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# $\sigma$-ACTIONS AND SYMMETRIC TRIADS

OSAMU IKAWA

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**Abstract.** For a given compact connected Lie group and an involution on it, we can define a hyperpolar action. We study the orbit space and the properties of each orbit of the action. The result is a natural extension of maximal torus theory.

**Introduction.** Let $\sigma$ be an automorphism of a compact connected Lie group $G$. The action of $G$ on itself defined by $g \cdot x = gx\sigma(g)^{-1}$ is called a $\sigma$-*action* ([7]). A $\sigma$-action is a hyperpolar action. In general an isometric action of a Lie group on a Riemannian manifold is *hyperpolar* if there exists a closed flat submanifold $A$ such that every orbit intersects $A$ orthogonally. Such a submanifold $A$ is called a *section*. It is known that a section is a totally geodesic submanifold ([3]). When $\sigma$ is identity, then $\sigma$-action is nothing but an adjoint action defined by $x \mapsto gxg^{-1}$. More generally when $\sigma$ is of inner type, then the $\sigma$-action is essentially the same as the adjoint action (Lemma 1.4). Since there are many studies on adjoint action, we focus our attention to the case where $\sigma$ is of outer type. Moreover we mainly deal with the $\sigma$-action when $G$ is simple and $\sigma$ is an involution of outer type. Further when $G$ is of classical type, we studied the $\sigma$-action in [5]. In this paper we study the orbit space of a $\sigma$-action and the properties of each orbit such as regular, singular, minimal, austere and totally geodesic when $G$ is a compact connected simple Lie group with a biinvariant Riemannian metric and $\sigma$ is an involution of outer type. These are a generalization of the results in [5]. Here the notion of an austere submanifold was introduced by Harvey-Lawson ([2]), which is a kind of minimal submanifold whose second fundamental form has a certain symmetry.

The organization of this paper is as follows: In Section 1 we mainly construct a symmetric triad $(\tilde{\Sigma}, \Sigma, W)$ from a compact connected simple Lie group $G$ and an involution $\sigma$ of outer type. Here the notion of a symmetric triad, which was introduced in [4], is a generalization of that of an irreducible root system. In Section 2 we describe the orbit space of a $\sigma$-action using the symmetric triad $(\tilde{\Sigma}, \Sigma, W)$. We also study the each orbit.

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**1. $\sigma$-action.** Let $G$ be a compact connected Lie group. Take a maximal torus $T$ of $G$. It is known that

$$G = \bigcup_{g \in G} gTg^{-1}. \tag{1.1}$$

In Subsection 1.1 we will show (1.2) in the below for a given automorphism $\sigma$. When $\sigma$ is equal to identity, the relation (1.2) reduced to (1.1). Thus (1.2) is a generalization of (1.1). In this section we review the definition of a $\sigma$-action, and we mainly study the $\sigma$-action when $\sigma$ is an involution.

**1.1. General case.** In this subsection let $G$ be a compact connected Lie group and $\sigma$ an automorphism of $G$. An action of $G$ on itself defined by $g \cdot x = gx\sigma(g)^{-1}$ $(g, x \in G)$ is called a *$\sigma$-action* [7]. We define two involutions $\theta_1$ and $\theta_2$ on $G \times G$ by

$$\theta_1(g, h) = (\sigma^{-1}(h), \sigma(g)), \quad \theta_2(g, h) = (h, g).$$

The two involutions $\theta_1$ and $\theta_2$ commute each other if and only if $\sigma$ is an involution, that is, $\sigma^2 = 1$. Denote by $F(\theta_i, G \times G)$ the fixed point set of $\theta_i$. Then

$$F(\theta_1, G \times G) = \{(g, \sigma(g)) \mid g \in G\}, \quad F(\theta_2, G \times G) = \triangle G = \{(g, g) \mid g \in G\}.$$

Thus a triple $(G \times G, F(\theta_1, G \times G), \triangle G)$ is a compact symmetric triad, that is, $(G \times G, F(\theta_1, G \times G))$ and $(G \times G, \triangle G)$ are compact symmetric pairs. Take a biinvariant Riemannian metric $\langle\ ,\ \rangle$ on $G$. Then we get an action of $F(\theta_1, G \times G)$ on $(G \times G)/\triangle G$ by

$$(g, \sigma(g))((a, b)\Delta G) = (ga, \sigma(g)b)\Delta G,$$

which is a kind of Hermann actions. If we identify $(G \times G)/\Delta G$ with $G$ by the map $(G \times G)/\Delta G \to G$; $(a, b)\triangle G \mapsto ab^{-1}$, then we can see that the Hermann action of this type is nothing but the $\sigma$-action. Since a Hermann action is hyperpolar, a $\sigma$-action is also hyperpolar. In order to study a section of the $\sigma$-action, we denote by $\mathfrak{g}$ the Lie algebra of $G$. Define a closed subgroup $K_\sigma$ of $G$ by $K_\sigma = F(\sigma, G)$. Then the Lie algebra $\mathfrak{k}_\sigma$ of $K_\sigma$ is given by

$$\mathfrak{k}_\sigma = F(\sigma, \mathfrak{g}) = \{X \in \mathfrak{g} \mid \sigma(X) = X\},$$

where we denote the differentiation of $\sigma$ by the same symbol $\sigma$. Denote by $\mathfrak{k}_i$ and $\mathfrak{m}_i$ the $(+1)$ and $(-1)$-eigenspace of $\theta_i$ in $\mathfrak{g} \times \mathfrak{g}$ respectively. Then

$$\mathfrak{k}_1 = \{(X, \sigma(X)) \mid X \in \mathfrak{g}\}, \quad \mathfrak{k}_2 = \{(X, X) \mid X \in \mathfrak{g}\},$$
$$\mathfrak{m}_1 = \{(X, -\sigma(X)) \mid X \in \mathfrak{g}\}, \quad \mathfrak{m}_2 = \{(X, -X) \mid X \in \mathfrak{g}\}.$$

