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On a Generalization of the Abe Groups.

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Introduction

In recent years various kinds of homotopy groups have been introduced as invariants of a topological space. Among others, M. Abe has introduced *a group* $\varkappa_n$, for every integer $n \geqq 2$, containing subgroups isomorphic to the n-dimensional homotopy group π_n and those isomorphic to the fundamental group π_1, in which the operation of π_1 on π_n forms an inner automorphism.

It is shown in this paper that this group $\varkappa_r$ can be extended in a certain way to obtain a *new group* $\sigma^{(r,n)}$, for $r \geqq n > 0$, containing a subgroup isomorphic to the r-dimensional homotopy group π_r. This group $\sigma^{(r,n)}$ will be called *the generalized Abe group of the type* (r, n). In a special case, this group corresponds with *the Abhomotopy group* $\varkappa^{(r,n)}$ introduced by S. T. Hu, which has a definite geometrical meaning: the homotopy classes consisting of mappings $f(S^r) \subset Z$ such that $f(S^n) = *$, where S^n denotes an n-dimensional subsphere of the r-dimensional sphere S^r, and $*$ is a base point of a topological space Z, constitute a group $\varkappa^{(r,n)}$ under a multiplication defined appropriately among them; while as was shown by M. Abe, $\varkappa_r$ is composed of the homotopy classes of mappings which transform S^r into Z and the subsphere S^0 (two points) into the base point $*$.

It has already been shown by Hu that the algebraic structure of the group $\varkappa^{(r,n)}$ is completely determined in terms of homotopy groups. My proof of this theorem is based upon Abe's arguments, and is simplified by utilization of the concept of the function space, which consists of all the Z-valued functions of r-variables with certain conditions.

In the latter part of this paper, I shall discuss several relations between the Abhomotopy group and the torus homotopy group; (i) the isomorphic imbedding of $\varkappa^{(r,n)}$ in *the r-dimensional torus homotopy group* τ_r, which was recently introduced by R. H. Fox[7] ii) the simplicity in the sense of Eilenberg.

I have to add here that a considerable part of the results in this paper happens to be duplicate to the results of S. T. Hu.[6] But my method is quite different from his.

§ 1. *The definition of the generalized Abe group of the type* (r, n), $\sigma^{(r, n)}(Z)$.

Let Z be a topological space and S^r the r-dimensional sphere. All the mappings of the cartesian product of S^{r-1} and the closed inteival I between 0 and 1, $S^{r-1}\times I$, into Z which transform the subset $\{S^{r-1}\times(0)+S^{r-1}\times(1)+K^{n-1}\times I\}$ of $S^{r-1}\times I$ into the base point $*$ of Z, where K^{n-1} is an $(n-1)$-dimensional fixed subcomplex of S^{r-1} such that the difference set $S^{r-1}-K^{n-1}$ is an open set, constitute a function space $\sigma^{(r, n)}$,[1] which is usually denoted by the symbol $Z^{S^{r-1}\times I}(S^{r-1}\times(0)+S^{r-1}\times(1)+K^{n-1}\times I, *)$. As usual two elements f and g of $\sigma^{(r, n)}$ are defined to be homotopic each other if there exists a homotopy h_t, for $1\geqq t\geqq 0$, belonging to $\sigma^{(r, n)}$ such that $h_0=f$ and $h_1=g$, and this relation of homotopy is, of course, an equivalence relation so that $\sigma^{(r, n)}$ is divided into homotopy classes, every one of which consists of homotopic elements. Two elements f and g, belonging to $\sigma^{(r, n)}$, are multiplied together by the rule:

$$(1.1)\qquad fg(x, t)=f(x, 2t) \quad \text{when} \quad x\in S^{r-1} \text{ and } \frac{1}{2}\geqq t\geqq 0$$
$$=g(x, 2t-1) \quad \text{when} \quad x\in S^{r-1} \text{ and } 1\geqq t\geqq\frac{1}{2}$$

and the resulting mapping $f\cdot g$ is again a member of the collection $\sigma^{(r, n)}$. The multiplication defined by the rule (1.1) induces a multiplication of homotopy classes, and these classes together with the induced multiplication constitute *a new group* $\sigma^{(r, n)}(Z)$, *which I call the generalized Abe group of the type* (r, n). The identity is represented by the mapping $f(x, t)=*$ for any $x\in S^{r-1}$ and $0\geqq t\geqq 1$, and the inverse of an element represented by a mapping f is also represented by a mapping f^{-1} defined by the formula $f^{-1}(x, t)=f(x, 1-t)$.

It is easily seen that when the fixed subcomplex K^{n-1} is the vacuous set of S^{r-1}, $\sigma^{(r, n)}(Z)$ corresponds with *the r-dimensional Abe group* $\varkappa_r(Z)$ and that if K^{n-1} is regarded as the $(n-1)$-dimensional subsphere S^{n-1} of S^{r-1}, it coincides with *the Abhomotopy group* $\varkappa^{(r, n)}(Z)$, which was nominated after M. Abe by its discoverer Hu. If $S^{r-1}\times(0)$ and $S^{r-1}\times(1)$ are identified to single points p_0 and p_1 respectively, $S^{r-1}\times I$ and $S^{n-1}\times I$ are reduced to be the r-dimensional sphere S^r

1) The function space is topologized by the compact-open topology due to R. H. Fox. Evidently $r\geqq n>0$ is assumed.

and its subsphere S^n respectively. This mapping φ of $S^{r-1}\times I$ onto S^r transforms the set $\{S^{r-1}(0)+S^{r-1}\times(1)\}$ on two points p_0 and p_1 continuously and elsewhere topological. Then for any mapping f of $\sigma^{(r,n)}$ φ induces a mapping $\bar{f}$ belonging to $Z^s(S^n, *)$ such $f=\bar{f}\varphi$, and this correspondence φ between $\sigma^{(r,n)}$ and $Z^{S^r}(S^n, *)$ is verified to be one-to-one and bicontinuous; two function spaces $Z^{S^r}(S^n, *)$ and $\sigma^{(r,n)}$ are homeomorphic. Thus the group $\varkappa^{(r,n)}(Z)$ may be said that it consists of the homotopy classes of mappings, which transform S^r into Z and the subsphere S^n into the base point $*$ of Z, together with a multiplication defined among them. As was mentioned in the introduction of this paper, this group is obviously a generalization of the r-dimensional Abe group $\varkappa_r(Z)$.

The main purpose of this paper consists in researching for some properties of the generalized Abe groups, especially interesting relations between these groups and homotopy groups.

