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OBSTACLE PROBLEM FOR MUSIELAK-ORLICZ DIRICHLET ENERGY INTEGRAL ON METRIC MEASURE SPACES

FUMI-YUKI MAEDA, TAKAO OHNO AND TETSU SHIMOMURA

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Abstract. We introduce Musielak-Orlicz Newtonian space on a metric measure space. After discussing properties of weak upper gradients of functions in such spaces and Poincaré inequalities for functions with zero boundary values in bounded open subsets, we prove the existence and uniqueness of a solution to an obstacle problem for Musielak-Orlicz Dirichlet energy integral.

1. Introduction.

Sobolev spaces on metric measure spaces have been studied during the last two decades, see [8, 9, 13, 26], etc.; systematic presentations are given in the book [4]. The theory was generalized to Orlicz-Sobolev spaces on metric measure spaces in [2, 3, 28] and further to very general quasi-Banach function lattices in [18, 19].

The p-Dirichlet energy integral in metric measure spaces has been investigated by Shanmugalingam [27]. She proved the existence of a minimizer in Newtonian space $N^{1,p}(X)$, a Sobolev type space, which is defined in terms of p-weak upper gradients of functions in a metric measure space (X, d, μ). Kinnunen and Martio [15] studied the existence and uniqueness of a solution to an obstacle problem for p-Dirichlet energy integrals in Newtonian spaces. To show the existence of solutions, Poincaré inequalities play important roles. In [22], Mocanu proved the existence and uniqueness of a solution to an obstacle problem for an energy integral in Orlicz-Sobolev spaces supporting a Poincaré inequality.

Variable exponent Lebesgue spaces and Sobolev spaces were introduced to discuss nonlinear partial differential equations with non-standard growth conditions (see [5, 6]). Acerbi and Mingione [1] have studied the existence and the regularity of minimizers of the $p(\cdot)$-Dirichlet energy integral on a bounded domain in $\mathbf{R}^N$. Harjulehto, Hästö, Koskenoja and Varonen [10] defined and studied variable exponent Sobolev spaces with zero boundary values in the Euclidean setting and proved that Dirichlet energy integral has a minimizer. Their results are based on a $p(\cdot)$-Poincaré inequality.

Variable exponent Sobolev spaces on metric measure spaces have been developed during the past decades (see e.g. [7, 11, 12, 21]). Recently, we defined Musielak-Orlicz-Sobolev space on a metric measure space X defined in terms of a function $\Phi(x, t) : X \times [0, \infty) \to [0, \infty)$.

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We proved basic properties of such spaces (see [24]) and studied Musielak-Orlicz-Sobolev spaces with zero boundary values on X (see [25]), as an extension of [10, 14].

In this paper, we develop the theories for obstacle problems in the framework of Musielak-Orlicz-Sobolev space on a metric measure space X. We prove a Poincaré inequality for Musielak-Orlicz Newtonian functions with zero boundary values in bounded open subsets of X. Using the Poincaré inequality we prove the existence and uniqueness of a solution to an obstacle problem for a Φ-Dirichlet energy integral on a bounded open set in X.

This present paper is organized as follows. In Section 2, we define Musielak-Orlicz spaces $L^{\Phi}(\Omega)$. In Section 3, we study Φ-weak upper gradients and introduce Musielak-Orlicz Newtonian space $N^{1,\Phi}(\Omega)$. In Section 4, we study a capacity defined in terms of Φ. In Section 5, we define Musielak-Orlicz Newtonian spaces with zero boundary values and we consider the Poincaré inequalities for such spaces. In Section 6, we solve the obstacle problem for Φ-Dirichlet energy integral (see Theorem 6.1).

2. Musielak-Orlicz spaces. Throughout this paper, let C denote various constants independent of the variables in question and $C(a, b, \dots)$ be a constant that depends on $a, b, \dots$.

We denote by (X, d, μ) a metric measure spaces, where X is a set, d is a metric on X and μ is a nonnegative complete Borel regular outer measure on X which is finite and positive for every open balls in X. For simplicity, we often write X instead of (X, d, μ). For $x \in X$ and $r > 0$, we denote by $B(x, r)$ the open ball centered at x with radius r and $d_E = \sup\{d(x, y) : x, y \in E\}$ for a set $E \subset X$. We denote by χ_E the characteristic function of $E \subset X$.

We consider a function

$$\Phi(x, t) : X \times [0, \infty) \to [0, \infty)$$

satisfying the following conditions $(\Phi1)$ – $(\Phi4)$:

- $(\Phi1)$ $\Phi(\,\cdot\,, t)$ is measurable on X for each $t \geq 0$ and $\Phi(x, \,\cdot\,)$ is continuous on $[0, \infty)$ for each $x \in X$;
- $(\Phi2)$ $\Phi(x, 0) = 0$ and $\Phi(x, \cdot)$ is a convex function on $[0, \infty)$ for every $x \in X$;
- $(\Phi3)$ $0 < \inf_{x \in B} \Phi(x, 1) \leq \sup_{x \in B} \Phi(x, 1) < \infty$ for every open ball B in X;
- $(\Phi4)$ there exists a constant $A_d \geq 2$ such that

$$\Phi(x, 2t) \leq A_d \Phi(x, t) \quad \text{for all } x \in X \text{ and } t > 0\,.$$

Note that $(\Phi2)$ and $(\Phi4)$ imply

$$a\Phi(x, t) \leq \Phi(x, at) \leq \frac{A_d}{2} a^{\log_2 A_d} \Phi(x, t) \quad \text{for } a \geq 1\,; \tag{2.1}$$

in particular,

$$t\Phi(x, 1) \leq \Phi(x, t) \leq \frac{A_d}{2} t^{\log_2 A_d} \Phi(x, 1) \quad \text{for } t \geq 1\,. \tag{2.2}$$

REMARK 2.1. Let $p_0 \geq 1$. Suppose $\Phi(x,t)$ satisfies ($\Phi1$), ($\Phi3$), ($\Phi4$) and

($\Phi2'; p_0$) $t \mapsto t^{-p_0}\Phi(x,t)$ is uniformly almost increasing, namely there exists a constant $A' \geq 1$ such that

$$t^{-p_0}\Phi(x,t) \leq A' s^{-p_0}\Phi(x,s) \quad \text{for all } x \in X \quad \text{whenever } 0 \leq t < s\,.$$

Then,

$$\overline{\Phi}(x,t) = t^{p_0-1}\int_0^t \left\{ \sup_{0\leq s\leq r} s^{-p_0}\Phi(x,s)\right\} dr$$

satisfies ($\Phi1$), ($\Phi2$), ($\Phi3$) and ($\Phi4$) with the same A_d; further

$$\Phi(x,t/2) \leq \overline{\Phi}(x,t) \leq A'\Phi(x,t)$$

for all $x \in X$ and $t > 0$.