Thus we have

$$\mathfrak{k}_1 \cap \mathfrak{k}_2 = \{(X, X) \mid X \in \mathfrak{k}_\sigma\}, \quad \mathfrak{m}_1 \cap \mathfrak{m}_2 = \{(X, -X) \mid X \in \mathfrak{k}_\sigma\},$$
$$\mathfrak{k}_1 \cap \mathfrak{m}_2 = \{(X, -X) \mid \sigma(X) = -X\}, \quad \mathfrak{k}_2 \cap \mathfrak{m}_1 = \{(X, X) \mid \sigma(X) = -X\}.$$

For a set $X$ and a map $\phi : X \to X$ we define $F(\phi, X) = \{x \in X \mid \phi(x) = x\}$. We use this notation throughout the paper.

Take a maximal torus $A$ in $K_\sigma$, and denote by $\mathfrak{a}$ the Lie algebra of $A$. Define a subspace $\hat{\mathfrak{a}}$ in $\mathfrak{m}_1 \cap \mathfrak{m}_2$ by

$$\hat{\mathfrak{a}} = \{(H, -H) \mid H \in \mathfrak{a}\}\,.$$

Then $\hat{\mathfrak{a}}$ is a maximal abelian subspace of $\mathfrak{m}_1 \cap \mathfrak{m}_2$. Hence $\exp \hat{\mathfrak{a}} \Delta G$ is a section of the $\sigma$-action, where $\exp \hat{\mathfrak{a}} = \{(a, a^{-1}) \mid a \in A\}$ ([3]). In particular

$$(G \times G)/\Delta G = \bigcup_{g \in G} (g, \sigma(g)) \exp \hat{\mathfrak{a}} \Delta G\,.$$

If we identify $(G \times G)/\Delta G$ with $G$, then we have

$$G = \bigcup_{g \in G} g A \sigma(g)^{-1}\,. \tag{1.2}$$

Denote by $\mathfrak{g}^{\mathbb{C}}$ the complexification of $\mathfrak{g}$. For $\alpha \in \mathfrak{a}$, we define a subspace $\mathfrak{g}(\mathfrak{a}, \alpha)$ of $\mathfrak{g}^{\mathbb{C}}$ by

$$\mathfrak{g}(\mathfrak{a}, \alpha) = \{X \in \mathfrak{g}^{\mathbb{C}} \mid [H, X] = \sqrt{-1}\langle \alpha, H \rangle X \quad (H \in \mathfrak{a})\}$$

and set

$$\tilde{\Sigma} = \{\alpha \in \mathfrak{a} - \{0\} \mid \mathfrak{g}(\mathfrak{a}, \alpha) \neq \{0\}\}\,.$$

Then we have

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}(\mathfrak{a}, 0) \oplus \sum_{\alpha \in \tilde{\Sigma}} \mathfrak{g}(\mathfrak{a}, \alpha)\,. \tag{1.3}$$

Denote by $^-$ the conjugation of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{g}$. If $\alpha$ is in $\tilde{\Sigma}$ then $-\alpha$ is also in $\tilde{\Sigma}$ since $\overline{\mathfrak{g}(\mathfrak{a}, \alpha)} = \mathfrak{g}(\mathfrak{a}, -\alpha)$. We denote the complex linear extension of $\sigma$ to $\mathfrak{g}^{\mathbb{C}}$ by the same symbol $\sigma$. The following two lemmas were proved in [5].

LEMMA 1.1 [5, Lemma 3].

(1) $[\mathfrak{g}(\mathfrak{a}, \alpha), \mathfrak{g}(\mathfrak{a}, \beta)] \subset \mathfrak{g}(\mathfrak{a}, \alpha + \beta)$.
(2) $\mathfrak{g}(\mathfrak{a}, \alpha)$ *is* $\sigma$*-invariant*.

Denote by $\mathfrak{z}$ the center of $\mathfrak{g}$.

LEMMA 1.2 [5, Lemma 4]. $\operatorname{span}(\tilde{\Sigma}) = \mathfrak{z}^{\perp} \cap \mathfrak{a}$.

Denote by $\Sigma$ the root system of $\mathfrak{k}_\sigma$ with respect to $\mathfrak{a}$. Then $\Sigma$ is a reduced root system. The multiplicity $m(\lambda)$ of each $\lambda \in \Sigma$ is equal to two. Denote by $W(\Sigma)$ the Weyl group of $\Sigma$.

For $(a, b) \in G \times G$ we denote by $\tau_{(a,b)}$ the inner automorphism defined by $(a, b)$: $\tau_{(a,b)}(x, y) = (a, b)(x, y)(a, b)^{-1}$.

DEFINITION 1.3 ([9]). Let $(\theta_1, \theta_2)$ and $(\theta_1', \theta_2')$ be two pairs of involutions of $G \times G$. Then $(\theta_1, \theta_2)$ and $(\theta_1', \theta_2')$ are *equivalent* if there exist an automorphism $\rho \in \operatorname{Aut}(G \times G)$ of $G \times G$ and $(a, b) \in G \times G$ such that

$$\theta_1' = \tau_{(a,b)} \rho \theta_1 \rho^{-1} \tau_{(a,b)}^{-1}, \quad \theta_2' = \rho \theta_2 \rho^{-1}\,.$$

In this case we write $(\theta_1, \theta_2) \sim (\theta_1', \theta_2')$. The relation $\sim$ is an equivalent relation.