§ **2** *The isomorphic imbedding of $\pi_r(Z)$ in $\sigma^{(r,n)}(Z)$*

First of all we prove the following theorem, the proof of which is not so easy as expected.

Theorem 1. The generalized Abe group $\sigma^{(r,n)}(Z)$, for $r\geqq n>0$, contains a subgroup isomorphic to the r-dimensional homotopy group $\pi_r(Z)$.

Proof. Let K^{r-1} be an $(r-1)$-dimensional contractible subcomplex of S^{r-1} such that $K^{n-1}\subset K^{r-1}\subsetneqq S^{r-1}$ and $S^{r-1}-K^{r-1}$ is an open set. The assumptions assigned to K^{n-1} assure the existence of such a complex K^{r-1}. Now let Π_r be the function space $Z^{S^{r-1}\times I}(S^{r-1}\times(0)+S^{r-1}\times(1)+K^{r-1}\times I, *)$. It is easily verified that the r-dimensional homotopy group is obtained if both homotopy and multiplication are defined as usual in Π_r, for K^{r-1} can be contracted to a point. From the considerations that Π_r is a subspace of $\sigma^{(r,n)}$, and two mappings α and β belonging to Π_r represent the same element of $\sigma^{(r,n)}(Z)$ if they are homotopic each other in Π_r, it follows that each of some homotopy classes of $\sigma^{(r,n)}$ contains at least one homotopy class of Π_r. In order to complete the proof it is sufficient to show that each of them includes at most one homotopy class of Π_r: two mappings α and β belonging to Π_r represent the same element of $\pi_r(Z)$ if they are homotopic in $\sigma^{(r,n)}$.

Now let P be an arbitrary point of K^{n-1}, then α is, of course, homotopic to β relative to $\{P\times\overset{s}{I}\}$, for they are homotopic each other relative to $\{K^{n-1}\times\overset{s}{I}\}$. Therefore there exists a homotopy $h(x, s, t)$ for $x\in S^{r-1}$, $s\in\overset{s}{I}$, and $t \in \overset{t}{\smile}{I}$ such that

$$(2.1)\qquad \begin{cases} \text{i)} & h(x, s, o)=\alpha(x, s), \\ \text{ii)} & h(x, s, 1)=\beta(x, s), \\ \text{iii)} & h(x, s, t)=* \quad \text{for } x \in K^{n-1},\ s \in \overset{s}{\breve{I}}, \text{ and } t \in \overset{t}{\breve{I}}. \end{cases}$$

Since P can be regarded as a deformation retract of K^{r-1}, a deformation $\rho_u(x)$, $1 \geqq u \geqq 0$, is defined such that

$$(2.2)\qquad \begin{cases} \text{i)} & \rho_u(x) \in K^{r-1} & \text{for } x \in K^{r-1} \text{ and } 1 \geqq u \geqq 0, \\ \text{ii)} & \rho_0(x)=x & \text{for } x \in K^{r-1}, \\ \text{iii)} & \rho_1(x)=P & \text{for every } x \in K^{r-1}. \end{cases}$$

As is easily seen $K^{r-1}\times \overset{s}{\breve{I}}\times \overset{t}{\breve{I}}$ can be pushed onto $\{P\}\times \overset{s}{\breve{I}}\times \overset{t}{\breve{I}}$ by a deformation $D(x, s, t, u)$ such that

$$(2.3)\qquad D(x, s, t, u)=(\rho_u(x), s, t).$$

Then we have from (2.2) and (2.3)

$$(2.4)\qquad \begin{cases} \text{i)} & D(x, s, t, o)=(x, s, t), \\ \text{ii)} & D(x, s, t, 1)=(P, s, t). \end{cases}$$

Now we define a mapping Ψ of $T=\{S^{r-1}\times \overset{s}{\breve{I}}\times \overset{t}{\breve{I}}\times (0)+K^{r-1}\times \overset{s}{\breve{I}}\times \overset{t}{\breve{I}}\times \overset{u}{\breve{I}}+ S^{r-1}\times \overset{s}{\breve{I}}\times (0)\times \overset{u}{\breve{I}}+S^{r-1}\times \overset{s}{\breve{I}}\times (1)\times \overset{u}{\breve{I}}\}$ into Z continuouly such that

$$(2.5)\qquad \begin{cases} \text{i)} & \Psi(x, s, t, u)=h\,D(x, s, t, u) \quad x \in K^{r-1},\ s \in \overset{s}{\breve{I}},\ t \in \overset{t}{\breve{I}} \text{ and } u \in \overset{u}{\breve{I}}, \\ \text{ii)} & \Psi(x, s, t, o)=h(x, s, t) \text{ for } x \in S^{r-1},\ s \in \overset{s}{\breve{I}}, \text{ and } t \in \overset{t}{\breve{I}}, \\ \text{iii)} & \Psi(x, s, o, u)=h(x, s, o) \quad x \in S^{r-1},\ s \in \overset{s}{\breve{I}}, \text{ and } u \in \overset{u}{\breve{I}}, \\ \text{iv)} & \Psi(x, s, 1, u)=h(x, s, 1) \quad x \in S^{r-1},\ s \in \overset{s}{\breve{I}}, \text{ and } u \in \overset{u}{\breve{I}}. \end{cases}$$

The continuity of the mapping Ψ defined on T can be verified from the following considerations; putting $u=o$ in (2.5) (i), $hD(x, s, t, o)=h(x, s, t)$ from (2.4) (i) and hence Ψ is defined continuously on $S^{r-1}\times \overset{s}{\breve{I}}\times \overset{t}{\breve{I}}\times (o)$; from (2.3), (2.1) (i), and (2.2) (i), $hD(x, s, o, u)=h(\rho_u(x), s, o)=\alpha(\rho_u(x), s)=*$, for $\alpha(x, s)=*$ if $x \in K^{r-1}$ and $s \in \overset{s}{\breve{I}}$, so that the continuity of

Ψ on $S^{r-1}\times\overset{s}{\breve{I}}\times(0)\times\overset{u}{\breve{I}}$ is verified together with (2.5) (iii); and similar is the case of (2.5) (iv).

Here we must point out a property of the mapping Ψ;

$$(2.6)\qquad \Psi(x,\ s,\ t,\ 1)=*\quad \text{for } x\in K^{r-1},\ s\in\overset{s}{\breve{I}},\ t\in\overset{t}{\breve{I}},$$

for when $x\in K^{r-1}$, $s\in\overset{s}{\breve{I}}$, and $t\in\overset{t}{\breve{I}}$, $\Psi(x,\ s,\ t,\ 1)=hD(x,\ s,\ t,\ 1)=h(P,\ s,\ t)=*$ from (2.5) (i), (2.4) (ii), and (2.1) (iii).

