations which send e_i to e_i, e_α to $e_{\bar\alpha}$ and hence $e_{\bar\alpha}$ to $-e_\alpha$ transform the vector $R(e_i, e_i \wedge e_\alpha)$ to the vector $R(e_i, e_i \wedge e_{\bar\alpha}) = R^\alpha{}_{ii\bar\alpha} e_\alpha + R^{\bar\alpha}{}_{ii\bar\alpha} e_{\bar\alpha}$ and the vector $R^\alpha{}_{ii\alpha} e_\alpha + R^{\bar\alpha}{}_{ii\alpha} e_{\bar\alpha}$ to the vector $R^\alpha{}_{ii\alpha} e_{\bar\alpha} - R^{\bar\alpha}{}_{ii\alpha} e_\alpha$. From this we obtain $R^{\bar\alpha}{}_{ii\alpha} = -R^\alpha{}_{ii\bar\alpha}$.

We next consider the vector $R(e_\alpha, e_{\bar\alpha} \wedge e_h)$ which is equal to $R^h{}_{\alpha\bar\alpha h} e_h + R^{\bar h}{}_{\alpha\bar\alpha h} J e_h$. Those unitary transformations which send e_α to $e_{\bar\alpha}, e_{\bar\alpha}$ to $-e_\alpha$ and e_h to e_h transform $R(e_\alpha, e_{\bar\alpha} \wedge e_h)$ to $-R(e_{\bar\alpha}, e_\alpha \wedge e_h) = -R^h{}_{\bar\alpha\alpha h} e_h - R^{\bar h}{}_{\bar\alpha\alpha h} J e_h$ which on the other hand is equal to $R^h{}_{\alpha\bar\alpha h} e_h + R^{\bar h}{}_{\alpha\bar\alpha h} J e_h$. This implies $R^h{}_{\alpha\bar\alpha h} = -R^h{}_{\bar\alpha\alpha h}$, $R^{\bar\beta}{}_{\alpha\bar\alpha\beta} = -R^{\bar\beta}{}_{\bar\alpha\alpha\beta}$, $R^\beta{}_{\alpha\bar\alpha\bar\beta} = -R^\beta{}_{\bar\alpha\alpha\bar\beta}$.

From the relation $R(e_\alpha, e_\beta \wedge e_{\bar\beta}) = R^\alpha{}_{\alpha\beta\bar\beta} e_\alpha + R^{\bar\alpha}{}_{\alpha\beta\bar\beta} e_{\bar\alpha}$ together with a unitary transformation which sends e_β to e_β, e_α to $e_{\bar\alpha}$ and hence $e_{\bar\alpha}$ to $-e_\alpha$, we find the equation $R^{\bar\alpha}{}_{\alpha\beta\bar\beta} = -R^\alpha{}_{\bar\alpha\beta\bar\beta}$.

From the relation $R(e_\alpha, e_\alpha \wedge e_{\bar\alpha}) = R^\alpha{}_{\alpha\alpha\bar\alpha} e_\alpha + R^{\bar\alpha}{}_{\alpha\alpha\bar\alpha} e_{\bar\alpha}$ together with such unitary transformation which sends e_α to $e_{\bar\alpha}$ and hence $e_{\bar\alpha}$ to $-e_\alpha$, we get the equation $R^\alpha{}_{\alpha\alpha\bar\alpha} = -R^{\bar\alpha}{}_{\bar\alpha\bar\alpha\alpha}$ and $R^{\bar\alpha}{}_{\alpha\alpha\bar\alpha} = R^\alpha{}_{\bar\alpha\bar\alpha\alpha}$.