The following lemma means that when $\sigma$ is of inner type, then the $\sigma$-action is essentially the same as adjoint action.

LEMMA 1.4 [5, Lemma 1]. *Set* $\theta_1(g,h) = (\sigma^{-1}(g), \sigma(g))$ *and* $\theta_2(g,h) = (h,g)$. *Then* $(\theta_1, \theta_2) \sim (\theta_2, \theta_2)$ *holds if and only if* $\sigma$ *is an inner automorphism of* $G$.

We are interested in the case when $\sigma$ is of outer type. The following lemma means that $\sigma$-action is essentially the same as $\tau_a \sigma \tau_a^{-1}$-action, where we set $\tau_a x = axa^{-1}$.

LEMMA 1.5 [5, Lemma 2]. *For* $\sigma \in \mathrm{Aut}(G)$ *and* $a \in G$ *we define* $\sigma' \in \mathrm{Aut}(G)$ *by* $\sigma' = \tau_a \sigma \tau_a^{-1}$. *Set* $\theta_1(g,h) = (\sigma^{-1}(h), \sigma(g))$, $\theta_1'(g,h) = (\sigma'^{-1}(h), \sigma'(g))$ *and* $\theta_2(g,h) = (h,g)$ *then* $(\theta_1, \theta_2) \sim (\theta_1', \theta_2)$.

**1.2. In the case when $\sigma$ is of finite order.** In the sequel we assume that the order $s$ of $\sigma$ is finite. We define a subgroup of $U(1)$ by $\{\varepsilon_1 = 1, \varepsilon_2, \dots, \varepsilon_s\} = \{\varepsilon \in U(1) \mid \varepsilon^s = 1\}$. Define a subspace $\mathfrak{g}(\mathfrak{a}, \alpha, \varepsilon_j)$ of $\mathfrak{g}(\mathfrak{a}, \alpha)$ by

$$\mathfrak{g}(\mathfrak{a}, \alpha, \varepsilon_j) = \{X \in \mathfrak{g}(\mathfrak{a}, \alpha) \mid \sigma X = \varepsilon_j X\}.$$

In particular

$$\mathfrak{g}(\mathfrak{a}, 0, 1) = \{X \in \mathfrak{k}_\sigma^{\mathbb{C}} \mid [\mathfrak{a}, X] = \{0\}\} = \mathfrak{a}^{\mathbb{C}}.$$

By Lemma 1.1,(2) we have

$$\mathfrak{g}(\mathfrak{a}, \alpha) = \sum_{j=1}^{s} \mathfrak{g}(\mathfrak{a}, \alpha, \varepsilon_j).$$

The following two lemmas were proved in [5].

LEMMA 1.6 [5, Lemma 5].

(1) $\mathfrak{g}(\mathfrak{a}, \alpha, \varepsilon_j)$ *is* $\sigma$*-invariant*.
(2) $\overline{\mathfrak{g}(\mathfrak{a}, \alpha, \varepsilon_j)} = \mathfrak{g}(\mathfrak{a}, -\alpha, \varepsilon_j^{-1})$.
(3) $[\mathfrak{g}(\mathfrak{a}, \alpha, \varepsilon_i), \mathfrak{g}(\mathfrak{a}, \beta, \varepsilon_j)] \subset \mathfrak{g}(\mathfrak{a}, \alpha+\beta, \varepsilon_i\varepsilon_j)$.

LEMMA 1.7 [5, Lemma 6]. $\tilde{\Sigma}$ *is a root system of* $\mathfrak{z}^{\perp} \cap \mathfrak{a}$.

In order to study the properties of $\tilde{\Sigma}$ it is necessary to recall the finite dimensional complex irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$. We define a basis $\{X, \bar{X}, H\}$ of $\mathfrak{sl}(2,\mathbb{C})$ as follows:

$$X = \sqrt{-1}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \bar{X} = \sqrt{-1}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = [X, \bar{X}] = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

LEMMA 1.8 [4, Lemma 4.37]. *Let* $(\rho, V_{n+1})$ *be an* $(n+1)$*-dimensional complex irreducible representation of* $\mathfrak{sl}(2,\mathbb{C})$. *Then there exists a basis* $\{f_k\}_{0 \le k \le n}$ *of* $V$ *such that*

$$\begin{aligned}
\rho(X) f_k &= \sqrt{-1}\sqrt{(n-k)(k+1)}\, f_{k+1}, \\
\rho(\bar{X}) f_k &= \sqrt{-1}\sqrt{k(n-k+1)}\, f_{k-1}, \\
\rho(H) f_k &= (n-2k) f_k.
\end{aligned}$$

In the lemma above, the set $\mathrm{Spec}(\rho(H))$ of eigenvalues of $\rho(H)$ is given by

$$\mathrm{Spec}(\rho(H)) = \{n - 2k \mid 0 \leq k \leq n\}.$$

Denote by $W_{n-2k}$ the eigenspace of $\rho(H)$ with eigenvalue $n-2k$. Then $W_{n-2k} = \mathbb{C} f_k$. When $0 \leq k \leq [n/2]$ then the mapping

$$\rho(X)^{n-2k} : W_{n-2k} \to W_{-(n-2k)} \tag{1.4}$$

is a linear isomorphism.