$\overline{\Phi}(x,\cdot)$ is strictly convex if $p_0 > 1$.

LEMMA 2.2. *For every $\varepsilon > 0$, there exists a constant $A(\varepsilon) > 0$ such that*

$$|\Phi(x,t_1) - \Phi(x,t_2)| \leq \varepsilon\{\Phi(x,t_1) + \Phi(x,t_2)\} + A(\varepsilon)\Phi(x,|t_1 - t_2|) \tag{2.3}$$

for all $x \in X$ and $t_1, t_2 \geq 0$.

PROOF. We may assume $t_1 > t_2$. If $\Phi(x,t_1) - \Phi(x,t_2) \leq \varepsilon\Phi(x,t_1)$, then (2.3) trivially holds. Thus, consider the case

$$\Phi(x,t_1) - \Phi(x,t_2) > \varepsilon\Phi(x,t_1)\,.$$

By ($\Phi2$) and ($\Phi4$), we see

$$\begin{aligned}\Phi(x,t_1) &\leq \frac{t_2}{t_1}\Phi(x,t_2) + \frac{t_1-t_2}{t_1}\Phi(x,t_1+t_2)\\ &\leq \Phi(x,t_2) + \frac{t_1-t_2}{t_1}A_d\Phi(x,t_1)\,.\end{aligned}$$

Hence

$$\varepsilon\Phi(x,t_1) < \Phi(x,t_1) - \Phi(x,t_2) \leq \frac{t_1-t_2}{t_1}A_d\Phi(x,t_1)\,,$$

which implies $t_1 < (A_d/\varepsilon)(t_1 - t_2)$. Thus,

$$\begin{aligned}|\Phi(x,t_1) - \Phi(x,t_2)| &\leq \Phi(x,t_1) \leq \Phi(x,(A_d/\varepsilon)(t_1-t_2))\\ &\leq A(\varepsilon)\Phi(x,t_1-t_2)\,.\end{aligned}$$

□

EXAMPLE 2.3. Let w be a positive measurable function on X such that $0 < \inf_{x\in B} w(x) \le \sup_{x\in B} w(x) < \infty$ for every open ball B in X. Let $p(\cdot)$ and $q_j(\cdot)$, $j = 1, \dots, k$, be measurable functions on X such that

(P1) $1 \le p^- := \inf_{x\in X} p(x) \le \sup_{x\in X} p(x) =: p^+ < \infty$

and

(Q1) $0 \le q_j^- := \inf_{x\in X} q_j(x) \le \sup_{x\in X} q_j(x) =: q_j^+ < \infty$

for all $j = 1, \dots, k$.

Set $L_c(t) = \log(c + t)$ for $c \ge e$ and $t \ge 0$, $L_c^{(1)}(t) = L_c(t)$, $L_c^{(j+1)}(t) = L_c(L_c^{(j)}(t))$ and

$$\Phi(x, t) = w(x)t^{p(x)} \int_0^t \left[\prod_{j=1}^k (L_c^{(j)}(s))^{q_j(x)} \right] ds.$$

Then, $\Phi(x, t)$ satisfies (Φ1), (Φ2), (Φ3) and (Φ4).

Let Ω be a measurable set in X. For $\Phi(x, t)$ satisfying (Φ1), (Φ2), (Φ3) and (Φ4), the associated Musielak-Orlicz space

$$L^\Phi(\Omega) = \left\{ f : \text{measurable function on } \Omega \text{ such that } \int_\Omega \Phi(y, |f(y)|)\, d\mu(y) < \infty \right\}$$

is a Banach space with respect to the norm

$$\|f\|_{L^\Phi(\Omega)} = \inf \left\{ \lambda > 0\,;\, \int_\Omega \Phi(y, |f(y)|/\lambda)\, d\mu(y) \le 1 \right\}$$

if we identify functions which are equal μ-a.e. (cf. [23]). Note that $L^\Phi(\Omega) \subset L^1(\Omega)$ if $\mu(\Omega) < \infty$ by (2.2).

For a measurable function f on Ω, we define the modular $\rho_{\Phi,\Omega}(f)$ by

$$\rho_{\Phi,\Omega}(f) = \int_\Omega \Phi(y, |f(y)|)\, d\mu(y)\,.$$

If $\Omega = X$, we denote $\rho_{\Phi,\Omega}(f)$ by $\rho_\Phi(f)$.

By convexity of $\Phi(x, \cdot)$ and (2.1), we see that

$$(2.4) \qquad \|f\|_{L^\Phi(\Omega)} \le \rho_{\Phi,\Omega}(f) \le \frac{A_d}{2} \|f\|_{L^\Phi(\Omega)}^\omega \quad \text{if } \|f\|_{L^\Phi(\Omega)} \ge 1$$

and

$$(2.5) \qquad 2(A_d)^{-1} \|f\|_{L^\Phi(\Omega)}^\omega \le \rho_{\Phi,\Omega}(f) \le \|f\|_{L^\Phi(\Omega)} \quad \text{if } \|f\|_{L^\Phi(\Omega)} \le 1\,,$$

where $\omega = \log_2 A_d$.

By (2.5), we have

LEMMA 2.4 (cf. [16, Lemma 2.2] and [23, Theorem 8.14]). *Let $\{f_i\}$ be a sequence in $L^\Phi(\Omega)$. Then $\rho_{\Phi,\Omega}(f_i)$ converges to 0 if and only if $\|f_i\|_{L^\Phi(\Omega)}$ converges to 0.*

LEMMA 2.5. *Let $\{f_i\}$ be a sequence in $L^\Phi(\Omega)$ and $f \in L^\Phi(\Omega)$. If $\rho_{\Phi,\Omega}(f_i - f)$ converges to 0, then $\rho_{\Phi,\Omega}(f_i)$ converges to $\rho_{\Phi,\Omega}(f)$.*

PROOF. First, note that $\{\rho_{\Phi,\Omega}(f_i)\}$ is bounded. In fact, by the above lemma, $\|f_i - f\|_{L^\Phi(\Omega)} \to 0$, and hence $\{\|f_i\|_{L^\Phi(\Omega)}\}$ is bounded, which implies that $\{\rho_{\Phi,\Omega}(f_i)\}$ is bounded by (2.4).