We now assume that $n > 1$ and consider the relation $R(e_\alpha, e_\alpha \wedge e_{\bar\alpha}) = R^\alpha{}_{\alpha\alpha\bar\alpha} e_\alpha + R^{\bar\alpha}{}_{\alpha\alpha\bar\alpha} e_{\bar\alpha}$. Let us apply to the above relation those unitary transformations which send e_α to $\frac{1}{\sqrt{2}}(e_\alpha + e_\beta)$ and e_β to $\frac{1}{\sqrt{2}}(e_{\bar\alpha} - e_{\bar\beta})$. Then $R^\alpha{}_{\alpha\alpha\bar\alpha} e_\alpha + R^{\bar\alpha}{}_{\alpha\alpha\bar\alpha} e_{\bar\alpha}$ transforms into $\frac{1}{\sqrt{2}}(R^\alpha{}_{\alpha\alpha\bar\alpha} e_\alpha + R^{\bar\alpha}{}_{\alpha\alpha\bar\alpha} e_{\bar\alpha} + R^\alpha{}_{\alpha\alpha\bar\alpha} e_\beta + R^{\bar\alpha}{}_{\alpha\alpha\bar\alpha} e_{\bar\beta})$, and $R(e_\alpha, e_\alpha \wedge e_{\bar\alpha})$ into

$\frac{1}{2\sqrt{2}} R(e_\alpha + e_\beta, (e_\alpha + e_\beta) \wedge (e_{\bar\alpha} + e_{\bar\beta})) = \frac{1}{2\sqrt{2}}(R(e_\alpha, e_\alpha \wedge e_{\bar\alpha}) + R(e_\alpha, e_\alpha \wedge e_{\bar\beta}) + R(e_\alpha, e_\beta \wedge e_{\bar\alpha}) + R(e_\alpha, e_\beta \wedge e_{\bar\beta}) + R(e_\beta, e_\alpha \wedge e_{\bar\alpha}) + R(e_\beta, e_\alpha \wedge e_{\bar\beta}) + R(e_\beta, e_\beta \wedge e_{\bar\alpha}) + R(e_\beta, e_\beta \wedge e_{\bar\beta}))$. Taking up the coefficients of e_α and of $e_{\bar\alpha}$ and remembering results of lemma 1, we find $R^\alpha{}_{\alpha\alpha\bar\alpha} = \frac{1}{2}(R^\alpha{}_{\alpha\alpha\bar\alpha} + R^\alpha{}_{\alpha\beta\bar\beta} + R^\alpha{}_{\beta\alpha\bar\beta} + R^\alpha{}_{\beta\beta\bar\alpha})$ and $R^{\bar\alpha}{}_{\alpha\alpha\bar\alpha} = \frac{1}{2}(R^{\bar\alpha}{}_{\alpha\alpha\bar\alpha} + R^{\bar\alpha}{}_{\alpha\beta\bar\beta} + R^{\bar\alpha}{}_{\beta\alpha\bar\beta} + R^{\bar\alpha}{}_{\beta\beta\bar\alpha})$. Equalities already obtained yield the announced equations

$$R^\alpha{}_{\alpha\alpha\bar\alpha} = -R^{\bar\beta}{}_{\alpha\alpha\beta} - R^\beta{}_{\alpha\bar\alpha\beta} + R^\alpha{}_{\alpha\beta\bar\beta}$$

and

$$R^{\bar\alpha}{}_{\alpha\alpha\bar\alpha} = R^\beta{}_{\alpha\alpha\beta} - R^{\bar\beta}{}_{\alpha\bar\alpha\beta} + R^{\bar\alpha}{}_{\alpha\beta\bar\beta} \qquad \text{for } n > 1\,.$$

This completes the proof of lemma 2.

4. On the basis of the results of lemmas 1 and 2 we now proceed to the proof of (1.d) and (1.d′). We define six tensors $\underset{a}{R}{}^h{}_{ijk} = -\underset{a}{R}{}^h{}_{ikj}$ $(a = 1, \cdots, 6)$ invariant under the unitary group as follows

$$\underset{1}{R}{}^h{}_{ijk} = \delta_k{}^h g_{ij} - \delta_j{}^h g_{ik}\,, \qquad R^h{}_{ijk} = \phi_k{}^h g_{ij} - \phi_j{}^h g_{ik}\,,$$

$$\underset{3}{R}{}^{h}{}_{ijk}=\delta_k{}^h\phi_{ij}-\delta_j{}^h\phi_{ik}\,,\qquad \underset{4}{R}{}^{h}{}_{ijk}=\phi_k{}^h\phi_{ij}-\phi_j{}^h\phi_{ik}\,,$$
$$\underset{5}{R}{}^{h}{}_{ijk}=\delta_i{}^h\phi_{jk}\,,\qquad \underset{6}{R}{}^{h}{}_{ijk}=\phi_i{}^h\phi_{jk}\,.$$