LEMMA 1.9. *Let $\alpha \in \tilde{\Sigma}$ and $X \in \mathfrak{g}(\mathfrak{a}, \alpha, \varepsilon_j) - \{0\}$. When $\beta \in \tilde{\Sigma}$ and $\langle \alpha, \beta \rangle < 0$, then $\|\beta\| \geq \|\alpha + \beta\|$ and the mapping*

$$(\mathrm{ad}X)^m : \mathfrak{g}(\mathfrak{a}, \beta) \to \mathfrak{g}(\mathfrak{a}, \beta + m\alpha)$$

*is a linear isomorphism, where we set $m = -\frac{2\langle \alpha, \beta \rangle}{\|\alpha\|^2} \in \mathbb{N}$. In particular the linear mapping*

$$\mathrm{ad}X : \mathfrak{g}(\mathfrak{a}, \beta) \to \mathfrak{g}(\mathfrak{a}, \beta + \alpha)$$

*is injective. Here $s_\alpha \beta := \beta - 2\frac{\langle \alpha, \beta \rangle}{\|\alpha\|^2}\alpha = \beta + m\alpha$.*

PROOF. Since $\|\beta\|^2 - \|\alpha + \beta\|^2 = \|\alpha\|^2(m-1) \geq 0$, we have $\|\beta\| \geq \|\alpha + \beta\|$. We denote by $\beta + n\alpha$ $(p \leq n \leq q)$ the $\alpha$-series containing $\beta$. Then $p + q = m$ and

$$\langle \alpha, \beta + n\alpha \rangle = -\frac{1}{2}\|\alpha\|^2(p + q - 2n) \quad (p \leq n \leq q).$$

If we set $H = \frac{2}{\langle X, \bar{X} \rangle \|\alpha\|^2}[X, \bar{X}]$ and $\mathfrak{l} = \mathbb{C}H \oplus \mathbb{C}X \oplus \mathbb{C}\bar{X}$, then $\mathfrak{l}$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ as Lie algebras and

$$\mathrm{ad}(H) = (p + q - 2n)\mathrm{id} = (m - 2n)\mathrm{id} \quad \text{on} \quad \mathfrak{g}(\mathfrak{a}, \beta + n\alpha).$$

Thus

$$\mathfrak{g}(\mathfrak{a}, \beta) = \left\{ Y \in \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}(\mathfrak{a}, \beta + n\alpha) \mid [H, Y] = mY \right\},$$

$$\mathfrak{g}(\mathfrak{a}, \beta + m\alpha) = \left\{ Y \in \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}(\mathfrak{a}, \beta + n\alpha) \mid [H, Y] = -mY \right\}.$$

Taking this into account, we decompose

$$\bigoplus_{n \in \mathbb{Z}} \mathfrak{g}(\mathfrak{a}, \beta + n\alpha) = \bigoplus_{n=p}^{q} \mathfrak{g}(\mathfrak{a}, \beta + n\alpha)$$

into $\mathfrak{l}$-irreducible representations. Then Lemma 1.8 and (1.4) imply the assertion. $\square$

We decompose $\tilde{\Sigma}$ into irreducible root systems, and denote it by $\tilde{\Sigma} = \tilde{\Sigma}_1 \cup \cdots \cup \tilde{\Sigma}_r$. Let $\mathfrak{g}_i^{\mathbb{C}}$ be a subalgebra of $\mathfrak{g}^{\mathbb{C}}$ generated by $\sum_{\alpha \in \tilde{\Sigma}_i} \mathfrak{g}(\mathfrak{a}, \alpha)$. Then

$$\mathfrak{g}_i^{\mathbb{C}} \subset \mathfrak{g}(\mathfrak{a}, 0) \oplus \sum_{\alpha \in \tilde{\Sigma}_i} \mathfrak{g}(\mathfrak{a}, \alpha).$$

LEMMA 1.10 [5, Lemma 7]. *$\mathfrak{g}_i^{\mathbb{C}}$ is an ideal of $\mathfrak{g}^{\mathbb{C}}$, which is not equal to $\{0\}$. When $i \neq j$, then $[\mathfrak{g}_i^{\mathbb{C}}, \mathfrak{g}_j^{\mathbb{C}}] = \{0\}$. In particular if $\mathfrak{g}$ is simple then $\tilde{\Sigma}$ is an irreducible root system of $\mathfrak{a}$.*

**1.3. In the case when $\sigma$ is an involution.** In the sequel we assume that $\sigma$ is an involution. Define a subspace $\mathfrak{m}_\sigma$ of $\mathfrak{g}$ by

$$\mathfrak{m}_\sigma = \{X \in \mathfrak{g} \mid \sigma(X) = -X\}.$$

Then $\mathfrak{g} = \mathfrak{k}_\sigma \oplus \mathfrak{m}_\sigma$. Since $\mathfrak{a} \subset \mathfrak{k}_\sigma$, we have $[\mathfrak{a}, \mathfrak{m}_\sigma] \subset \mathfrak{m}_\sigma$. We denote by $W$ the set of nonzero weights of $\mathfrak{m}_\sigma$ with respect to $\mathfrak{a}$. Denote by $n(\alpha)$ the multiplicity of $\alpha \in W$. Define subspaces $V(\mathfrak{m}_\sigma)$ and $V^\perp(\mathfrak{m}_\sigma)$ of $\mathfrak{m}_\sigma$ by

$$V(\mathfrak{m}_\sigma) = \{X \in \mathfrak{m}_\sigma \mid [\mathfrak{a}, X] = \{0\}\},$$
$$V^\perp(\mathfrak{m}_\sigma) = \{X \in \mathfrak{m}_\sigma \mid X \perp V(\mathfrak{m}_\sigma)\}.$$

LEMMA 1.11. *If we set $\mathfrak{t} = \mathfrak{a} \oplus V(\mathfrak{m}_\sigma)$, then $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$.*

PROOF. Since a maximal abelian subalgebra in $\mathfrak{g}$ containing $\mathfrak{a}$ is $\sigma$-invariant, and is contained in $\mathfrak{t}$, it is sufficient to prove that $\mathfrak{t}$ is abelian. By the definition of $\mathfrak{t}$ we have

$$[\mathfrak{t}, \mathfrak{t}] = [V(\mathfrak{m}_\sigma), V(\mathfrak{m}_\sigma)] \subset \mathfrak{k}_\sigma.$$

