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On a conformal mapping with certain boundary correspondences.

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Given any set of points E on the unit circle C of z-plane, we shall treat in this paper the problem to map the interior of C conformally on a schlicht domain D so that the set E corresponds to a set of accessible boundary points of D all lying on one and the same point of the plane.

In order to simplify the wording, we call a half straight-line on w-plane : $\arg w=$const., $\infty \geqq |w| \geqq$ const. >0, an infinite radial slit.

First, we consider the case where the set E is *finite*.

Theorem 1. *Let $z_1,\ldots,z_n$ be n points on C associated with n positive numbers $a_1,\ldots,a_n$, whose sum is equal to 1. Then, there exists a function $w=w(z)$, which maps the interior of C conformally on a domain D, so that:* 1. *D is the whole w-plane cut along n infinite radial slits,* 2. *z_k corresponds to the accessible boundary point of D lying on $w=\infty$, which is determined by an angular domain between two of these slits including an angle $2\pi a_k$ at $w=\infty$, and* 3. *$w(0)=0$, $w'(0)=1$. Under these conditions the mapping function is uniquely determined and is given by*

$$w=w(z)=z\Big/\prod_{k=1}^{n}\left(1-\frac{z}{z_k}\right)^{2a_k}. \tag{1}$$

Proof. We construct a potential function $u(z)$ on z-plane, whose singularities are

$$\log|z| \quad \text{at} \quad z=0,$$

$$\log\frac{1}{|z|} \quad \text{at} \quad z=\infty,$$

and

$$2a_k \log\frac{1}{|z-z_k|} \quad \text{at} \quad z=z_k.$$

Denoting the conjugate potential of $u(z)$ by $v(z)$, we put

$$w(z)=\text{const.}\ \exp.\ \{u(z)+iv(z)\}.$$

$w(z)$ is regular in $|z| \leqq 1$ except the n points z_k, and vanishes only at $z=0$.

On account of symmetry of singularities, $u(z)$ takes one and the same value at z and its reflection $1/\bar{z}$ in C. Hence, we have, putting $z=re^{i\theta}$, at any point on C except $z_1, \ldots\ldots, z_n$,

$$\frac{\partial u}{\partial r}=0 \quad \text{consequently} \quad \frac{\partial v}{\partial \theta}=0.$$

which means that $v(z)$ is equal to a constant on each of n arcs of C separated by the n points z_k. On the other hand, $u(z)$ is bounded below and unbounded above on each of these arcs. Hence the image of each of these arcs by $w=w(z)$ is an infinite radial slit.

The angle at $w=\infty$ between the images of two arcs forming an angle π at z_k is equal to $2\pi a_k$, since $w(z)$ has an expansion of the form

$$(2) \qquad (z-z_k)^{-2a_k}\left\{C_0+C_1\ (z-z_k)+\ldots\ldots\right\} \quad (C_0 \neq 0)$$

in a neighbourhood of $z=z_k$.

Further, $w(z)$ takes each value w_0, which does not belong to the n slits, once and only once in $|z|<1$. This follows easily from the facts that $1/w(z)$ has only one pole in $|z|<1$ and that

$$\arg\left\{\frac{1}{w(z)}-\frac{1}{w_0}\right\}$$

remains unchanged, when z moves on C once around and returns to the original value. Hence, by suitable determination of constant factor, $w(z)$ constructed above provides the required mapping.

Since the potential function $u(z)$ with required singularities is explicitly given by

$$u(z)=\sum_{k=1}^{n} 2a_k \log \frac{1}{|z-z_k|}+\log|z|+\text{const.},$$

we have the mentioned expression (1) for $w(z)$.

The uniqueness of the mapping function is proved as follows. Let $w_1(z)$ be another mapping function with the properties 1, 2, 3. Since $|w_1|$ remains unchanged by reflection in a radial slit, we obtain a one-valued harmonic function $\log |w_1(z)|$ with isolated singularities at $z=0$, ∞ and $z_1, \ldots\ldots, z_n$, while continuing $w_1(z)$ analytically across the unit circle by the principle of reflection. Then, by the expansion (2), $\log |w_1(z)|$ must be identical with $\log |w(z)|$, save for an additive constant.