The components of tensors g_{ij}, ϕ_{ij} and $\phi_j{}^i$ with respect to the orthonormal base $\{e_1,\cdots,e_n,e_{\bar 1},\cdots,e_{\bar n}\}$ under consideration are the elements of i-th row and j-th column of the following three matrices respectively

$$\begin{pmatrix} E_n & 0 \\ 0 & E_n \end{pmatrix},\quad \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix},\quad \begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}.$$

We see immediately

$$\underset{1}{R}{}^{2}{}_{112}=1,\quad \underset{a}{R}{}^{2}{}_{112}=0 \quad\text{for}\quad a\neq 1,\qquad \underset{2}{R}{}^{\bar 2}{}_{112}=1,\quad \underset{a}{R}{}^{\bar 2}{}_{112}=0 \quad\text{for}\quad a\neq 2,$$
$$\underset{3}{R}{}^{2}{}_{1\bar 12}\neq 1,\quad \underset{a}{R}{}^{2}{}_{1\bar 12}=0 \quad\text{for}\quad a\neq 3,\qquad \underset{4}{R}{}^{\bar 2}{}_{1\bar 12}=1,\quad \underset{a}{R}{}^{\bar 2}{}_{1\bar 12}=0 \quad\text{for}\quad a\neq 4,$$
$$\underset{5}{R}{}^{1}{}_{12\bar 2}=1,\quad \underset{a}{R}{}^{1}{}_{12\bar 2}=0 \quad\text{for}\quad a\neq 5,\qquad \underset{6}{R}{}^{\bar 1}{}_{12\bar 2}=1,\quad \underset{a}{R}{}^{\bar 1}{}_{12\bar 2}=0 \quad\text{for}\quad a\neq 6.$$

If we define the tensor $T^h{}_{ijk}=-T^h{}_{ikj}$ for $n>1$ as follows

$$T^h{}_{ijk}=R^h{}_{ijk}-R^2{}_{112}\underset{1}{R}{}^h{}_{ijk}-R^{\bar 2}{}_{112}\underset{2}{R}{}^h{}_{ijk}-R^2{}_{1\bar 12}\underset{3}{R}{}^h{}_{ijk}$$
$$-R^{\bar 2}{}_{1\bar 12}\underset{4}{R}{}^h{}_{ijk}-R^1{}_{11\bar 2}\underset{5}{R}{}^h{}_{ijk}-R^{\bar 1}{}_{12\bar 2}\underset{6}{R}{}^h{}_{ijk}\,,$$

then the tensor is invariant under the unitary group and we have

$$T^2{}_{112}=T^{\bar 2}{}_{112}=T^2{}_{1\bar 12}=T^{\bar 2}{}_{1\bar 12}=T^1{}_{12\bar 2}=T^{\bar 1}{}_{12\bar 2}=0\,.$$

This together with lemmas 1 and 2 shows $T^h{}_{ijk}=0$ for $n>1$, i.e. there exist six constants $\underset{1}{c},\underset{2}{c},\cdots,\underset{6}{c}$ such that

$$R^h{}_{ijk}=\underset{1}{c}\,\underset{1}{R}{}^h{}_{ijk}+\underset{2}{c}\,\underset{2}{R}{}^h{}_{ijk}+\cdots+\underset{6}{c}\,\underset{6}{R}{}^h{}_{ijk}\qquad \text{for } n>1\,.$$