By the Jacobi identity, we have

$$[\mathfrak{a}, [\mathfrak{t}, \mathfrak{t}]] = [[\mathfrak{a}, V(\mathfrak{m}_\sigma)], V(\mathfrak{m}_\sigma)] + [V(\mathfrak{m}_\sigma), [\mathfrak{a}, V(\mathfrak{m}_\sigma)]] = \{0\}.$$

The maximality of $\mathfrak{a}$ implies that $[\mathfrak{t}, \mathfrak{t}] \subset \mathfrak{a}$. Hence $\mathfrak{t}$ is a subalgebra of $\mathfrak{g}$. Since $\mathfrak{g}$ is compact, $\mathfrak{t}$ is also compact. The semisimple part $[\mathfrak{t}, \mathfrak{t}]$ of $\mathfrak{t}$ satisfies

$$[[\mathfrak{t}, \mathfrak{t}], [\mathfrak{t}, \mathfrak{t}]] \subset [\mathfrak{a}, \mathfrak{a}] = \{0\}.$$

Hence $[\mathfrak{t}, \mathfrak{t}]$ is abelian. Thus we get $[\mathfrak{t}, \mathfrak{t}] = \{0\}$. □

For $\alpha \in \mathfrak{a}$ we define a subspace $V_\alpha^\perp(\mathfrak{m}_\sigma)$ of $V^\perp(\mathfrak{m}_\sigma)$ by

$$V_\alpha^\perp(\mathfrak{m}_\sigma) = \{X \in V^\perp(\mathfrak{m}_\sigma) \mid (\mathrm{ad}H)^2 X = -\langle \alpha, H\rangle^2 X \quad (H \in \mathfrak{a})\}.$$

Then we have $W = \{\alpha \in \mathfrak{a} \mid V_\alpha^\perp(\mathfrak{m}_\sigma) \neq \{0\}\}$, which is invariant under the multiplication by $-1$ since $V_{-\alpha}^\perp(\mathfrak{m}_\sigma) = V_\alpha^\perp(\mathfrak{m}_\sigma)$. For $\alpha \in W$ we have $n(\alpha) = \dim V_\alpha^\perp(\mathfrak{m}_\sigma)$. By the definitions of $\tilde{\Sigma}$, $\Sigma$ and $W$ we get $\tilde{\Sigma} = \Sigma \cup W$.

We recall the definition of a symmetric triad.

DEFINITION 1.12 ([4, Definition 2.2]). Let $\mathfrak{a}$ be a finite dimensional vector space over $\mathbb{R}$ with an inner product $\langle\ ,\ \rangle$. For $\alpha, \beta \in \mathfrak{a}$ set

$$s_\alpha \beta = \beta - 2\frac{\langle \alpha, \beta\rangle}{\|\alpha\|^2}\alpha.$$

A triple $(\tilde{\Sigma}, \Sigma, W)$ is a *symmetric triad* of $\mathfrak{a}$, if it satisfies the following six conditions:

(1) $\tilde{\Sigma}$ is an irreducible root system of $\mathfrak{a}$.

(2) $\Sigma$ is a root system of $\mathrm{span}(\Sigma)$.

(3) $W$ is a nonempty subset of $\mathfrak{a}$, which is invariant under the multiplication by $-1$, and $\tilde{\Sigma} = \Sigma \cup W$.
(4) $\Sigma \cap W$ is a nonempty subset. If we put $l = \max\{\|\alpha\| \mid \alpha \in \Sigma \cap W\}$, then $\Sigma \cap W = \{\alpha \in \tilde{\Sigma} \mid \|\alpha\| \leq l\}$.
(5) For $\alpha \in W, \lambda \in \Sigma - W$,

$$2\frac{\langle \alpha, \lambda \rangle}{\|\alpha\|^2} \text{ is odd if and only if } s_\alpha \lambda \in W - \Sigma\,.$$

(6) For $\alpha \in W, \lambda \in W - \Sigma$,

$$2\frac{\langle \alpha, \lambda \rangle}{\|\alpha\|^2} \text{ is odd if and only if } s_\alpha \lambda \in \Sigma - W\,.$$

REMARK 1.13. When $(\tilde{\Sigma}, \Sigma, W)$ is a symmetric triad of $\mathfrak{a}$, then $\operatorname{span}(\Sigma) = \mathfrak{a}$.

In fact the set of shortest roots of an irreducible root system $\tilde{\Sigma}$ spans $\mathfrak{a}$. The condition (4) of Definition 1.12 implies that

$$\operatorname{span}(\Sigma) \supset \operatorname{span}(\Sigma \cap W) = \mathfrak{a}\,.$$

In the sequel we assume that $G$ is a compact connected simple Lie group. The purpose of this section is to show the following theorem, which is a generalization of Theorem 1 in [5]:

THEOREM 1.14. *Let $G$ be a compact connected simple Lie group, $\sigma$ an involution of $G$ of outer type. Then the triple $(\tilde{\Sigma}, \Sigma, W)$ defined above is a symmetric triad, and $\Sigma$ is a reduced root system of $\mathfrak{a}$. $m(\lambda) = n(\alpha) = 2$ for any $\lambda \in \Sigma$ and $\alpha \in W$.*

We need some lemmas to prove the theorem above.