$\overline{X}=\{K^{r-1}\times\overset{s}{\breve{I}}\times\overset{t}{\breve{I}}+S^{r-1}\times\overset{s}{\breve{I}}\times(0)+S^{r-1}\times\overset{s}{\breve{I}}\times(1)\}$ is a subcomplex of

$$X=S^{r-1}\times\overset{s}{\breve{I}}\times\overset{t}{\breve{I}},$$

and $T=X\times(0)+\overline{X}\times\overset{u}{\breve{I}}$ is a deformation retract of $X\times\overset{u}{\breve{I}}$. Hence the mapping Ψ of T into Z can be extended continuously onto $X\times\overset{u}{\breve{I}}$. Let $\Phi(x,s,t)\equiv\Psi(x,s,t,1)$, where Ψ is now the extended mapping, then we have

$$(2.7)\quad\begin{cases}\text{i)}\quad \Phi(x,\ s,\ 0)=\alpha(x,\ s)\ \text{ for } x\in S^{r-1},\ \text{and } s\in\overset{s}{\breve{I}},\\ \text{ii)}\quad \Phi(x,\ s,\ 1)=\beta(x,\ s)\ \text{ for } x\in S^{r-1},\ \text{and } s\in\overset{s}{\breve{I}},\\ \text{iii)}\quad \Phi(x,\ s,\ t)=*\ \text{ for } x\in K^{r-1},\ s\in\overset{s}{\breve{I}},\ \text{and } t\in\overset{s}{\breve{I}}\end{cases}$$

For $\Phi(x,\ s,\ o)=\Psi(x,\ s,\ o,\ 1)=h(x,\ s,\ o)=\alpha(x,\ s)$ from (2.5) (iii) and (2.1) (i); $\Phi(x,s,t)=\Psi(x,s,t,1)=*$, for $x\in K^{r-1}$, $s\in\overset{s}{\breve{I}}$, and $t\in\overset{t}{\breve{I}}$,from(2.6).

It follows from (2.7) that α and β are homotopic each other in Π_r if they represents the same element of $\sigma^{(r,n)}(Z)$. Thus the theorem has been established.

As was mentioned in the previous paragraph, both the r-dimensional Abe group $\varkappa_r(Z)$ and the Abhomotopy group $\varkappa^{(r,n)}(Z)$ are contained, as special cases, in the generalized Abe group $\sigma^{(r,n)}(Z)$, and hence we have the following corollaries.

Corollary 1.1. The r-dimensional Abe group $\varkappa_r(Z)$, for $r>o$, contains a subgroup isomorphic to the r-dimensional homotopy group $\pi_r(Z)$.

Corollary 1.2. The Abhomotopy group $\varkappa^{(r,n)}(Z)$, for $r\geqq n>o$, contains a subgroup isomorphic to the r-dimensional homotopy group $\pi_r(Z)$.

In order to determine the algebraic structure of the group $\varkappa^{(r,n)}(Z)$ more accurately, we intend to follow M. Abe's arguments in some respects, and before this we make some preliminary arrangements for function spaces in the two following paragraphs.

§ **3.** *Some preliminary arrangements for function spaces.*

Let Z be a topological space with a base point $*$ and S^r the r-dimensional sphere. All the mappings of S^r into Z constitute the points of the function space which is usually designated by the symbol Z^{S^r}. The space is topologized by the compact-open topology introduced by R. H. Fox. $Z_0^{S^r}(x_0, *)$ denotes the subspace of Z^{S^r} and of all the inessential mappings which transform a fixed point $x_0 \in S^r$ into the base point $*$ of Z. Then according to Hurewicz's definition *the r-dimensional homotopy group*

$$\pi_r(Z) \equiv \pi_1(Z_0^{S^{r-1}}(x_0, *)). \tag{3.1}$$

As a generalization of this formula the following theorem holds true

Theorem 2.

$$\pi_r(Z) \cong \pi_n(Z_0^{S^{r-n}}(x_0, *)) \quad r \geqq n > o.$$

Proof. Let E^r be the r-dimensional cube $E^r = E^n \times E_n^r$ 1), then the boundary $E^r, \dot{E}^r$ is the union of $\dot{E}^n \times E_n^r$ and $E^n \times \dot{E}_n^r$. Now a mapping f representing an element of $\pi_n(Z_0^{S^{r-n}}(x_0, *))$ satisfies the conditions;

$$\begin{aligned} f(x^r) &= * \quad \text{when} \quad x^r \in \dot{E}^n \times E_n^r \\ &= * \quad \text{when} \quad x^r \in E^n \times \dot{E}_n^r \end{aligned} \tag{3.2}$$

Then f represents an element of $\pi_r(Z)$, and the proof is practically established.

Now that the group $\pi_n(Z_0^{S^{r-n}}(x_0, *))$ is analysed to be a generalization of Hurewicz's formula, it is quite natural that the algebraic structure and the geometric structure of the groups $\pi_n(Z_0^{S^{r-n}})$, $\varkappa_n(Z_0^{S^{r-n}}(x_0, *))$, and $\varkappa_n(Z_0^{S^{r-n}})$ should be also examined. Judging from the fact that M. Abe has defined *the r-dimensional Abe group* as

$$\varkappa_r(Z) \equiv \pi_1(Z_0^{S^{r-1}}), \tag{3.3}$$

we can imagine with ease that they may be regarded as generalizations of Abe groups. The reasonable conjectures like this will be verified and related to Abhomotopy groups in the following paragraph.

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1) $E^n = \{x^n = (x_0, x_1, \cdots, x_{n-1}) ; 1 \geqq x_i \geqq 0, n-1 \geqq i \geqq 0\}$
$E^r{}_n = \{x^r_n = (x^r_n, x_{n+1}, \cdots, x_{r-1}) ; 1 \geqq x_i \geqq 0, r-1 \geqq i \geqq n\}$
and also $E^n \times E^r_n$ denotes the cartesian product of E^n and E^r_n.

§4. *The geometric structures of the groups* $\pi_n(Z_0^{S^{r-n}})$ *and* $\varkappa_n(Z_0^{S^{r-n}}(x_0, *))$.

It can be seen from the following theorem that the groups $\pi_n(Z_0^{S^{r-n}})$ and $\varkappa_n(Z_0^{S^{r-n}}(x_0, *))$ may be regarded to be the same in the algebraic point of view.