Let $\varepsilon > 0$ be arbitrarily given. By Lemma 2.2,

$$|\rho_{\Phi,\Omega}(f_i) - \rho_{\Phi,\Omega}(f)| \le \int_\Omega |\Phi(x, |f_i(x)|) - \Phi(x, |f(x)|)| \, d\mu(x)$$
$$\le \varepsilon\{\rho_{\Phi,\Omega}(f_i) + \rho_{\Phi,\Omega}(f)\} + A(\varepsilon)\rho_{\Phi,\Omega}(f_i - f),$$

so that

$$\limsup_{i\to\infty} |\rho_{\Phi,\Omega}(f_i) - \rho_{\Phi,\Omega}(f)| \le \varepsilon\left[\limsup_{i\to\infty} \rho_{\Phi,\Omega}(f_i) + \rho_{\Phi,\Omega}(f)\right].$$

Since $\{\rho_{\Phi,\Omega}(f_i)\}$ is bounded and $\varepsilon > 0$ is arbitrary, it follows that $\rho_{\Phi,\Omega}(f_i) \to \rho_{\Phi,\Omega}(f)$. □

LEMMA 2.6. Let B be an open ball in X. Then

$$\|1\|_{L^\Phi(B)} \le \max\left\{1, \mu(B) \sup_{x\in B} \Phi(x, 1)\right\}.$$

PROOF. Let $\lambda = \mu(B) \sup_{x\in B} \Phi(x, 1)$.

If $\lambda \ge 1$, then by convexity

$$\int_B \Phi(x, 1/\lambda) \, d\mu(x) \le (1/\lambda) \int_B \Phi(x, 1) \, d\mu(x) \le 1 \, ,$$

so that $\|1\|_{L^\Phi(B)} \le \lambda$.

If $\lambda \le 1$, then $\int_B \Phi(x, 1) \, d\mu(x) \le \lambda \le 1$, so that $\|1\|_{L^\Phi(B)} \le 1$. □

3. Φ-weak upper gradient and Musielak-Orlicz Newtonian space $N^{1,\Phi}(\Omega)$. Let $\Gamma(\Omega)$ be the family of all rectifiable curves in a set $\Omega \subset X$. Each $\gamma \in \Gamma(X)$ is a nonconstant continuous map $\gamma : [0, \ell_\gamma] \to X$, where ℓ_γ is the length of γ. For $\Gamma \subset \Gamma(X)$, we denote by $F(\Gamma)$ the set of all Borel measurable functions $h : X \to [0, \infty]$ such that

$$\int_\gamma h \, ds \ge 1$$

for every $\gamma \in \Gamma$, where ds represents integration with respect to arc length. We define the Φ-modulus of $\Gamma \subset \Gamma(X)$ by

$$M_\Phi(\Gamma) = \inf_{h\in F(\Gamma)} \rho_\Phi(h) \, .$$

If $F(\Gamma) = \emptyset$, then we set $M_\Phi(\Gamma) = \infty$.

For a set $\Omega \subset X$, we say that a property holds for M_Φ-a.e. $\gamma \in \Gamma(\Omega)$, if it holds on $\gamma \in \Gamma(\Omega) \setminus \Gamma$ for a family $\Gamma \subset \Gamma(X)$ with $M_\Phi(\Gamma) = 0$.

REMARK 3.1. In [19], in a general setting of quasi-Banach function lattices, Malý defined modulus of curves in terms of norms instead of modular. His definition applied to our case is:

$$\mathrm{Mod}_{L^\Phi(X)}(\Gamma) = \inf_{h\in F(\Gamma)} \|h\|_{L^\Phi(X)} \, .$$

This plays almost the same roles as our M_Φ; in particular, since

$$M_\Phi(\Gamma) = 0 \quad \text{if and only if} \quad \mathrm{Mod}_{L^\Phi(X)}(\Gamma) = 0\,,$$

in view of Lemma 2.4, the notions "M_Φ-a.e." and "$\mathrm{Mod}_{L^\Phi(X)}$-a.e." coincide. Further, proofs of the results in [18] and [19] are often applicable to the proofs of corresponding results in this paper.

LEMMA 3.2 ([19, Lemma 4.10]). *Let Ω be a measurable set in X and h be a nonnegative measurable function on Ω. Then $\int_\gamma h\,ds$ is well-defined for M_Φ-a.e. $\gamma \in \Gamma(\Omega)$; in fact, if h_1 is a nonnegative Borel functions on X such that $h_1 = h$ μ-a.e. in Ω, then*

$$\int_\gamma h\,ds = \int_\gamma h_1\,ds$$

for M_Φ-a.e. $\gamma \in \Gamma(\Omega)$.

Let Ω be a measurable set in X and let u be a function $\Omega \to [-\infty, \infty]$. A nonnegative measurable function h on Ω is said to be a Φ-weak upper gradient of u in Ω if

$$|u(\gamma(0)) - u(\gamma(\ell_\gamma))| \le \int_\gamma h\,ds \tag{3.1}$$

holds for M_Φ-a.e. $\gamma \in \Gamma(\Omega)$. Here, by saying that (3.1) holds, we understand that $\int_\gamma h\,ds$ is well-defined and $\int_\gamma h\,ds = \infty$ in case $|u(\gamma(0))| = \infty$ or $|u(\gamma(\ell_\gamma))| = \infty$ (cf. [4]).

REMARK 3.3. Let $\Omega' \subset \Omega$ be a measurable set. If h is a Φ-weak upper gradient of u in Ω, then $h|_{\Omega'}$ is a Φ-weak upper gradient of $u|_{\Omega'}$ in Ω'.

The Musielak-Orlicz Newtonian space $N^{1,\Phi}(\Omega)$ is defined to be the family of all $u \in L^\Phi(\Omega)$ having a Φ-weak upper gradient $h \in L^\Phi(\Omega)$ in Ω. For $u \in N^{1,\Phi}(\Omega)$ we define

$$\|u\|_{N^{1,\Phi}(\Omega)} = \|u\|_{L^\Phi(\Omega)} + \inf \|h\|_{L^\Phi(\Omega)}\,,$$

where the infimum is taken over all Φ-weak upper gradients of u in Ω.

We say that u is absolutely continuous on a curve γ, if $u \circ \gamma$ is absolutely continuous on $[0, \ell_\gamma]$. Let $ACC_\Phi(\Omega)$ be the family of measurable functions on Ω each of which is absolutely continuous on M_Φ-a.e. $\gamma \in \Gamma(\Omega)$.

LEMMA 3.4 ([19, Theorem 6.7]). *If $u \in N^{1,\Phi}(\Omega)$, then $u \in ACC_\Phi(\Omega)$.*

LEMMA 3.5 ([19, Lemma 6.8]). *Let $u \in ACC_\Phi(\Omega)$ and let g be a nonnegative measurable function on Ω. If, for M_Φ-a.e. $\gamma \in \Gamma(\Omega)$,*

$$|(u \circ \gamma)'(t)| \le g(\gamma(t)) \quad \textit{for a.e. } t \in [0, \ell_\gamma], \tag{3.2}$$

then g is a Φ-weak upper gradient of u in Ω.