LEMMA 1.15 [5, Lemma 8]. *For $\alpha \in W$, the subspace $V_\alpha^\perp(\mathfrak{m}_\sigma)$ is $\mathfrak{a}$-invariant, and $n(\alpha)$ is even for any $\alpha \in W$. If we denote by $W(\Sigma)$ the Weyl group of $\Sigma$, then $W$ is invariant under the action of $W(\Sigma)$. For $s \in W(\Sigma)$ and $\alpha \in W$, we have $s(V_\alpha^\perp(\mathfrak{m}_\sigma)) = V_{s\alpha}^\perp(\mathfrak{m}_\sigma)$ and $n(s\alpha) = n(\alpha)$.*

LEMMA 1.16. *When $\Sigma \cap W = \emptyset$, then for $\alpha, \beta, \alpha + \beta \in \tilde{\Sigma}$ there exist $\varepsilon_1 = \pm 1$ and $\varepsilon_2 = \pm 1$ such that*

$$\mathfrak{g}(\mathfrak{a}, \alpha) = \mathfrak{g}(\mathfrak{a}, \alpha, \varepsilon_1)\,, \quad \mathfrak{g}(\mathfrak{a}, \beta) = \mathfrak{g}(\mathfrak{a}, \beta, \varepsilon_2)$$

*and that*

$$\mathfrak{g}(\mathfrak{a}, \alpha + \beta) = \mathfrak{g}(\mathfrak{a}, \alpha + \beta, \varepsilon_1 \varepsilon_2)\,.$$

PROOF. Since $\Sigma \cap W = \emptyset$ there exist $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{\pm 1\}$ such that

$$\mathfrak{g}(\mathfrak{a}, \alpha) = \mathfrak{g}(\mathfrak{a}, \alpha, \varepsilon_1)\,, \quad \mathfrak{g}(\mathfrak{a}, \beta) = \mathfrak{g}(\mathfrak{a}, \beta, \varepsilon_2)\,, \quad \mathfrak{g}(\mathfrak{a}, \alpha + \beta) = \mathfrak{g}(\mathfrak{a}, \alpha + \beta, \varepsilon_3)\,.$$

If $\langle \alpha, \beta \rangle < 0$ then Lemma 1.9 implies that

$$\{0\} \neq [X, \mathfrak{g}(\mathfrak{a}, \beta, \varepsilon_2)] \subset \mathfrak{g}(\mathfrak{a}, \alpha + \beta, \varepsilon_1 \varepsilon_2)$$

for $X \in \mathfrak{g}(\mathfrak{a}, \alpha, \varepsilon_1) - \{0\}$. Hence $\varepsilon_3 = \varepsilon_1\varepsilon_2$. When $\langle \alpha, \beta \rangle \geq 0$ then

$$\langle \alpha, -(\alpha+\beta) \rangle = -\|\alpha\|^2 - \langle \alpha, \beta \rangle \leq -\|\alpha\|^2 < 0\,.$$

Since there exists $\mu \in \{\pm 1\}$ such that

$$\mathfrak{g}(\mathfrak{a}, -(\alpha+\beta)) = \mathfrak{g}(\mathfrak{a}, -(\alpha+\beta), \mu)\,,$$

using Lemma 1.9 we have

$$\{0\} \neq [X, \mathfrak{g}(\mathfrak{a}, -(\alpha+\beta), \mu)] \subset \mathfrak{g}(\mathfrak{a}, -\beta, \varepsilon_1\mu)$$

for $X \in \mathfrak{g}(\mathfrak{a}, \alpha, \varepsilon_1) - \{0\}$. Thus

$$\mathfrak{g}(\mathfrak{a}, -\beta) = \mathfrak{g}(\mathfrak{a}, -\beta, \varepsilon_2) = \mathfrak{g}(\mathfrak{a}, -\beta, \varepsilon_1\mu)\,,$$

which implies that $\mu = \varepsilon_1\varepsilon_2$. Hence

$$\mathfrak{g}(\mathfrak{a}, \alpha+\beta) = \mathfrak{g}(\mathfrak{a}, \alpha+\beta, \varepsilon_1\varepsilon_2)\,. \qquad \square$$

We extend the inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ to a complex symmetric nondegenerate bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{g}^{\mathbb{C}}$.

LEMMA 1.17. *$\langle \mathfrak{g}(\mathfrak{a}, \alpha, \varepsilon), \mathfrak{g}(\mathfrak{a}, -\alpha, -\varepsilon) \rangle = \{0\}$ for $\alpha \in \tilde{\Sigma}$ and $\varepsilon = \pm 1$.*

PROOF. For $X \in \mathfrak{g}(\mathfrak{a}, \alpha, \varepsilon)$ and $Y \in \mathfrak{g}(\mathfrak{a}, -\alpha, -\varepsilon)$ we have

$$\langle X, Y \rangle = \langle \sigma X, \sigma Y \rangle = -\varepsilon^2 \langle X, Y \rangle = -\langle X, Y \rangle\,.$$

Thus $\langle X, Y \rangle = 0$. $\square$

LEMMA 1.18. *If $\alpha, \beta \in \tilde{\Sigma} \cup \{0\}$ and $\alpha + \beta \neq 0$ then $\langle \mathfrak{g}(\mathfrak{a}, \alpha), \mathfrak{g}(\mathfrak{a}, \beta) \rangle = \{0\}$.*

PROOF. For $H \in \mathfrak{a}$, $X \in \mathfrak{g}(\mathfrak{a}, \alpha)$ and $Y \in \mathfrak{g}(\mathfrak{a}, \beta)$ we have

$$0 = \langle [H, X], Y \rangle + \langle X, [H, Y] \rangle = \sqrt{-1}\langle \alpha+\beta, H \rangle \langle X, Y \rangle\,.$$

Take $H$ such that $\langle \alpha+\beta, H \rangle \neq 0$. Then we have $\langle X, Y \rangle = 0$. $\square$