Theorem 3. $\varkappa_n(Z_0^{S^{r-n}}(x_0, *))\cong\pi_{r-n+1}(Z_0^{S^{n-1}})$ *for* $r\geqq n>0$.

Proof. An element of $\varkappa_n(Z_0^{S^{r-n}}(x_0, *))$ is represented by a continuous Z-valued function f of the r-real variables x_i, $1\geqq x_i\geqq 0$ for $i=0, 1,\cdots, n-1$, which satisfies the condition:

$$(4.1)\qquad \begin{aligned} f(x^r) &= * \quad \text{when} \quad x_0(x_0-1)\prod_{i=n}^{r-1} x_i\ (x_i-1)=0, \\ &=P(x_0, x_n^r) \quad \text{when} \quad \prod_{n=1}^{n-1} x_i(x_i-1)=0, \end{aligned}$$

where $P(x_0, x_n^r)$ denotes a point in Z which depends only on the partial coordinates x_0 and x_n^r of x^r; while a mapping representing an element of $\pi_{r-n+1}(Z_0^{S^{n-1}})$ is also characterized by the condition (4.1). Hence by further simple considerations the theorem is established.

Now we intend to discuss the close relations between $\pi_n(Z_0^{S^{r-n}})$ and the Abhomotopy, and the geometric structures of the groups $\pi_n(Z_0^{S^{r-n}})$ and $\varkappa_n(Z_0^{S^{r-n}}(x_0, *))$ are clearly explained as soon as the relations are given in the following. It is no exaggeration to say that these groups are worth while investigating merely because they can be treated in the background of the generalized Abe groups.

Theorem 4. $\pi_n(Z_0^{S^{r-n}})\cong\pi_{n-\sigma}(Z_0^{S^{r-n+\sigma}}(S^{\sigma-1}, *))$ *for* $r\geqq n>\sigma\geqq 0$.

Especially, if $\sigma=n-1$, $\pi_n(Z_0^{S^{r-n}})\cong\pi_1(Z_0^{S^{r-1}}(S^{n-2}, *))$.

Proof. The proof shown here is the same as Hu's one.
Let $Y=Z_0^{S^{r-n}}$, then from Theorem 2 $\pi_n(Y)\cong\pi_{n-1}(Y_0^{S^1}(y_0, \xi_0))$, where $y_0 \in S^1$ and $\xi_0(S^{r-n})=*$. An element $f\in Y_0^{S^1}(y_0, \xi_0)$ is a mapping $f(S^{r-n}\times I)\subset Z$, such that $f(S^{r-n}\times(0)+S^{r-n}\times(1))=*$. If $S^{r-n}\times(0)$ and $S^{r-n}\times(1)$ are identified to single points p_0 and p_1 respectively, $S^{r-n}\times I$ becomes the $(r-n+1)$-dimensonal sphere S^{r-n+1}. This mapping θ of $S^{r-n}\times I$ onto S^{r-n+1} transforms the set $\{S^{r-n}\times(0)+S^{r-n}\times(1)\}$ on two points p_0 and p_1 continuously and elsewhere topological. Let $f=\bar{f}\ \theta$, then $\bar{f}\in Z_0^{S^{r-n+1}}(S^0,*)$; two spaces $Y_0^{S^{r-n}}(y_0, \xi_0)$ and $Z_0^{S^{r-n+1}}(S^0, *)$ are homeomorphic. Thus we

have

$$\pi_n(Z_0^{S^{r-n}})\cong\pi_{n-1}(Z_0^{S^{r-n+1}}(S^0,\ *)). \tag{4.2}$$

Again, let $Z_0^{S^{r-n+1}}(S^0,\ *)=X$, then $\pi_{n-1}(X)\cong\pi_{n-2}(X_0^{S^1}(x_0,\ \eta_0))$ where $x_0\in S^1$ and η_0 is a constant mapping $\eta_0(S^{r-n+1})=*$. An element $f\in X^{S^1}(x_0,\ \eta_0)$ is a mapping of $S^{r-n+1}\times I$ into Z such that $f(S^{r-n+1}\times(0)+S^{r-n+1}\times(1)+S^0\times I)=*$. If we identify $S^{r-n+1}\times(0)$ and $S^{r-n+1}\times(1)$ to single points q_0 and q_1 respectively, then $S^{r-n+1}\times I$ and $S^0\times I$ are reduced to S^{r-n+2} and S^1 respectively. If we denote this mapping of $S^{r-n+1}\times I$ onto S^{r-n+2} by ρ, then $f=\bar{f}\cdot\rho$ such that $\bar{f}$ belongs to $Z_0^{S^{r-n+2}}(S^1,\ *)$ and by this correspondence ρ two spaces $X_0^{S^1}(x_0,\ \eta_0)$ and $Z_0^{S^{r-n+2}}(S^1,\ *)$ are homeomorphic. Thus we have

$$\pi_{n-1}(Z_0^{S^{r-n+1}}(S^0,\ *))\cong\pi_{n-2}(X_0^{S^1}(x_0,\ \eta_0))\cong\pi_{n-2}(Z_0^{S^{r-n+2}}(S',\ *)), \tag{4.3}$$

and from (4.2) and (4.3)

$$\pi_n(Z_0^{S^{r-n}})\cong\pi_{n-2}\ (Z_0^{S^{r-n+2}}(S^1_,\ *)) \tag{4.4}$$

Repeating the same process, we have the following general formula

$$\pi_n(Z_0^{S^{r-n}})\cong\pi_{n-\sigma}\cong Z_0^{S^{r-n+\sigma}}(S^{\sigma-1},\ *)\ \text{for}\ n>\sigma\geqq 1. \tag{4.5}$$

In a special case $\sigma=o$, if we agree that S^{-1} is vacuous, the right side of (4.5) is equal to $\pi_n(Z_0^{S^{r-n}})$. Thus (4.5) holds true for $r\geqq n>\sigma\geqq o$.

Theorem 5. $\pi_n(Z_0^{S^{r-n}})\cong\kappa^{(r,\ n-1)}(Z)$ *for* $r\geqq n>o$.

Proof. From Theorem 4 $\pi_n(Z_0^{S^{r-n}})\cong\pi_1(Z_0^{S^{r-1}}(S^{n-2},\ *))$. It is evident from the definition of $\kappa^{(r,\ n-1)}(Z)$ that $\kappa^{(r,\ n-1)}(Z)\cong\pi_1(Z_0^{S^{r-1}}(S^{n-2},\ *))$, and hence the Theorem has been established.

Theorem 6. $\kappa_n(Z_0^{S^{r-n}}(x_0,\ *))\cong\kappa^{(r,\ r-n)}(Z)$ *for* $r\geqq n>0$.