Conversely, let $u \in ACC_\Phi(\Omega)$ and let $g \in L^\Phi(\Omega)$ be a Φ-weak upper gradient of u in Ω. Then (3.2) *holds for M_Φ-a.e. $\gamma \in \Gamma(\Omega)$.*

We say that $h_u \in L^\Phi(\Omega)$ is a minimal Φ-weak upper gradient of $u \in N^{1,\Phi}(\Omega)$ in Ω if h_u is a Φ-weak upper gradient of u in Ω and $h_u \le h$ μ-a.e. in Ω for all Φ-weak upper gradients $h \in L^\Phi(\Omega)$ of u in Ω.

LEMMA 3.6 (cf. [18, Theorem 4.6]). *Let $u \in N^{1,\Phi}(\Omega)$. Then there exists a minimal Φ-weak upper gradient h_u of u in Ω.*

Moreover h_u is unique up to sets of measure zero.

LEMMA 3.7 (cf. [4, Corollary 2.20]). *Let $u, v \in N^{1,\Phi}(\Omega)$ and let h_u and h_v be minimal Φ-weak upper gradients of u and v in Ω respectively. Then $h_u\chi_{\{u>v\}} + h_v\chi_{\{v\ge u\}}$ is a minimal Φ-weak upper gradient of* $\max\{u, v\}$ *in Ω and $h_v\chi_{\{u>v\}} + h_u\chi_{\{v\ge u\}}$ is a minimal Φ-weak upper gradient of* $\min\{u, v\}$ *in Ω.*

LEMMA 3.8 (cf. [4, Corollary 2.21]). *Let $u, v \in N^{1,\Phi}(\Omega)$ and let h_u and h_v be minimal Φ-weak upper gradients of u and v in Ω respectively. Then $h_u = h_v$ μ-a.e. on $\{x \in \Omega : u(x) = v(x)\}$.*

LEMMA 3.9 (cf. [4, Lemma 2.23]). *Let $E \subset \Omega$ be an open set. If $u \in N^{1,\Phi}(\Omega)$ and h_u is a minimal Φ-weak upper gradient of u in Ω, then $h_u|_E$ is a minimal Φ-weak upper gradient of $u|_E$ in E.*

LEMMA 3.10 ([19, Proposition 6.10]). *Let $u, v \in N^{1,\Phi}(\Omega)$ and let h_u and h_v be minimal Φ-weak upper gradients of u and v in Ω respectively. Then $|u|h_v + |v|h_u$ is a Φ-weak upper gradient of uv in Ω.*

4. Capacity c_Φ. For $u \in N^{1,\Phi}(\Omega)$, we set

$$\hat\rho_{\Phi,\Omega}(u) = \rho_{\Phi,\Omega}(u) + \inf \rho_{\Phi,\Omega}(h)\,,$$

where the infimum is taken over all Φ-weak upper gradients of u in Ω.

For $E \subset \Omega$, we denote

$$s_\Phi(E;\Omega) = \{u \in N^{1,\Phi}(\Omega) : u \ge 1 \text{ on } E\}$$

and define the Φ-capacity with respect to Ω by

$$c_\Phi(E;\Omega) = \inf_{u\in s_\Phi(E;\Omega)} \hat\rho_{\Phi,\Omega}(u)\,.$$

In case $s_\Phi(E;\Omega) = \emptyset$, we set $c_\Phi(E;\Omega) = \infty$. If $X = \Omega$, we denote $s_\Phi(E;\Omega)$ and $c_\Phi(E;\Omega)$ by $s_\Phi(E)$ and $c_\Phi(E)$ respectively.

$c_\Phi(\,\cdot\,;\Omega)$ is an outer measure; in particular, it is countably subadditive (see [24, Proposition 4.5]).

REMARK 4.1. For $E \subset \Omega$, $c_\Phi(E;\Omega) \le c_\Phi(E)$.

REMARK 4.2. In [19], Malý defined a capacity in terms of norms instead of modular. As remarked for the notion of modulus of curves in Remark 3.1, our capacity c_Φ plays almost the same roles as the capacity defined in [19].

LEMMA 4.3. *Let B be an open ball with radius r in X. Then*

$$c_\Phi(B) \le \left(1 + \max\{r^{-1}, A_d r^{-\omega}/2\}\right)\mu(2B)\sup_{x\in 2B}\Phi(x,1)\,,$$

where $\omega = \log_2 A_d$.

PROOF. Set $u(x) = \max\{1 - d(x,B)/r, 0\}$ and

$$h(x) = \begin{cases} 1/r & \text{for } x \in 2B \\ 0 & \text{for } x \in X \setminus 2B\,. \end{cases}$$

Then $u \in L^\Phi(X)$, $u = 1$ on B and h is a Φ-weak upper gradient of u in X, so that $u \in s_\Phi(B)$. Hence we have by (Φ2) and (2.1)

$$\begin{aligned} c_\Phi(B) &\le \int_{2B} \Phi(x,1)\, d\mu(x) + \int_{2B} \Phi\left(x, \frac{1}{r}\right) d\mu(x) \\ &\le \mu(2B)\sup_{x\in 2B}\Phi(x,1) + \mu(2B)\sup_{x\in 2B}\Phi(x,1)\max\{r^{-1}, A_d r^{-\omega}/2\}, \end{aligned}$$

as required. □

For a set $E \subset \Omega$, we say that a property holds $c_\Phi(\cdot;\Omega)$-q.e. in E, if it holds on E except of a set $F \subset E$ with $c_\Phi(F;\Omega) = 0$, where q.e. stands for quasi-everywhere.

LEMMA 4.4 ([19, Corollary 5.11]). *If $u = v$ $c_\Phi(\cdot;\Omega)$-q.e. in Ω and h is a Φ-weak upper gradient of u with respect to Ω, then h is also a Φ-weak upper gradient of v in Ω.*

LEMMA 4.5 ([19, Proposition 6.12]). *If $u, v \in N^{1,\Phi}(\Omega)$ and $u = v$ μ-a.e. in Ω, then $u = v$ $c_\Phi(\cdot;\Omega)$-q.e. in Ω.*

Moreover, if Ω is an open set in X, then $u = v$ c_Φ-q.e. in Ω.

LEMMA 4.6 ([18, Proposition 5.6]). *Let Ω be an open set in X. Let $h_i \in L^\Phi(\Omega)$ be a Φ-weak upper gradient of $u_i \in N^{1,\Phi}(\Omega)$ in Ω for $i = 1, 2, \ldots$. Suppose $\{u_i\}$ converges to a function u in $L^\Phi(\Omega)$ and $\{h_i\}$ converges to a nonnegative function h in $L^\Phi(\Omega)$. Then there exists a measurable function $\tilde{u}$ such that $\tilde{u} = u$ μ-a.e. in Ω and h is a Φ-weak upper gradient of $\tilde{u}$ in Ω, and there exists a subsequence $\{u_{i_k}\}$ which converges to $\tilde{u}$ pointwise c_Φ-q.e. in Ω.*

Moreover, if there exists a subsequence $\{u_{i_k}\}$ which converges to u pointwise c_Φ-q.e. in Ω, then we may choose $\tilde{u} = u$ in Ω.