LEMMA 1.19. *$n(\alpha) = 2$ for any $\alpha \in W$.*

PROOF. Since $\alpha$ is in $W$, the dimension of $\mathfrak{g}(\mathfrak{a}, \alpha, -1)$ is greater than or equal to one. Combining Lemmas 1.17 and 1.18 with the nondegeneracy of $\langle\ ,\ \rangle$ we have $\langle \mathfrak{g}(\mathfrak{a}, \alpha, -1), \mathfrak{g}(\mathfrak{a}, -\alpha, -1) \rangle \neq \{0\}$. Thus we can take $E_{\pm\alpha} \in \mathfrak{g}(\mathfrak{a}, \pm\alpha, -1)$ such that $\langle E_\alpha, E_{-\alpha} \rangle = 1$. Then

$$[E_\alpha, E_{-\alpha}] \in [\mathfrak{g}(\mathfrak{a}, \alpha, -1)\,, \mathfrak{g}(\mathfrak{a}, -\alpha, -1)] \subset \mathfrak{g}(\mathfrak{a}, 0, 1) = \mathfrak{a}^{\mathbb{C}}\,.$$

For any $H \in \mathfrak{a}$ we have

$$\langle H, [E_\alpha\,, E_{-\alpha}] \rangle = \langle [H, E_\alpha], E_{-\alpha} \rangle = \sqrt{-1}\langle \alpha, H \rangle \langle E_\alpha, E_{-\alpha} \rangle = \sqrt{-1}\langle \alpha, H \rangle\,,$$

which implies that $[E_\alpha, E_{-\alpha}] = \sqrt{-1}\alpha$. Take $D_\alpha \in \mathfrak{g}(\mathfrak{a}, \alpha, -1)$ such that $\langle D_\alpha, E_{-\alpha} \rangle = 0$ and set

$$D_{-1} = 0\,, \quad D_n = (\mathrm{ad}E_\alpha)^n D_\alpha \in \mathfrak{g}(\mathfrak{a}, (n+1)\alpha) \quad (n = 0, 1, 2, \dots)\,.$$

We will show that

$$[E_{-\alpha}, D_n] = \frac{n(n+1)}{2}\|\alpha\|^2 D_{n-1} \quad (n = 0, 1, 2, \dots)$$

by the induction with respect to $n$. When $n = 0$, then $D_0 = D_\alpha$ and

$$[E_{-\alpha}, D_0] = [E_{-\alpha}, D_\alpha] \in [\mathfrak{g}(\mathfrak{a}, -\alpha, -1), \mathfrak{g}(\mathfrak{a}, \alpha, -1)] \subset \mathfrak{g}(\mathfrak{a}, 0, 1) = \mathfrak{a}^{\mathbb{C}}.$$

For any $H \in \mathfrak{a}^{\mathbb{C}}$ we have

$$\langle [E_{-\alpha}, D_0], H\rangle = -\langle E_{-\alpha}, [H, D_\alpha]\rangle = -\sqrt{-1}\langle \alpha, H\rangle\langle E_{-\alpha}, D_\alpha\rangle = 0.$$

Thus $[E_{-\alpha}, D_0] = 0$. Assume that the assertion holds until $n$. By the Jacobi identity we have

$$\begin{aligned}
[E_{-\alpha}, D_{n+1}] &= [E_{-\alpha}, [E_\alpha, D_n]] \\
&= [[E_{-\alpha}, E_\alpha], D_n] + [E_\alpha, [E_{-\alpha}, D_n]] \\
&= -\sqrt{-1}[\alpha, D_n] + \frac{n(n+1)}{2}\|\alpha\|^2 [E_\alpha, D_{n-1}] \\
&= (n+1)\|\alpha\|^2 D_n + \frac{n(n+1)}{2}\|\alpha\|^2 D_n \\
&= \frac{(n+1)(n+2)}{2}\|\alpha\|^2 D_n .
\end{aligned}$$

Hence $[E_{-\alpha}, D_n] = \frac{n(n+1)}{2}\|\alpha\|^2 D_{n-1}$. If the dimension of $\mathfrak{g}(\mathfrak{a}, \alpha, -1)$ were greater than or equal to 2, we could take $D_\alpha \neq 0$ such that $\langle D_\alpha, E_{-\alpha}\rangle = 0$. Thus $D_n \in \mathfrak{g}(\mathfrak{a}, (n+1)\alpha) - \{0\}$, which would be a contradiction. Hence $\dim \mathfrak{g}(\mathfrak{a}, \alpha, -1) = 1$. □

By Lemma 1.19 we get, for any $\alpha \in \tilde{\Sigma}$,

$$\dim \mathfrak{g}(\mathfrak{a}, \alpha) = \begin{cases} 2 & (\alpha \in \Sigma \cap W), \\ 1 & (\alpha \in \tilde{\Sigma} - \Sigma \cap W), \end{cases} \tag{1.5}$$

since $m(\lambda) = 2$ for any $\lambda \in \Sigma$. We can take a fundamental system $\tilde{\Pi}$ of $\tilde{\Sigma}$ since $\tilde{\Sigma}$ is a root system of $\mathfrak{a}$ by Lemma 1.10. We denote by $\tilde{\Sigma}^+$ the set of positive roots in $\tilde{\Sigma}$ with respect to $\tilde{\Pi}$. Set $\Sigma^+ = \Sigma \cap \tilde{\Sigma}^+$ and $W^+ = W \cap \tilde{\Sigma}^+$.

LEMMA 1.20. *The following three conditions are equivalent.*

(1) *$\sigma$ is an inner type involution of* $\mathfrak{g}$.
(2) $\Sigma \cap W = \emptyset$.
(3) $\Sigma \cap W \cap \tilde{\Pi} = \emptyset$.