Proof. From Theorem 3 $\kappa_n(Z_0^{S^{r-n}}(x_0,\ *))\cong\pi_{r-n+1}(Z_0^{S^{n-1}})$ and from Theorem 5 $\pi_{r-n+1}(Z_0^{S^{n-1}})\cong\kappa^{(r,\ r-n)}(Z)$.

Remark. It is worthy of note that $\kappa^{(r,\ n)}(Z)$ is abelian if $r\geqq n>o$ though $\kappa_r(Z)$ is not generally abelian.

§ 5. *The algebraic structure of* $\kappa^{(r,\ n)}(Z)$

The subsequent discussions will be developed in parallel with M. Abe's arguments; M. Abe has shown that the r-dimensional Abe group $\kappa_r(Z)$

is a group extension of $\pi_r(Z)$ by the fundamental group $\pi_1(Z)$, and that $\varkappa_r(Z)$ is actually the direct product of $\pi_r(Z)$ and $\pi_1(Z)$ if and only only if Z is r-simple in the sense of Eilenberg. By analogy with this statement we obtain the following theorem.

Theorem 7. $\pi_n(Z_0^{s^{r-n}})$, *for* $r \geqq n > 1$, *is isomorphic to the direct sum of the groups* $\pi_r(Z)$ *and* $\pi_n(Z)$; $\pi_n(Z_0^{s^{r-n}}) \cong \pi_r(Z) \oplus \pi_n(Z)$.

Proof. From Corollary 2 of Theorem 1 and Theorem 5 $\pi_n(Z_0^{s^{r-n}})$ contains a normal subgroup $\bar{\pi}_r(Z)$ isomorphic to $\pi_r(Z)$. At first a slight relaxed statement is proved; $\pi_n(Z_0^{s^{r-n}})$ is a group extension of $\pi_r(Z)$ by $\pi_n(Z)$; namely the difference group $\pi_n(Z_0^{s^{r-n}})-\pi_r(Z)$ is isomorphic to $\pi_n(Z)$. This extension is shown to be of special type, which I call split, meaning that $\pi_n\ (Z_0^{s^{r-n}})$ contains a normal subgroup $\bar{\pi}_n(Z)$ isomorphic to $\pi_n(Z)$. For $r \geqq n > 1$ the groups $\pi_n(Z_0^{s^{r-n}})$, $\pi^n(Z)$ and $\pi_r(Z)$ are all abelian, so that $\pi_n(Z_0^{s^{r-n}})$, being an abelian split extension of an abelian group by an abelian group, is actually the direct sum of $\pi_r(Z)$ and $\pi_n(Z)$.

Let f be a mapping representing an element α of $\pi_n(Z_0^{s^{r-n}})$. As was considered in (4.1),

$$f(x^r) = * \qquad \text{when } x^r \in \dot{E}^n \times E_n^r \ \left(\prod_{i=0}^{n-1} x_i(x_i-1)=0\right),$$

$$(5.1) \qquad\qquad = P(x^n) \quad \text{when } x^r \in E^n \times \dot{E}_n^r \ \left(\prod_{i=n}^{r-1} x_i(x_i-1)=o\right).$$

Putting $f(x^n,\ 0\ldots0)=\bar{f}(x^n)$, then $\bar{f}(x^n)=*$ for $x^n \in E^n$, and hence $\bar{f}$ represents an element of $\pi_n(Z)$. If two elements f and g represent the same element of $\pi_n(Z_0^{s^{r-n}})$, $\bar{f}$ is homotopic to g; it may also be verified that $\overline{f\cdot g}=\bar{f}\cdot\bar{g}$, and from these two facts it follows immediately that the rransformation f into $\bar{f}$ induces a homomorphism Γ of $\pi_n\ (Z_0^{s^{r-n}})$ onto $\pi_n(Z)$. Now let α be any element of $\pi_n(Z_0^{s^{r-n}})$, which belongs to the kernel of Γ, so that $\Gamma\alpha=0$. Thus α must be represented by a mapping f for which $\bar{f}(x^n)=f(x_n,0\cdots0)=*$ for any $x^n \in E^n$, so that according to the condition (2.6) $f(x^r)=*$ whenever x^r belongs to the boundary $\dot{E}^r$. Therefore the kernel of Γ is involved in the subgroup $\pi_r(Z)$ of $\bar{\pi}_n(Z_0^{s^{r-n}})$. Conversely any mapping g representing an element β of $\bar{\pi}_r\ (Z)$ satisfies the condition $\bar{g}(x^n)=*$ for $x^n \in \dot{E}^n$, so that $\Gamma\beta=0$ for every $\beta \in \bar{\pi}_r(Z)$. It follows from these verifications that the kernel of Γ is $\pi_r(Z)$. Thus $\pi_n(Z_0^{s^{r-n}}\ )$ is a group extension of $\bar{\pi}_r(Z)$ by the group $\pi_n(Z)$. Now it remains not to be shown that this extension is split. Let f be a mapping representing an element a of $\pi_n(Z)$. To every mapping f there is assigned a mapping f^{φ} which is defined by the rule

$$f^{\varphi}(x^r) \equiv f(x^n)$$

It is obvious that f^{φ} satisfies the condition (5.1) and hence represents an element of $\pi_n(Z_0^{S^{r-n}})$. That this correspondence between mappings induces a correspondence between homotopy classes of mappings and that this homotopy class correspondence is really a homomorphism Φ of the group π_n into the group $\pi_n(Z_0^{S^{r-n}})$ are immediate from the definition. Since $\Gamma\Phi a = a$ for any element $a \in \pi_n(Z)$, $\Gamma\Phi$ is the identity transformation of $\pi_n(Z)$ into itself, and hence Φ is an isomorphism of $\pi_n(Z)$ into the group $\pi_n(Z_0^{S^{r-n}})$. Now it is concluded from these considerations that $\pi_n(Z)$ is imbedded isomorphically in the group $\pi_n(Z_0^{S^{r-n}})$. The proof has been completed.

Now from Theorem 5 and Theorem 7 just obtained above the algebraic structure of the group $\varkappa^{(r,n)}(Z)$, for $r \geqq n > 0$, is completely determined in terms of homotopy groups. As a theorem we have.

Theorem 8. $\varkappa^{(r,n)}(Z)$, *for* $r \geqq n > o$, *is isomorphic to the direct sum of the groups* $\pi_r(Z)$ *and* $\pi_{n+1}(Z)$; $\varkappa^{(r,n)}(z) \cong \pi_r(z) \oplus \pi_{n+1}(Z)$.

