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On the uniform continuity of Wiener process

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It is the purpose of this note to ameliorate Lévy's result concerning the uniform continuity of Wiener process. Let $\varphi(h)$ be a continuous and monotone increasing function which tends to zero with h. After P. Lévy we say that a function $f(t)$ verifies "*Hölder's weak condition*" relative to $\varphi(t)$, if there exists a positive number ε such that $|t'-t| = h \leqq \varepsilon$ yields the relation

$$|f(t')-f(t)| \leqq \varphi(h).$$

Let us put

$$\varphi_c(h) = \{h(2\log 1/h + c\log\log 1/h)\}^{1/2}.$$

Then we obtain the following theorem.

THEOREM. *If $c>5$, Wiener process $\{X(t,\omega)\,;\ 0\leqq t\leqq 1\}$*[1] *verifies "Hölder's weak condition" relative to $\varphi_c(t)$ and if $c<-1$, it does not verify the condition, with probability one.*

PROOF. Let us put

$$\alpha(h) = Pr\{|\triangle X(t)| > \varphi_c(h)\}, \tag{1}$$

where $\triangle X(t)$ is the difference of $X(t+h)$ and $X(t)$. Since $\triangle X(t)$ is a normal random variable satisfying the conditions $E(\triangle X(t))=0$ and $V(\triangle X(t))=h$[2] we have the following asymptotic relation

$$\alpha(h)/h \sim (1/\pi)^{1/2}(\log 1/h)^{-(c+1)/2}. \tag{2}$$

If $c<-1$, we obtain

$$\alpha(h)/h \to \infty \quad \text{as} \quad h \to 0, \tag{3}$$

and therefore

$$1-(1-\alpha(1/n))^n \to 1 \quad \text{as} \quad n \to \infty. \tag{4}$$

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1) ω is the probability parameter.

2) E and V denote the expectation and the variance respectively.

The left side of (4) is equal to the probability that there exists at least one k ($k=0, 1, 2, \cdots, n-1$), such that $|X((k+1)/n)-X(k/n)|$ is larger than $\varphi_c(1/n)$. So (4) proves the second part of the theorem.

To prove the first part of the theorem we put $c=5+\varepsilon$ with a positive number ε. We shall define $\alpha_{n,m,l}$ as the probability that the event

$$|X((m+l)/2^n)-X(l/2^n)|>\varphi_c(m/2^n) \tag{5}$$

holds,

$$n=1, 2, 3, \cdots, \quad m=1, 2, \cdots, n, \quad l=0, 1, 2, \cdots, 2^n-1 .$$

Since $\alpha_{n,m,l}$ is independent of l, we may omit l hereafter. Then we have for a sufficiently large n

$$\alpha_{n,1}\leqq\alpha_{n,2}\leqq \cdots \leqq\alpha_{n,n}=\alpha(n/2^n)=O(1)/(n^{2+\varepsilon/2}2^n) \tag{6}$$

and

$$\sum_n n2^n\alpha(n/2^n)=O(1)\sum_n n^{-(1+\varepsilon/2)} . \tag{7}$$

Since $\alpha(h)$ is monotone increasing function for small $h\,(>0)$ the n-th term of the left side of (7) is not less than the probability that there exists at least one pair (m, l) for which (5) holds. Since the right side of (7) converges, we can see, by Borel-Cantelli's theorem, that there exists $n(\omega)$ such that $n>n(\omega)$ implies the following inequalities

$$|X((m+l)/2^n)-X(l/2^n)|\leqq\varphi_c(m/2^n) , \tag{8}$$

$$m=1, 2, \cdots, n ,$$

$$l=0, 1, 2, \cdots, 2^n-1 ,$$

with probability one. Thus the first part of the theorem is true for the special cases that both t and t' take the values of the form $k/2^n$.

If $t=k/2^n$ and $t<t'<t+1/2^n$ ($n>n(\omega)\geqq 2$), we may write $t'-t$ as

$$t'-t=\sum_{\nu=1}^{\infty}\varepsilon_\nu/2^{n+\nu} ,$$

where $\varepsilon_\nu=0$ or 1. Using the monotony of $\varphi_c(h)$ for small $h\,(>0)$, by (8), we have

$$|X(t')-X(t)|\leqq\sum_{\nu=1}^{\infty}\varepsilon_\nu\varphi_c(1/2^{n+\nu}) \tag{9}$$

$$\leqq \varphi_c(1/2^{n+l}) \sum_{\nu=1}^{\infty} (2\nu/2^{\nu})^{1/2}$$

$$\leqq C\varphi_c(t'-t),$$

where l is the smallest ν for which $\varepsilon_\nu=1$ and C is a constant. In the same way we can see that (9) holds also for the cases of $t'=k/2^n$ and $t'-1/2^n<t<t'$ $(n>n(\omega)\geqq 2)$.

For any pair of (t, t'), if $(t_\nu-t)$ is sufficiently small, there exists n such that

$$(10) \qquad (n+1)/2^{n+1}<t'-t\leqq n/2^n, \qquad n>n(\omega)\geqq 2.$$

Then we may take t_1 and t_1' as follows:

$$(11) \qquad k/2^n<t\leqq t_1=(k+1)/2^n<t_1'=k'/2^n\leqq t'<(k'+1)/2^n.$$

From (10) and (11) it follows that (8) holds for (t_1, t_1') and (9) holds for (t, t_1) and (t_1', t'). Then we have

$$(12) \qquad |X(t')-X(t)|\leqq|X(t')-X(t_1')|+|X(t_1')-X(t_1)|+|X(t_1)-X(t)|$$

$$\leqq \varphi_c(t'-t)+2C\varphi_c(1/2^n).$$

But we have, by (10),

$$(13) \qquad \varphi_c(1/2^n)=\{2^{-n}(2\log 2^n+c\log\log 2^n)\}^{1/2}$$

$$\leqq [2(t'-t)/n\{2\log(n/(t'-t))$$

$$+c\log\log(n/(t'-t))\})]^{1/2},$$

which implies

$$(14) \qquad \varphi_c(1/2^n)\leqq[2(t'-t)/n\{4\log(1/(t'-t))+c\log\log(1/(t'-t))$$

$$+c\log 2\}]^{1/2},$$

as $(t'-t)$ is less than $1/n$. Moreover, by (10), we get

$$(15) \qquad n\log 2-\log n\leqq\log(1/(t'-t))<(n+1)\log 2-\log n,$$

and therefore

$$(16) \qquad \log(1/(t'-t))/n=o\{(\log\log(1/(t'-t)))\} \quad \text{as} \quad (t'-t)\to 0.$$

Combining (12), (14) and (16) we obtain for a sufficiently small $(t'-t)$

(17) $$|X(t')-X(t)| \leqq \varphi_{c+\varepsilon/2}(t'-t).$$

Since the above discussion is available for $c'=5+\varepsilon/2$, (17) proves the first part of the theorem.

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Reference

[1] P. Lévy, Théorie de l'Addition des Variables Aléatoires, Paris, 1937.