From the definition of $\underset{1}{R}{}^h{}_{ijk}$ and $\underset{2}{R}{}^h{}_{ijk}$ we see immediately

$$\underset{2}{R}{}^1{}_{11\bar 1}=0,\quad \underset{2}{R}{}^1{}_{11\bar 1}=-1,\quad \underset{2}{R}{}^{\bar 1}{}_{11\bar 1}=1,\quad \underset{2}{R}{}^{\bar 1}{}_{11\bar 1}=0\,.$$

If we define $T^h{}_{ijk}$ for $n=1$ as follows

$$T^h{}_{ijk}=R^h{}_{ijk}-R^{\bar 1}{}_{11\bar 1}\underset{1}{R}{}^h{}_{ijk}+R^1{}_{11\bar 1}\underset{2}{R}{}^h{}_{ijk}\,,$$

then we get $T^h{}_{ijk}=0$ from lemma 2, i.e. there exist two constants c', c' such that

$$R^h{}_{ijk}=\underset{1}{c'}\,\underset{1}{R}{}^h{}_{ijk}+\underset{2}{c'}\,\underset{2}{R}{}^h{}_{ijk}\,.$$

This completes the proof of (1.d) and (1.d′).

(1.e) By alternation we obtain from (1.d)

$$3R^h{}_{[ijk]}=(2\underset{3}{c}+\underset{5}{c})(\delta_k{}^h\phi_{ij}+\delta_j{}^h\phi_{ki}+\delta_i{}^h\phi_{jk})$$
$$+(2\underset{4}{c}+\underset{6}{c})(\phi_k{}^h\phi_{ij}+\phi_j{}^h\phi_{ki}+\phi_i{}^h\phi_{jk})\,.$$

Since the tensors $\delta_k{}^h\phi_{ij}+\delta_j{}^h\phi_{ki}+\delta_i{}^h\phi_{jk}$ and $\phi_k{}^h\phi_{ij}+\phi_j{}^h\phi_{ki}+\phi_i{}^h\phi_{jk}$ are linearly independent, the condition $R^h{}_{[ijk]}=0$ implies $2\underset{3}{c}+\underset{5}{c}=0$ and $2\underset{4}{c}+\underset{6}{c}=0$.

(1.f) From the relation (1.d) we get

$$g_{ai}R^a{}_{hjk}+g_{ha}R^a{}_{ijk}=(\underset{2}{c}+\underset{3}{c})(\phi_{ki}g_{hj}-\phi_{ji}g_{hk}+\phi_{kh}g_{ij}-\phi_{jh}g_{ik})+2\underset{5}{c}\,g_{hi}\phi_{jk}\,.$$

The tensors $\phi_{ki}g_{hj}-\phi_{ji}g_{hk}+\phi_{kh}g_{ij}-\phi_{jh}g_{ik}$ and $g_{hi}\phi_{jk}$ are linearly independent. Therefore the condition $g_{ai}R^a{}_{hjk}+g_{ha}R^a{}_{ijk}=0$ implies $\underset{2}{c}+\underset{3}{c}=0$ and $\underset{5}{c}=0$.

(1.f′) From the relation (1.d′) we have

$$g_{ai}R^a{}_{hjk}+g_{ha}R^a{}_{ijk}=\underset{2}{c}'(\phi_{ki}g_{hj}-\phi_{ji}\,g_{hk}+\phi_{kh}\,g_{ij}-\phi_{jh}g_{ik})\,.$$

The condition $g_{ai}R^a{}_{hjk}+g_{ha}R^a{}_{ijk}=0$ yields $\underset{2}{c}'=0$.

(1.g) From the relation (1.d) we obtain

$$\begin{aligned}\phi_i{}^aR^h{}_{ajk}-\phi_a{}^hR^a{}_{ijk}=&(\underset{2}{c}-\underset{3}{c})(\delta_k{}^hg_{ij}-\delta_j{}^hg_{ik}+\phi_k{}^h\phi_{ij}-\phi_j{}^h\phi_{ik})\\&-(\underset{1}{c}+\underset{4}{c})(\phi_k{}^hg_{ij}-\phi_j{}^hg_{ik}-\delta_k{}^h\phi_{ij}+\delta_j{}^h\phi_{ik})\,.\end{aligned}$$

Hence the condition $\phi_i{}^aR^h{}_{ajk}-\phi_a{}^hR^a{}_{ijk}=0$ implies $\underset{2}{c}-\underset{3}{c}=0$, $\underset{1}{c}+\underset{4}{c}=0$, which proves (1.g).