Proof. $\varkappa^{(r,n)}(Z) \cong \pi_{n+1}(Z_0^{S^{r-n-1}}) \cong \pi_r(Z) \oplus \pi_{n+1}(Z)$.

Theorem 9. $\varkappa_n(Z_0^{S^{r-n}}(x_0, *)) \cong \pi_r(Z) \oplus \pi_{r-n+1}(Z)$ *if* $r > n > o$.

Proof. $\varkappa_n(Z_0^{S^{r-n+1}}(x_0, *)) \cong \pi_{r-n+1}(Z_0^{S^{n-1}}) \cong \pi_r(Z) \oplus \pi_{r-n+1}(Z)$ from Theorem 3 and Theorem 7.

Corollary 9. 1. The function space $Z_0^{S^{r-n}}(x_0, *)$, *for* $r > n > o$, *is n-simple in the sense of Eilenberg.*

Proof. Utilizing Abe's result, we have Corollary 9.1. because $\varkappa_n(Z_0^{S^{r-n}}(x_0, *)) \cong \pi_r(Z) \oplus \pi_{r-n+1}(Z) \cong \pi_n(Z_0^{S^{r-n}}(x_0, *)) \oplus \pi_1(Z_0^{S^{r-n}}(x_0, *))$; namely $\varkappa_n(Y) \cong \pi_n(Y) \oplus \pi_1(Y)$ if we put $Y = Z_0^{S^{r-n}}(x_0, *)$.

§ 6. *The determination of the algebraic structures of the groups* $\varkappa_n(Z_0^{S^{r-n}})$ *and* $\tau_{r-n}(Z_0^{S^n}(x_0, *))$.

In this paragraph I want to research for determinating the algebraic structures of $\varkappa_n(Z_0^{S^{r-n}})$ and $\tau_{r-n}(Z_0^{S^n}(x_0, *))$.

Tedious as this work is, it seems to me that it can not be neglected. Thus I intend to summarize this paragraph in the following theorem.

Theorem 10. The group $\varkappa_n(Z_0^{S^{r-n}})$, for $r \geqq n > 0$, has several properties.

i) *The group contains normal subgroups isomorphic to the groups $\pi_n(Z_0^{S^{r-n}})$ and $\varkappa_{r-n+1}(Z)$.*

ii) *$\varkappa_n(Z_0^{S^{r-n}})$ is a group extension of $\pi_n(z_0^{S^{r-n}})$ by $\varkappa_{r-n+1}(Z)$.*

iii) *If $Z_0^{S^{r-n}}$ is n-simple in the sense of Eilenberg, $\varkappa_n(Z_0^{S^{r-n}}) \cong (Z_0^{S^{r-n}}) \otimes \varkappa_{r-n+1}(Z)$*

iv) *If Z is $(r-n+1)$-simple and $Z_0^{S^{r-n}}$ is n-simple, $\varkappa_n(Z_0^{S^{r-n}}) \cong \pi_r(Z) \otimes \pi_n(Z) \otimes \pi_1(Z) \otimes \pi_{r-n+1}(Z)$.*

Proof: According to M. Abe's result, it is immediate that $\varkappa_n(Z_0^{S^{r-n}})$ contains a normal subgroup isomorphic to $\pi_n(Z_0^{S^{r-n}})$. Now we proceed to show that $\varkappa_{r-n+1}(Z)$ is imbedded isomorphically in $\varkappa_n(Z_0^{S^{r-n}})$. Let f be a mapping representing any element of $\varkappa_n(Z_0^{S^{r-n}})$. Then the Z-valued function f of r-variables satisfies the conditions:

$$
\begin{aligned}
f(x^r) &= * && \text{when } x_0(x_0-1)=0 \\
&= P(x_0, x_n^r) && \text{when } \prod_{i=1}^{n-1} x_i(x_i-1)=0 \\
&\doteq P(x^n) && \text{when } \prod_{i=1}^{r-1} x_i(x_i-1)=0,
\end{aligned} \tag{6.1}
$$

while any mapping g representing an element a of $\varkappa_{r-n+1}(Z)$ satisfies the condition:

$$
\begin{aligned}
g(x_0, x_n^r) &= * && \text{when } x_0(x_0-1)=0, \\
&= P(x_0) && \text{when } \prod_{i=n}^{r-1} x_i(x_i-1)=0.
\end{aligned} \tag{6.2}
$$

Put $f^{\varphi}(x^r) = f(x_0, x_n^r)$. It is verified that f^{φ} satisfies (6.1) and hence represents an element of $\varkappa_n(Z_0^{S^{r-n}})$ and that this correspondence φ induces a homomorphism Φ of the group $\varkappa_{r-n+1}(Z)$ into the group $\varkappa_n(Z_0^{S^{r-n}})$. In order to complete the proof it is sufficient to prove that Φ is an isomorphism. To every function f representing an element a of $\varkappa_n(Z_0^{S^{r-n}})$ there corresponds an element f^{ψ}, which represents an element of $\varkappa_{r-n+1}(Z)$, defined by the rule

$$
f^{\psi}(x_0, x_n^r) \equiv f(x_0, 0 \cdots\cdots 0, x_n^r). \tag{6.3}
$$

This correspondence induces the homomorphism Ψ of $\varkappa_n(Z_0^{S^{r-n}})$ into $\varkappa_{r-n+1}(Z)$. Since $\Psi\Phi a=a$ for any $a \in \varkappa_{r-n+1}(Z)$, $\Psi\Phi$ is an identity transformation of $\varkappa_{r-n+1}(Z)$ into itself. It follows that Φ is an isomorphism of $\varkappa_{r-n+1}(Z)$ into $\varkappa_n(Z_0^{S^{r-n}})$, and hence the proof has been completed.

According to M. Abe's result $\varkappa_n(Z_0^{S^{r-n}})\cong\pi_n(Z_0^{S^{r-n}})\otimes\pi_1(Z_0^{S^{r-n}})$, if $Z_0^{S^{r-n}}$ is n-simple in the sense of Eilenberg. From the definition of Abe group $\pi_1(Z_0^{S^{r-n}})=\pi_1(Z_0^{S^{(r-n+1)-1}})\equiv\varkappa_{r-n+1}(Z)$, and hence the third statement of this theorem has been established.