We know, if $w(z)$ is schlicht and star-shaped in $|z|<1$ with respect to $w(0)=0$ being normalised by $w'(0)=1$, it is expressed in the form

$$w(z)=z \cdot \exp. \ 2\int_C \log \frac{\zeta}{\zeta - z} \, d\mu(\zeta) \tag{3}$$

and vice versa, where μ is a positive distribution of total mass 1 on C determined by the non-decreasing function of θ

$$\frac{1}{2\pi} \lim_{r \to 1} \arg w(re^{i\theta}).$$

This can easily be proved by applying Herglotz' formula to zw'/w and integrating it.

The mapping function (1) of Theorem 1 is in fact merely a special case of this formula, where μ vanishes outside the n points $z_1, \ldots\ldots, z_n$.

Next, we consider the case where the given set E is *infinite*. In case E consists of an enumerable infinity of points, we can construct a mapping function analogous to that of Theorem 1 by taking the limes of functions of the form (1), or, simply by (3), while we give a positive μ-mass to each point of E. But then, the boundary of the resulting image domain is in general very much complicated.

No matter whether E be enumerable or not, we have in the following case an image domain whose boundary is fairly simple.

Theorem 2. *If the closure $\bar{E}$ of E is of logarithmic capacity zero, and only in such a case, there exists a function $w(z)$, which maps the interior of C conformally on a domain D, so that:* 1. *D is the whole w-plane cut along an enumerable infinity of infinite radial slits, which cluster to $w=\infty$ only,* 2. *every point of E corresponds to an accessible boundary point of D lying on $w=\infty$, and* 3. $w(0)=0$.

Proof. If $\bar{E}$ is of logarithmic capacity zero, there exists, by Evans' theorem,[1] a positive distribution μ of total mass 1 on $\bar{E}$, such that the logarithmic potential

$$\int_{\bar{E}} \log \frac{1}{|\zeta - z|} \, d\mu(\zeta)$$

tends to $+\infty$, when z tends to any point of $\bar{E}$. Then the star-shaped function (3) constructed with this μ tends to ∞, whenever z tends to $\bar{E}$, and provides the required mapping.

On the contrary, if $\bar{E}$ is of positive capacity, there exists, for any positive mass distribution μ on C, at least one point ζ_0 on $\bar{E}$, such that

$$\underline{\lim}_{z\to\zeta_0} \int_C \log \frac{1}{|\zeta - z|}\, d\mu(\zeta) < +\infty,$$

and we have

$$\underline{\lim}_{z\to\zeta_0} |w(z)| < +\infty.$$

On the other hand, if $w(z)$ satisfies the condition 2, we have

$$\overline{\lim}_{z\to\zeta_0} |w(z)| = +\infty$$

for any point ζ_0 on $\bar{E}$, so that it can not satisfy the condition 1.

Let $\varDelta$ be a simply connected schlicht domain, and e be a closed set of accessible boundary points of $\varDelta$, which is of logarithmic capacity zero. Then M. Tsuji[2] proved the following extention of Beurling's theorem[3] on exceptional sets: if we map $\varDelta$ conformally on the interior of the unit circle C, then the set E of points on C, which corresponds to the set e, is of logarithmic capacity zero.

By this theorem, we have from Theorem 2 the following

Theorem 3. *If each primary end of $\varDelta$ in Carathéodory's sense, which contains a point of e, consists of only one point, then we can map $\varDelta$ conformally on a domain D satisfying the condition 1 of Theorem 2, so that each point of e corresponds to the point at infinity.*

Proof. From the hypothesis we see easily that E is closed, so that the result follows from Tsuji's theorem and Theorem 2.

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References.

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3) A. Beurling: Ensembles exceptionnels. Acta Math. 72 (1940).