This completes the proof of all the assertion of theorem 1.

If a tensor $R^h{}_{ijk}=-R^h{}_{ikj}$ which is invariant under the unitary group satisfies the conditions $R^h{}_{[ijk]}=0$, $g_{ai}R^a{}_{hjk}+g_{ha}R^a{}_{ijk}=0$ and $\phi_i{}^aR^h{}_{ajk}-\phi_a{}^hR^a{}_{ijk}=0$, then theorem 1 shows that the tensor has the form

$$R^h{}_{ijk}=\underset{1}{c}(\delta_k{}^hg_{ij}-\delta_j{}^hg_{ik}-\phi_k{}^h\phi_{ij}+\phi_j{}^h\phi_{ik}+2\phi_i{}^h\phi_{jk})\qquad\text{for } n>1\,.$$

5. As an application of (1.d) we consider the tensor

$$e_{hijk}=\sqrt{g}\;\varepsilon_{hijk}$$

for $2n=4$, where g is the determinant of g_{ij} and ε_{hijk} is equal to $1, -1, 0$ according as $hijk$ is an even, odd or no permutation of 1, 2, 3, 4 respectively. If we set

$$R^h{}_{ijk}=g^{hr}e_{rijk}\,,$$

then $R^h{}_{ijk}=-R^h{}_{ikj}$ is invariant under the unitary group. We have with respect to an orthonormal base $\{e_1, e_2, Je_1, Je_2\}$

$$R^2{}_{112}=0,\quad R^{\bar 2}{}_{112}=0,\quad R^2{}_{1\bar 1 2}=0,\quad R^{\bar 2}{}_{1\bar 1 2}=e_{\bar 2 1\bar 1 2}=1\,,$$
$$R^1{}_{12\bar 2}=0,\quad R^{\bar 1}{}_{12\bar 2}=e_{\bar 1 12\bar 2}=1\,.$$

From this we have

$$g^{hr}e_{rijk}=\phi_k{}^h\phi_{ij}-\phi_j{}^h\phi_{ik}+\phi_i{}^h\phi_{jk}\,,$$

and hence ([4], Chap. VI)

$$e_{hijk}=\phi_{kh}\phi_{ij}+\phi_{jh}\phi_{ki}+\phi_{ih}\phi_{jk}\,.$$

Examining the proof of (1.c), we find the following

LEMMA 3. *Let M be a differentiable manifold and T be a tensor field on M of type (m, n). We assume that the value of the tensor field at a point of M is invariant under the linear transformation $-\delta_j^i$ of the tangent space at the point. Then the value of T at the point is zero provided $m+n$ is an odd number.*

THEOREM 2. *Let M be a $2n$-dimensional almost complex manifold and ϕ be the tensor field defining the almost complex structure ($\phi^2=-$identity). Let G be a transitive Lie group of transformations of M leaving the almost complex structure invariant. If at a point p of M and hence at every point of M the linear isotropy group contains the linear transformation $-\delta_j^i$, then the manifold M is complex analytic. There exists at most one linear connection on M invariant under the group G.*

PROOF. Since the tensor field ϕ is invariant under the group G, so is Nijenhuis' tensor field $N^h{}_{ij}$ of ϕ. From lemma 3 we find $N^h{}_{ij}=0$. The manifold M is real analytic as a homogeneous space of Lie group. Hence the manifold M is complex analytic. Let $\Gamma_i{}^h{}_j$ and $\Gamma'_i{}^h{}_j$ be two linear connections on M invariant under the group G. The tensor $T^h{}_{ij}=\Gamma_i{}^h{}_j-\Gamma'_i{}^h{}_j$ is invariant under the group G. Consequently we have $T^h{}_{ij}=0$. Hence theorem 2 is proved.