LEMMA 4.7 (cf. [4, Lemma 6.2]). *Let Ω be an open set in X. Assume that $L^\Phi(\Omega)$ is reflexive. Suppose $\{u_i\}$ and $\{h_i\}$ are bounded sequences in $L^\Phi(\Omega)$ such that h_i is a Φ-weak upper gradient of u_i in Ω for $i = 1, 2, \ldots$. Then there exist $u, h \in L^\Phi(\Omega)$, subsequences $\{u_{i_k}\}$ and $\{h_{i_k}\}$ and convex combinations $v_k = \sum_{n=k}^{N_k} a_{k,n} u_{i_n}$ and $g_k = \sum_{n=k}^{N_k} a_{k,n} h_{i_n}$ such that*

(1) *$\{v_k\}$ and $\{g_k\}$ converge to u and h in $L^\Phi(\Omega)$ respectively;*

(2) *there exists a subsequence $\{v_{k_i}\}$ which converges pointwise to u c_Φ-q.e. in Ω;*

(3) *h is a Φ-weak upper gradient of u in Ω.*

5. Musielak-Orlicz Newtonian spaces with zero boundary values $N_0^{1,\Phi}(E)$ and Poincaré inequality. For $E \subset X$, we define

$$N_0^{1,\Phi}(E) = \{f|_E : f \in N^{1,\Phi}(X) \text{ and } f = 0 \text{ in } X \setminus E\}.$$

By Lemma 4.4, we have

$$N_0^{1,\Phi}(E) = \{f|_E : f \in N^{1,\Phi}(X) \text{ and } f = 0 \ c_\Phi\text{-q.e. in } X \setminus E\}.$$

LEMMA 5.1 (cf. [4, Lemma 2.37]). *Let $u \in N^{1,\Phi}(\Omega)$ and let $v, w \in N_0^{1,\Phi}(\Omega)$ be such that $v \le u \le w$ c_Φ-q.e. in Ω. Then $u \in N_0^{1,\Phi}(\Omega)$.*

LEMMA 5.2. *Let $\Omega \subset X$ be an open set. Let $u_1 \in N_0^{1,\Phi}(\Omega)$ and let h_1 be a Φ-weak upper gradient of u_1 in Ω. Set*

$$u = \begin{cases} u_1 & \text{on } \Omega \\ 0 & \text{on } X \setminus \Omega \end{cases} \quad \text{and} \quad h = \begin{cases} h_1 & \text{on } \Omega \\ 0 & \text{on } X \setminus \Omega . \end{cases}$$

Then h is a Φ-weak upper gradient of u in X.

PROOF. Since $u \in N^{1,\Phi}(X)$ by definition, there exists a minimal Φ-weak upper gradient h_u of u in X by Lemma 3.6. Then, by Lemma 3.8, we may assume that h_u is identically zero outside Ω. On the other hand, $h_u|_\Omega$ is a minimal Φ-weak upper gradient of u_1 in Ω by Lemma 3.9, and hence $h_u \le h_1$ μ-a.e. in Ω, so that $h_u \le h$ μ-a.e. in X. Hence, we obtain the required result by Lemma 3.2. □

We say that X supports a Φ-Poincaré inequality if, for every open ball B in X, there exist constants $C_P(B) > 0$ and $\lambda \ge 1$ such that

$$\|u - u_B\|_{L^\Phi(B)} \le C_P(B)\|h\|_{L^\Phi(\lambda B)}$$

holds whenever h is a Φ-weak upper gradient of u on λB and u is integrable on B, where $u_B = ⨍_B u \, d\mu$ is the mean-value of u on B.

EXAMPLE 5.3. $\mathbf{R}^N$ supports a Φ-Poincaré inequality if $\Phi(x,t)$ satisfies $(\Phi 2'; p_0)$ for $p_0 > 1$ and the following condition for $0 < \nu < p_0/N$:

$(\Phi 5; \nu)$ For every $\gamma > 0$, there exists a constant $B_{\gamma,\nu} \ge 1$ such that

$$\Phi(x,t) \le B_{\gamma,\nu}\Phi(y,t)$$

whenever $|x - y| \le \gamma t^{-\nu}$ and $t \ge 1$.

We give a proof of this fact in the Appendix (Section 7).

PROPOSITION 5.4 (cf. [4, Lemma 5.53]). *Assume that X supports a Φ-Poincaré inequality. Let $B = B(x_0, r)$ be an open ball in X. Then there exists a constant $C = C(\sup_{x \in 2B} \Phi(x,1), A_d, C_P(B), \mu(2B), r) > 0$ such that*

$$c_\Phi(B \cap S)\|u\|_{L^\Phi(2B)} \le C\|h_u\|_{L^\Phi(2\lambda B)}$$

for all $u \in N^{1,\Phi}(X)$*, where* λ *is the constant in the* Φ*-Poincaré inequality,* $S = \{x \in X : u(x) = 0\}$ *and* $h_u \in L^{\Phi}(X)$ *is a minimal* Φ*-weak upper gradient of* u *in* X*.*

PROOF. Denote by h_g a minimal Φ-weak upper gradient of g in X. Let $u \in N^{1,\Phi}(X)$. First note from the Φ-Poincaré inequality that u is a constant μ-a.e. in $2B$ if $\|h_u\|_{L^\Phi(2\lambda B)} = 0$, so that it is sufficient to prove that there exists a constant $C = C(\sup_{x\in 2B} \Phi(x,1), A_d, C_P(B), \mu(2B), r) > 0$ such that

$$c_\Phi(B \cap S)\|u\|_{L^\Phi(2B)} \le C \tag{5.1}$$

for all $u \in N^{1,\Phi}(X)$ with $\|h_u\|_{L^\Phi(2\lambda B)} = 1$. Further, we may assume that u is nonnegative on $2B$ by Lemma 3.7.