Moreover, if Z is $(r-n+1)$-simple, $\varkappa_{r-n+1}(Z)\cong\pi_{r-n+1}(Z)\otimes\pi_1(Z)$ and from Theorem 7 $\pi_n(Z_0^{S^{r-n}})\cong\pi_n(Z)\otimes\pi_r(Z)$. Taking into consideration of the third proposition, we have the last one;

$$\varkappa_n(Z_0^{S^{r-n}})\cong\pi_{r-n+1}\otimes\pi_1\otimes\pi_n\otimes\pi_r.$$

Thus the proof of this Theorem has been completed.

In the next place we shall analyse the algebraic structure of the group $\tau_{r-n}(Z_0^{S^n}(x_0, *), *)$, for $r\geqq n>o$, which is the $(r-n)$-dimensional absolute torus homotopy group of the function space $Z_0^{S^n}(x_0, *)$. In some sense this group has been investigated by R. H. Fox (See Fox's paper (7)). Therefore we shall give only an outline of the group in this paper.

Theorem 11.

$$\tau_{r-n}(Z_0^{S^n}(x_0, *), *) \textit{ is abelian if } r\geqq n>0.$$

Proof: Refer to the proof of the Theorem (8.1) in Fox's paper (7).

Theorem 12.

$$\tau_{r-n}(Z_0^{S^n}(x_0, *), *)\cong\pi_{n+1}(Z)\oplus\pi_{n+2}(Z)\oplus\cdots\cdots\oplus\pi_r(Z).$$

Proof: Let f be a mapping representing an element of the group $\tau_{r-n}(Z_0^{S^n}(x_0, *), *)$. Then f satisfies the condition:

$$\begin{aligned} f(x_0, x_1^n, x_n^r) &= * \quad \text{when } x_0(x_0-1)\prod_{i=n}^{r-1} x_i(x_i-1)=0,\\ &=P(x_0\cdots\cdots\hat{x}_i\cdots\cdots x_{n-1}, x_n^r) \text{ when } x_i(x_i-1)=0 \ (i=1\cdots n\text{-}1). \end{aligned}$$

Put $f^\varphi(x_0, x_2^r)=f(x_0, 0, x_2^r)$. It is easily verified that f^φ represents an element of $\tau_{r-n-1}(Z_0^{S^n}(x_0, *), *)$ and that this correspondence induces the homomorphism Φ of $\tau_{r-n}(Z_0^{S^n}(x_0, *), *)$ onto $\tau_{r-n-1}(Z_0^{S^n}(x_0, *), *)$. The kernel of Φ is $\tau_{r-n-1}(Z_0^{S^{n+1}}(x_0, *), *)$, so that $\tau_{r-n}(Z_0^{S^n}(x_0, *), *)$ is a group extension of the group $\tau_{r-n-1}(Z_0^{S^{n+1}}(x_0, *)*)$ by the group $\tau_{r-n-1}(Z_0^{S^n}(x_0, *),*)$. Moreover, it can be proved that $\tau_0^{S^n}(x_0, *), *)$ is actually the direct sum of the groups $\tau_{r-n-1}(Z_0^{S^{n+1}}(x_0, *), *)$ and $\tau_{r-n-1}(Z_0^{S^n}(x_0, *))$ Continuing the similar process, we have the desired result.

Corollary 12. 1.

$\tau_{r-n}(Z_0^{S^n}(x_0, *), *)$ *imbeds isomorphically* $\varkappa_{r-n}(Z_0^{S^n}(x_0, *))$.

Proof: Evident.

§7. *The isomorphic imbedding of* $\varkappa^{(r, n)}(Z)$ *in the torus homotopy group* $\tau_r(Z)$. R. H. Fox has shown in his paper (7) "that the r-dimensional torus homotopy group $\tau_r(Z)$ imbeds isomorphically the n-dimensional homotopy group $\pi_n(r \geqq n > o)$ and the r-dimensional Abe group $\varkappa_r(Z)$. But the isomorphic imbedding of the Abhomotopy groups $\varkappa^{(r, n)}(Z)$ in the torus homotopy group $\tau_r(Z)$ was not treated by Fox. This isomorphism is proved in an analogoùs way as the case of the Abe groups.

Theorem 13. The r-dimensional torus homotopy group $\tau_r(Z)$ *contains a subgroup isomorphic to* $\varkappa^{(r, n)}(Z)$.

Proof: From Theorem 5 $\varkappa^{(r, n)}(Z) \cong \pi_{n+1}(Z_0^{S^{r-n-1}})$. As in (5.1) a mapping f representing an element of $\varkappa^{(r, n)}(Z)$ satisfies the conditions :

$$
(7.1) \qquad \begin{aligned} f(x^r) &= * && \text{when } \prod_{i=0}^{n} x_i(x_i-1)=0, \\ &= P(x^{n+1}) && \text{when } \prod_{i=n+1}^{r-1} x_i(x_i-1)=0. \end{aligned}
$$

Such a mapping represents an element of r-dimensional torus homotopy group τ_r ; if two mappings f and g represent the same element of $\varkappa^{(r, n)}(Z)$, then f and g are equivalent elements of $Z^{T^{r,1}}$ where $T^{r,1}$ is a r-dimensional pinched torus, and hence the transformation $\varkappa^{(r, n)}(Z)$ into $\tau_r(Z)$, which is induced by making correspondence to every function f satisfying (7.1) the element defined by the same formula, is a homomorphism. Let us denote this homomorphism by Γ.

In order that Γ may be an isomorphism, we utilize the homomorphism Ψ of τ_r onto τ_{r-1} which is defined by the rule ; given a representatve f of $\beta \in \tau_r$, a repesentative of $\Psi\,\beta \in \tau_{r-1}$ is the mapping $f^{\Psi} = |f|\{x_{r-1}=0\}$. By iteration of this homomorphism we may define a homomorphism of τ_r onto

τ_{n+1}. Let us denote this homomorphism also by Ψ. Now a denotes any element of $\chi^{(r,n)}(Z)$ which belongs to the kernel of Γ so that $\Gamma a=1$, and hence $\Psi\Gamma a=1$. Thus a must be represented by a mapping for which $f(x^{n+1}, 0\cdots\cdots 0)=*$ so that according to the condition (7.1) $f(x^r)=*$ whenever $x^r \in \prod_{i=0}^{r-1} x_i(x_i-1)=0$. Hence a belongs to $\pi_r(Z)$. That $\pi_r(Z)$ is normal in $\chi^{(r,n)}(Z)$, was shown in Corollary 1.2. Since $\Gamma|\pi_r(Z)$ was proved to be an isomorphism by Fox in his paper, it follows that $a=1$ if $\Gamma a=1$. This proves that Γ is an isomorphism.