THEOREM 3. *Let M be a connected $2n$-dimensional almost Hermitian space. Let G be an (n^2+2n)-dimensional effective group of automorphisms of M. Then G is transitive on M and a linear connection on M invariant under the group G is Levi-Civita's connection. The manifold M is a Kaehlerian manifold of constant holomorphic curvature.*

The linear group of isotropy at any point coincides with the unitary group of the tangent space at the point. Since Levi-Civita's connection is invariant under the group G, theorem 2 shows that a linear connection invariant under the group G is Levi-Civita's connection. Other parts of the theorem were already proved by S. Ishihara [2].

The curvature tensor of the manifold has the form [5]

$$R^h{}_{ijk}=\frac{k}{4}\,(\delta_k^h g_{ij}-\delta_j^h g_{ik}-\phi_k{}^h\phi_{ij}+\phi_j{}^h\phi_{ik}+2\phi_i{}^h\phi_{jk})\,,$$

where the scalar k is an absolute constant.

THEOREM 4. *Let M be an almost Hermitian manifold and $\Gamma_i{}^h{}_j$ be a linear connection in which the almost complex structure $\phi_j{}^i$ has null covariant derivative. If the manifold M admits a group of affine transformations and the isotropy group in the tangent space at any point contains the unitary group, then the torsion tensor is zero and the curvature tensor field has the form*

$$R^h{}_{ijk}=(\underset{1}{c}\delta_a{}^h+\underset{2}{c}\phi_a{}^h)(\delta_k{}^a g_{ij}-\delta_j{}^a g_{ik}-\phi_k{}^a\phi_{ij}+\phi_j{}^a\phi_{ik}+2\phi_i{}^a\phi_{jk})$$

with scalars $\underset{1}{c}$ *and* $\underset{2}{c}$.

PROOF. The value of the torsion tensor field at any point is invariant under the linear isotropy group at the point which contains the unitary group. It follows from theorem 1 (1.c) that the torsion tensor is zero. We see from (1.e) and (1.g) that the curvature tensor field has the form

$$
\begin{aligned}
R^h{}_{ijk} &= \underset{1}{c}(\delta_k{}^h g_{ij} - \delta_j{}^h g_{ik} - \phi_k{}^h \phi_{ij} + \phi_j{}^h \phi_{ik} + 2\phi_i{}^h \phi_{jk}) \\
&\quad + \underset{2}{c}(\phi_k{}^h g_{ij} - \phi_j{}^h g_{ik} + \delta_k{}^h \phi_{ij} - \delta_j{}^h \phi_{ik} - 2\delta_i{}^h \phi_{jk}) \\
&= (\underset{1}{c}\delta_a{}^h + \underset{2}{c}\phi_a{}^h)(\delta_k{}^a g_{ij} - \delta_j{}^a g_{ik} - \phi_k{}^a \phi_{ij} + \phi_j{}^a \phi_{ik} + 2\phi_i{}^a \phi_{jk})
\end{aligned}
$$

with scalars $\underset{1}{c}$ and $\underset{2}{c}$.

THEOREM 5. *Let M be a Kaehlerian manifold with Levi-Civita's connection. We assume that the homogeneous holonomy group of M at a point p of M is the unitary group. If the curvature tensor field has null covariant derivative, then the manifold is of constant holomorphic curvature.*

PROOF. Since the curvature tensor field has null covariant derivative, the value of the tensor field at the point p is invariant under the homogeneous holonomy group which is the unitary group. It follows from theorem 1 (1.e), (1.f) and (1.g) that the curvature tensor field has the form

$$R^h{}_{ijk} = c(\delta_k{}^h g_{ij} - \delta_j{}^h g_{ik} - \phi_k{}^h \phi_{ij} + \phi_j{}^h \phi_{ik} + 2\phi_i{}^h \phi_{jk}) \qquad \text{for } n>1$$

at any point of the manifold. Since the tensor field in the parentheses is parallel, the scalar c is absolute constant, i.e. the manifold is of constant holomorphic curvature. Hence the theorem is proved.

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