If $\|u\|_{L^\Phi(2B)} \le \|1\|_{L^\Phi(2B)}$, then we see that (5.1) holds by Lemmas 2.6 and 4.3. Thus, assume that $\|u\|_{L^\Phi(2B)} > \|1\|_{L^\Phi(2B)}$ and set $\alpha = \|u\|_{L^\Phi(2B)}/\|1\|_{L^\Phi(2B)} (> 1)$. Let $\eta(x) = \max\{1 - \mathrm{dist}(x,B)/r, 0\}$. Then $h_\eta \le (1/r)\chi_{2B}$. Set $v = \eta(1 - u/\alpha)$. By Lemma 3.10, we see that $(h_\eta|u-\alpha| + h_u)/\alpha$ is a Φ-weak upper gradient of v in X, so that $v \in N^{1,\Phi}(X)$. Since $v = 1$ in $B \cap S$, we have

$$c_\Phi(B \cap S) \le \rho_\Phi(v) + \rho_\Phi(h_v)\,. \tag{5.2}$$

Since $\alpha > 1$,

$$\rho_\Phi(v) \le \rho_{\Phi,2B}(1 - u/\alpha) \le \frac{1}{\alpha}\rho_{\Phi,2B}(u - \alpha)\,.$$

By (Φ4) and convexity of $\Phi(x,\cdot)$,

$$\rho_{\Phi,2B}(u-\alpha) \le \frac{A_d}{2}(\rho_{\Phi,2B}(u - u_{2B}) + \rho_{\Phi,2B}(u_{2B} - \alpha))\,.$$

Since

$$\begin{aligned} |u_{2B} - \alpha| &= \frac{\big|u_{2B}\|1\|_{L^\Phi(2B)} - \|u\|_{L^\Phi(2B)}\big|}{\|1\|_{L^\Phi(2B)}} \\ &\le \frac{\|u - u_{2B}\|_{L^\Phi(2B)}}{\|1\|_{L^\Phi(2B)}} \le \frac{C_P(B)}{\|1\|_{L^\Phi(2B)}} \end{aligned}$$

by the Φ-Poincaré inequality, we see that

$$\rho_{\Phi,2B}(u_{2B} - \alpha) \le C_1\rho_{\Phi,2B}(1/\|1\|_{L^\Phi(2B)}) \le C_1$$

by (2.1), where $C_1 = \max\{C_P(B), A_dC_P(B)^\omega/2\}$. By (2.4), (2.5) and the Φ-Poincaré inequality,

$$\rho_{\Phi,2B}(u - u_{2B}) \le C_1\,.$$

Hence,

$$\rho_{\Phi,2B}(u - \alpha) \le A_dC_1, \tag{5.3}$$

so that $\rho_\Phi(v) \le A_dC_1/\alpha$.

Since

$$h_v \le \frac{h_\eta |u-\alpha| + h_u}{\alpha} \chi_{2B} \le \frac{1}{\alpha}\left(\frac{1}{r}|u-\alpha| + h_u\right)\chi_{2B}$$

and $\alpha > 1$, we see that

$$\begin{aligned}\rho_\Phi(h_v) &\le \frac{1}{\alpha}\rho_{\Phi,2B}\left(\frac{1}{r}|u-\alpha| + h_u\right)\\ &\le \frac{A_d}{2\alpha}\left\{\rho_{\Phi,2B}\left(\frac{1}{r}(u-\alpha)\right) + \rho_{\Phi,2B}(h_u)\right\}\\ &\le \frac{A_d}{2\alpha}\left\{\max\{1/r, A_d/(2r^\omega)\}\rho_{\Phi,2B}(u-\alpha) + 1\right\}\\ &\le \frac{C_2(A_d, C_P(B), r)}{\alpha}\end{aligned}$$

in view of (5.3). Since $1/\alpha \le C_3(\sup_{x\in 2B}\Phi(x,1), \mu(2B))/\|u\|_{L^\Phi(2B)}$, we finally obtain (5.1) from (5.2). □

By Lemma 3.8 and Proposition 5.4, we have the following Poincaré inequalities for $N_0^{1,\Phi}(E)$.

COROLLARY 5.5 (cf. [4, Corollary 5.54]). *Assume that X supports a Φ-Poincaré inequality. Let Ω be a bounded set in X with $c_\Phi(X \setminus \Omega) > 0$. Then there exists a constant $C > 0$ such that*

$$\|u\|_{L^\Phi(X)} \le C\|h_u\|_{L^\Phi(X)}$$

for all $u \in N_0^{1,\Phi}(\Omega)$, where $h_u \in L^\Phi(X)$ is a minimal Φ-weak upper gradient of u in X (by considering as $u = 0$ on $X \setminus \Omega$).

PROOF. Let $u \in N_0^{1,\Phi}(\Omega)$. Then we may assume that $u \in N^{1,\Phi}(X)$ and $u = 0$ on $X \setminus \Omega$. Let $h_u \in L^\Phi(X)$ be a minimal Φ-weak upper gradient of u in X. By Lemma 3.8, we have $h_u = 0$ μ-a.e. in $X \setminus \Omega$. Since Ω is a bounded set in X with $c_\Phi(X \setminus \Omega) > 0$, there exists an open ball $B \supset \Omega$ such that $c_\Phi(B \setminus \Omega) > 0$. By Proposition 5.4, we find

$$\|u\|_{L^\Phi(X)} = \|u\|_{L^\Phi(2B)} \le \frac{C}{c_\Phi(B\setminus\Omega)}\|h_u\|_{L^\Phi(2\lambda B)} = C\|h_u\|_{L^\Phi(X)},$$

as required. □

6. Obstacle problem in $N^{1,\Phi}(\Omega)$. From now on, we assume that Ω is an open bounded set with $c_\Phi(X \setminus \Omega) > 0$. We denote by h_g a minimal Φ-weak upper gradient of g in Ω.

For $f \in N^{1,\Phi}(\Omega)$ and $\psi : \Omega \to [-\infty, \infty]$, we define

$$\mathcal{K}_{\psi,f}(\Omega) = \{u \in N^{1,\Phi}(\Omega) : u - f \in N_0^{1,\Phi}(\Omega) \text{ and } u \ge \psi \ c_\Phi\text{-q.e. in } \Omega\}.$$

A function $u \in \mathcal{K}_{\psi,f}(\Omega)$ is called a solution of the $\mathcal{K}_{\psi,f}(\Omega)$-obstacle problem in $N^{1,\Phi}(\Omega)$ if

$$\int_\Omega \Phi(x, h_u(x))\, d\mu(x) \le \int_\Omega \Phi(x, h_v(x))\, d\mu(x)$$

for all $v \in \mathcal{K}_{\psi,f}(\Omega)$.

THEOREM 6.1 (cf. [4, Theorem 7.2]). *Assume that $L^{\Phi}(\Omega)$ is reflexive and X supports a Φ-Poincaré inequality. Let $f \in N^{1,\Phi}(\Omega)$ and $\psi : \Omega \to [-\infty, \infty]$. If $\mathcal{K}_{\psi,f}(\Omega) \neq \emptyset$, then there exists a solution of the $\mathcal{K}_{\psi,f}(\Omega)$-obstacle problem in $N^{1,\Phi}(\Omega)$.*

Further, if $\Phi(x,\cdot)$ is strictly convex for μ-a.e. $x \in \Omega$, then the solution of the $\mathcal{K}_{\psi,f}(\Omega)$-obstacle problem in $N^{1,\Phi}(\Omega)$ is unique (up to sets of c_{Φ}-capacity zero).