Remark: It is easily seen from the definition of Γ that there are $\binom{r-1}{n}$ different isomorphisms of $\chi^{(r,n)}(Z)$ into τ_r and that the isomorphisms are induced by certain automorphisms of τ_r— those automorphisms are induced by the permutations of the indices 1, 2,$\cdots\cdots$,$r-1$.

§ 8. *The simplicity of a topological space.*

The simplicity of the function spaces $Z_0^{S^{r-n}}(x_0, *)$ and $Z_0^{S^{r-n}}$ was considered in this paper. Needless to say we need not confine ourselves within the concept of the simplicity in the sense of Eilenberg. We intend to introduce an idea "r-simplicity in the sense of the torus homotopy group". Let T^r be an r-dimensional torus and $T^{r,1}$ an r-dimensional pinched torus. $Z^{T^{r,1}}$ denotes the function space consisting of all the Z-valued functions of r variables x_i, $1\geq x_i\geq 0$ $(i=o,\cdots\cdots, r-1)$, which satisfy the condition;

$$(8.1)\qquad \begin{cases} f(0, x_1^r)=f(1, x_1^r)=* \\ f(x_0, 0, x_2^r)=f(x_0, 1, x_2^r) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f(x^{r-1}, 0)=f(x^{r-1}, 1) \end{cases}$$

while Z^{T^r} is a collection of all the mappings which can be extended with period 1 in each of r variables $x_0,\cdots\cdots,x_{r-1}$.

Two elements f and g belonging to Z^{T^r} are defined to be "freely homotopic" each other if there exsists a homotopy $h_t \in Z^{T^r}(1\geq t\geq 0)$ such that $h_0=f$, and $h_1=g$; two elements f and g of $Z^{T^{r,1}}$ are "homotopic" or "equivalent" if there exsists a homotopy $h_t \in Z_t^{T^{r,1}}$ $(1\geq t\geq 0)$ such that $h_0=f$ and $h_1=g$. It is evident that any two equivalent elements belonging to $Z^{T^{r,1}}$ are freely homotopic. But the converse of this statement is not always true. Thus if Z is arcwise connected, each of some homotopy classes of Z^{T^r} contains at least one homotopy class of $Z^{T^r, 1}$. An arcwise connected topological space is called "*r-simple in the sense of the torus homotopy group*" when each of homotopy classes of Z^{T^r}, if it contains homotopy chasses of $Z^{T^r, 1}$, contains just one homotopy class of $Z^{T^r, 1}$.

Theorem 14. *Two elements f and g belonging to $Z^{T^r, 1}$ are freely homotopic if and only if there exsists an element $h \in Z^{T^r, 1}$ such that $[f]=[h][g][h]^{-1}$, where $[f]$, $[g]$, and $[h]$ denote the homotopy classes represented by f, g, and h respeetiz ely..*

Proof: Since f and g are freely homotopic, there exsists a homotopy $\alpha(t, x^r) \in Z^{T^r}$ for $1 \geqq t \geqq o$ such that $\alpha(0, x^r) = f(x^r)$ and $\alpha(1, x^r) = g(x^r)$. Now let $\alpha(t, o, x_1^r,) = h(t, x_1^r)$, then $h(t, x_1^r)$ represents an element of $Z^{T^r, 1}$. For $h(t, x_1^r)$ satisfies the condition (8.1);

$$\begin{cases} h(o, x_1^r) = h(1, x_1^r) = * \\ h(t, o, x_2^r) = h(t, 1, x_2^r) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ h(t, x_1^{r-1}, o) = h(t, x_1^{r-1}, 1) \end{cases}$$

If the point sets drawn in dotted lines (see Fig. 1) are mapped by the homotopy α, it is verified that f and $h\,g\,h^{-1}$ are equivalent elements of $Z^{T^r, 1}$ and hence $[f]=[h][g][h]^{-1}$. Necessity of the theorem has been established.

Conversely, it is shown that f and g are freely homotopic, if f is equivalent to $h\,g\,h^{-1}$. Since f and $h\,g\,h^{-1}$ are equivalent, there exsists a homotopy $\beta(x^r, t) \in Z^{T^r, 1}$ for $1 \geqq t \geqq 0$ such that $\beta(x^r, 1) = f(x^r)$ and $\beta(x^r, 0) = h\,g\,h^{-1}(x^r)$, namely

$$\beta(x^r, 0) = \begin{cases} h(3x_0, x_1^r) & \text{for } \frac{1}{3} \geqq x_0 \geqq 0 \\ h(3x_0 - 1, x_1^r) & \text{for } \frac{2}{3} \geqq x_0 \geqq \frac{1}{3} \\ h(3 - 3x_0, x_1^r) & \text{for } 1 \geqq x_0 \geqq \frac{2}{3} \end{cases}$$

Again, if the figures drawn in dotted lines (See Fig. 2) are mapped by the mapping β, it can be easily seen that f and g are freely homotopic, noting that $\beta(0, x_1^r, t) = \beta(1, x_1^r, t) = *$. The proof has been completed.

From Theorem 14 the following theorem is obtained.

Theorem 15. A topological space Z is r-simple in the sense of the torus homotopy group, if and only if r-dimensional torus homotopy group τ_r is abelian.

Proof: Let $[f]$ and $[g]$ be any elements of τ_r which are represented by the mappings f and g respectively. Two elements f and gfg^{-1} are freely homotopic, for $f\sim g^{-1}(g\,f\,g^{-1})\ g$. From the simplicity of the space $[f]=[g]\ [f]\ [g]^{-1}$ and hence $[f]\ [g]=[g]\ [f]$. This proves that τ_r is abelian. Conversely it is obvious that if τ_r is abelian, Z is r-simple in the sense of the torus homotopy group.

In the last part of this paragraph we remark the relation between the simplicity of a space in the sense of Eilenberg and the simplicity in the sense of the torus homotopy group.

Theorem 16. If a space is r-simple in the sense of the torus homotopy group, it is r-simple in the ordinary sense.

Proof: In order to prove this theorem it is sufficient to show that if r-dimensional torus homotopy group $\tau_r(Z)$ is abelian, the r-dimensional Abe group $\varkappa_r(Z)$ is isomorphic to the direct product of the r-dimensional homotopy group π_r and the fundamental group π_1. According to M. Abe's results, $\varkappa_r$ is a split extension of π_r by π_1. Now since τ_r is abelian, π_r and π_1 are all abelian. Hence $\varkappa_r$ is an abelian split group extension of an abelian group π_r by an abelian group π_1, so that $\varkappa_r$ is actually the direct sum of π_r and π_1. This proves the theorem.

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