PROOF. Set

$$I = \inf_{v \in \mathcal{K}_{\psi,f}(\Omega)} \int_{\Omega} \Phi(x, h_v(x))\, d\mu(x)\,.$$

Then $0 \le I < \infty$ since $\mathcal{K}_{\psi,f}(\Omega) \neq \emptyset$. Take $\{v_j\} \subset \mathcal{K}_{\psi,f}(\Omega)$ such that $\int_{\Omega} \Phi(x, h_{v_j}(x))\, d\mu(x)$ converges to I as $j \to \infty$. Here note that $\{h_{v_j}\}$ is bounded in $L^{\Phi}(\Omega)$. By Corollary 5.5 and Lemmas 3.8 and 3.9, we have

$$\|v_j - f\|_{L^{\Phi}(\Omega)} \le C\|h_{v_j - f}\|_{L^{\Phi}(\Omega)} \le C\left(\|h_{v_j}\|_{L^{\Phi}(\Omega)} + \|h_f\|_{L^{\Phi}(\Omega)}\right)\,.$$

Hence $\{v_j\}$ is bounded in $N^{1,\Phi}(\Omega)$.

By Lemma 4.7, there exist sequences $\{u_j\}, \{h_j\} \subset L^{\Phi}(\Omega)$ and functions $u, h \in L^{\Phi}(\Omega)$ such that $\{u_j\}$ and $\{h_j\}$ converge to u and h in $L^{\Phi}(\Omega)$ respectively, $\{u_j\}$ converges pointwise to u c_{Φ}-q.e. in Ω, h_j and h are Φ-weak upper gradients of u_j and u in Ω respectively, where u_j, h_j are convex combinations of subsequences of $\{v_k\}_{k \ge j}$, $\{h_{v_k}\}_{k \ge j}$ respectively. It follows that $u \in N^{1,\Phi}(\Omega)$. Further, $u_j \ge \psi$ c_{Φ}-q.e. in Ω, which implies $u \ge \psi$ c_{Φ}-q.e. in Ω. Also, we see that $u_j - f \in N_0^{1,\Phi}(\Omega)$. Let $w_j \in N^{1,\Phi}(X)$ be such that $w_j = u_j - f$ on Ω and $w_j = 0$ on $X \setminus \Omega$. Then, w_j converges to w in $L^{\Phi}(X)$, where $w = u - f$ on Ω and $w = 0$ on $X \setminus \Omega$. We consider $g_j := h_j + h_f$ and $g := h + h_f$ to be identically zero outside Ω. Since g_j is a Φ-weak upper gradient of w_j in X by Lemma 5.2 and $\{w_j\}$ and $\{g_j\}$ converge to w and g in $L^{\Phi}(X)$ respectively, we have $w \in N^{1,\Phi}(X)$ by Lemma 4.6, so that $u - f \in N_0^{1,\Phi}(\Omega)$. Therefore $u \in \mathcal{K}_{\psi,f}(\Omega)$. By convexity of $\Phi(x,\cdot)$,

$$\int_{\Omega} \Phi(x, h_j(x))\, d\mu(x) \le \sup_{k \ge j} \int_{\Omega} \Phi(x, h_{v_k}(x))\, d\mu(x)\,,$$

so that

$$\lim_{j \to \infty} \int_{\Omega} \Phi(x, h_j(x))\, d\mu(x) \le I\,.$$

Hence

$$I \le \int_{\Omega} \Phi(x, h_u(x))\, d\mu(x) \le \int_{\Omega} \Phi(x, h(x))\, d\mu(x) = \lim_{j \to \infty} \int_{\Omega} \Phi(x, h_j(x))\, d\mu(x) \le I$$

by Lemma 2.5, which shows that u is the desired minimizer.

We next prove the uniqueness. Assume that u_1 and u_2 are solutions of the $\mathcal{K}_{\psi,f}(\Omega)$-obstacle problem. Then, since $u_3 = (u_1 + u_2)/2 \in \mathcal{K}_{\psi,f}(\Omega)$, we have by strictly convexity of Φ

$$\begin{aligned}\int_\Omega \Phi(x, h_{u_1}(x))\, d\mu(x) &\le \int_\Omega \Phi(x, h_{u_3}(x))\, d\mu(x)\\ &\le \int_\Omega \Phi\left(x, \frac{h_{u_1}(x)+h_{u_2}(x)}{2}\right) d\mu(x)\\ &< \frac{1}{2}\left(\int_\Omega \Phi(x, h_{u_1}(x))\, d\mu(x) + \int_\Omega \Phi(x, h_{u_2}(x))\, d\mu(x)\right)\\ &= \int_\Omega \Phi(x, h_{u_1}(x))\, d\mu(x)\end{aligned}$$

if $\mu(\{x \in \Omega : h_{u_1}(x) \neq h_{u_2}(x)\}) > 0$. Hence, $h_{u_1} = h_{u_2}$ μ-a.e. in Ω.

For $c \in \mathbf{R}$, set

$$u_c = \max\{u_1, \min\{u_2, c\}\}\,.$$

Then $u_c \in N^{1,\Phi}(\Omega)$ and $u_c \ge \psi$ c_Φ-q.e. in Ω. Since

$$u_c - f \le \max\{u_1 - f, u_2 - f\} \in N_0^{1,\Phi}(\Omega)$$

and $u_c - f \ge u_1 - f \in N_0^{1,\Phi}(\Omega)$, we have $u_c - f \in N_0^{1,\Phi}(\Omega)$ by Lemma 5.1, so that $u_c \in \mathcal{K}_{\psi,f}(\Omega)$. Let

$$V_c = \{x \in \Omega : u_1(x) < c < u_2(x)\}\,.$$

Then note that $V_c \subset \{x \in \Omega : u_c(x) = c\}$, so that $h_{u_c} = 0$ μ-a.e. in V_c by Lemma 3.8. The minimizer property of h_{u_1} implies

$$\begin{aligned}\int_\Omega \Phi(x, h_{u_1}(x))\, d\mu(x) &\le \int_\Omega \Phi(x, h_{u_c}(x))\, d\mu(x)\\ &= \int_{\Omega\setminus V_c} \Phi(x, h_{u_c}(x))\, d\mu(x) = \int_{\Omega\setminus V_c} \Phi(x, h_{u_1}(x))\, d\mu(x)\end{aligned}$$

since $h_{u_1} = h_{u_2} = h_{u_c}$ μ-a.e. in $\Omega \setminus V_c$ by Lemma 3.7. Hence, we have $h_{u_1} = h_{u_2} = 0$ μ-a.e. in V_c for all $c \in \mathbf{R}$. Since

$$\{x \in \Omega : u_1(x) < u_2(x)\} \subset \bigcup_{c\in\mathbf{Q}} V_c\,,$$

we see that $h_{u_1} = h_{u_2} = 0$ μ-a.e. in $\{x \in \Omega : u_1(x) < u_2(x)\}$. Similarly, $h_{u_1} = h_{u_2} = 0$ μ-a.e. in $\{x \in \Omega : u_1(x) > u_2(x)\}$. It follows that

$$h_{u_1-u_2}(x) \le (h_{u_1}(x) + h_{u_2}(x))\chi_{\{x\in\Omega: u_1(x)\neq u_2(x)\}} = 0$$

for μ-a.e in Ω. In view of Lemma 3.9, we find

$$\|u_1 - u_2\|_{L^\Phi(\Omega)} \le C\|h_{u_1-u_2}\|_{L^\Phi(\Omega)} = 0$$

by Corollary 5.5. Hence we have $u_1 = u_2$ c_Φ-q.e. in Ω by Lemma 4.5, as required. □

REMARK 6.2. If $f \in N^{1,\Phi}(\Omega)$ and $\max\{\psi - f, 0\} \in N_0^{1,\Phi}(\Omega)$, then $u = \max\{f, \psi\} \in \mathcal{K}_{\psi,f}(\Omega)$. Conversely, if $\mathcal{K}_{\psi,f}(\Omega) \neq \emptyset$ for $f \in N^{1,\Phi}(\Omega)$ and $\psi \in N^{1,\Phi}(\Omega)$, then we see that $\max\{\psi - f, 0\} \in N_0^{1,\Phi}(\Omega)$ by Lemma 5.1; cf. [4, Proposition 7.4].

REMARK 6.3. A solution u of the $\mathcal{K}_{\psi,f}(\Omega)$-obstacle problem is a superminimizer of the Φ-Dirichlet energy integral on Ω, namely

$$\int_{\{y\in\Omega:\varphi(y)\neq 0\}} \Phi(x, h_u(x))\, d\mu(x) \leq \int_{\{y\in\Omega:\varphi(y)\neq 0\}} \Phi(x, h_{u+\varphi}(x))\, d\mu(x), \tag{6.1}$$

for all nonnegative $\varphi \in N_0^{1,\Phi}(\Omega)$.

In fact, since $u + \varphi \in \mathcal{K}_{\psi,f}(\Omega)$,

$$\int_\Omega \Phi(x, h_u(x))\, d\mu(x) \leq \int_\Omega \Phi(x, h_{u+\varphi}(x))\, d\mu(x).$$

Since $h_{u+\varphi} = h_u$ μ-a.e. on $\{y \in \Omega : \varphi(y) = 0\}$ by Lemma 3.8, we have (6.1).

7. Appendix: Φ-Poincaré inequality for $N^{1,\Phi}(\mathbf{R}^N)$.

LEMMA 7.1 ([20, Lemma 1.50]). *Let B be an open ball in $\mathbf{R}^N$ and $u \in W^{1,1}(B)$. Then*

$$|u(x) - u_B| \leq C \int_B \frac{|\nabla u(y)|}{|x-y|^{N-1}}\, dy \quad \text{for a.e. } x \in B$$

with a constant $C > 0$ depending only on N.

The Hardy-Littlewood maximal function Mf of $f \in L^1_{loc}(\mathbf{R}^N)$ is defined by

$$Mf(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy\,,$$

where $|B(x,r)|$ is the Lebesgue measure of $B(x,r)$.

LEMMA 7.2 (cf. [20, Lemma 1.32]). *Let B be an open ball in $\mathbf{R}^N$ and $f \in L^1(B)$. Then, for $x \in B$,*

$$\int_B \frac{f(y)}{|x-y|^{N-1}}\, dy \leq C d_B M\tilde{f}(x)$$

with a constant $C > 0$ depending only on N, where d_B denotes the diameter of B and $\tilde{f}$ is the function f extended by 0 outside B.

As to the boundedness of the maximal operator M, we have shown (see [17, Theorem 7 and Remark 1]):

LEMMA 7.3. *Assume that $\Phi(x,t)$ satisfies $(\Phi2'; p_0)$ and $(\Phi5; \nu)$ given in Example 5.3 for $p_0 > 1$ and $0 < \nu < p_0/N$. Then, for every open ball B in $\mathbf{R}^N$, there is a constant $C(B) \geq 1$ such that*

$$\|M\tilde{f}\|_{L^\Phi(B)} \leq C(B)\|f\|_{L^\Phi(B)}$$

for all $f \in L^\Phi(B)$.

LEMMA 7.4 (cf. [28, Theorem 6.19]). *Let Ω be an open set in $\mathbf{R}^N$. Then $N^{1,\Phi}(\Omega) \subset W^{1,\Phi}(\Omega)$ and if $u \in N^{1,\Phi}(\Omega)$ and $h \in L^{\Phi}(\Omega)$ is a Φ-weak upper gradient of u in Ω, then $|\nabla u| \leq \sqrt{N}h$ a.e. in Ω.*

PROOF. Let $u \in N^{1,\Phi}(\Omega)$. Then, $u \in ACC_{\Phi}(\Omega)$ by Lemma 3.4. It follows that $u \in ACL(\Omega)$, namely u is absolutely continuous along almost every compact line segment in Ω (cf. [28, Lemma 4.7]); here note that $L^{\Phi}(\Omega) \subset L^1_{loc}(\Omega)$ by (2.2).

Hence u has partial derivatives $\partial_j u$ a.e. in Ω. Furthermore, $|\partial_j u| \leq h$ a.e. in Ω for every Φ-weak upper gradient h of u by Lemma 3.5. It then follows that $u \in W^{1,1}_{loc}(\Omega)$ and $|\nabla u| \leq \sqrt{N}h$ a.e. in Ω. It in turn follows that $|\nabla u| \in L^{\Phi}(\Omega)$, namely $u \in W^{1,\Phi}(\Omega)$.

□

Combining these lemmas, we obtain Poincaré inequality for $N^{1,\Phi}(\mathbf{R}^N)$:

THEOREM 7.5. *If $\Phi(x,t)$ satisfies $(\Phi2'; p_0)$ and $(\Phi5; \nu)$ with $p_0 > 1$ and $0 < \nu < p_0/N$, then for every open ball B in $\mathbf{R}^N$ there is a constant $C(B) > 0$ such that*

$$||u - u_B||_{L^{\Phi}(B)} \leq C(B)||h||_{L^{\Phi}(B)}$$

for all $u \in N^{1,\Phi}(B)$ and Φ-weak upper gradients h in B.

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4–24 FURUE-HIGASHI-MACHI, NISHI-KU
HIROSHIMA 733–0872
JAPAN

E-mail address: fymaeda@h6.dion.ne.jp

DEPARTMENT OF MATHEMATICS
GRADUATE SCHOOL OF EDUCATION
HIROSHIMA UNIVERSITY
HIGASHI-HIROSHIMA 739–8524
JAPAN

E-mail address: tshimo@hiroshima-u.ac.jp

FACULTY OF EDUCATION AND WELFARE SCIENCE
OITA UNIVERSITY
DANNOHARU OITA-CITY 870–1192
JAPAN

E-mail address: t-ohno@oita-u.ac.jp