*In the case above* $\mathfrak{a} = \mathfrak{t}$ *holds.*

PROOF. (1)⇒ (2): By the assumption there exists $H \in \mathfrak{g}$ such that $\sigma = \mathrm{Ad}(\exp H)$. Take a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ containing $H$. Then $\mathfrak{t} \subset \mathfrak{k}_\sigma$. Thus $\mathfrak{a} = \mathfrak{t}$. Since the dimension of $\mathfrak{g}(\mathfrak{a}, \alpha)$ is equal to one for $\alpha \in \tilde{\Sigma}$, and $\mathfrak{g}(\mathfrak{a}, \alpha)$ is $\sigma$-invariant by Lemma 1.1, (2), we have $\mathfrak{g}(\mathfrak{a}, \alpha) \subset \mathfrak{k}_\sigma^{\mathbb{C}}$ or $\mathfrak{g}(\mathfrak{a}, \alpha) \subset \mathfrak{m}_\sigma^{\mathbb{C}}$. Hence $\Sigma \cap W = \emptyset$.

(2)⇒(1): By assumption for $\alpha \in \tilde{\Sigma}$ there exists $\varepsilon_\alpha = \pm 1$ such that

$$\mathfrak{g}(\mathfrak{a}, \alpha) = \mathfrak{g}(\mathfrak{a}, \alpha, \varepsilon_\alpha).$$

Select $H \in \mathfrak{a}$ as follows: For $\alpha \in \tilde{\Pi}$,

$$\langle \alpha, H \rangle = \begin{cases} 0 & (\varepsilon_\alpha = 1)\,, \\ \pi & (\varepsilon_\alpha = -1)\,. \end{cases}$$

Since $(\mathrm{ad}H)X = \sqrt{-1}\langle \alpha, H \rangle X$ for $\alpha \in \tilde{\Pi}$ and $X \in \mathfrak{g}(\mathfrak{a}, \alpha)$, we have

$$\mathrm{Ad}(\exp H)X = e^{\sqrt{-1}\langle \alpha, H \rangle} X = \varepsilon_\alpha X = \sigma X\,.$$

Hence $\sigma = \mathrm{Ad}(\exp H)$ on $\sum_{\alpha \in \tilde{\Pi}} \mathfrak{g}(\mathfrak{a}, \alpha)$. Similarly we have $\sigma = \mathrm{Ad}(\exp H)$ on $\sum_{\alpha \in -\tilde{\Pi}} \mathfrak{g}(\mathfrak{a}, \alpha)$. By Lemma 1.16 we have $\sigma = \mathrm{Ad}(\exp H)$ on $\sum_{\alpha \in \tilde{\Sigma}} \mathfrak{g}(\mathfrak{a}, \alpha)$. The subalgebra generated by $\sum_{\alpha \in \tilde{\Sigma}} \mathfrak{g}(\mathfrak{a}, \alpha) (\neq \{0\})$ is an ideal of $\mathfrak{g}^{\mathbb{C}}$ by (1.3). Hence it coincides with $\mathfrak{g}^{\mathbb{C}}$ since $\mathfrak{g}^{\mathbb{C}}$ is simple. Hence $\sigma = \mathrm{Ad}(\exp H)$ on $\mathfrak{g}^{\mathbb{C}}$.

It is clear that (2) implies (3). We show that the negative of (2) implies the negative of (3). Assume that $\Sigma \cap W \neq \emptyset$. Let $\alpha$ be in $\Sigma^+ \cap W^+$. We will show that there exists $\beta \in \tilde{\Pi}$ such that $\langle \alpha, \beta \rangle > 0$ when $\alpha \notin \tilde{\Pi}$. We were to assume that $\langle \alpha, \beta \rangle \leq 0$ for any $\beta \in \tilde{\Pi}$. Express $\alpha$ as $\alpha = \sum_{\beta \in \tilde{\Pi}} m_\beta \beta$ $(m_\beta \geq 0)$ then

$$\|\alpha\|^2 = \sum_{\beta \in \tilde{\Pi}} m_\beta \langle \alpha, \beta \rangle \leq 0\,.$$

Hence we would have $\alpha = 0$, which would be a contradiction. Thus when $\alpha \notin \tilde{\Pi}$ there exists $\beta \in \tilde{\Pi}$ such that $\langle \alpha, \beta \rangle > 0$. Considering $\alpha$-series containing $\beta$ we get $\alpha - \beta \in \tilde{\Sigma}^+$. We will show that $\alpha - \beta \in \Sigma^+ \cap W^+$. The mapping

$$\mathrm{ad}X : \mathfrak{g}(\mathfrak{a}, \alpha) \to \mathfrak{g}(\mathfrak{a}, \alpha - \beta)$$

is injective for $X \in \mathfrak{g}(\mathfrak{a}, -\beta) - \{0\}$ by Lemma 1.9. In particular

$$\mathrm{ad}X : \mathfrak{g}(\mathfrak{a}, \alpha, \pm 1) \to \mathfrak{g}(\mathfrak{a}, \alpha - \beta)$$

is also injective. Since $\dim \mathfrak{g}(\mathfrak{a}, \alpha - \beta) \leq 2$ by (1.5), we have

$$\mathfrak{g}(\mathfrak{a}, \alpha - \beta) = [X, \mathfrak{g}(\mathfrak{a}, \alpha, 1)] \oplus [X, \mathfrak{g}(\mathfrak{a}, \alpha, -1)], \quad [X, \mathfrak{g}(\mathfrak{a}, \alpha, \pm 1)] \neq \{0\}\,.$$

Hence $\alpha - \beta \in \Sigma^+ \cap W^+$ by Lemma 1.6, (3) in both cases when $\beta$ is in $\Sigma^+$ or $\beta$ is in $W^+$. By iteration we have $\Sigma^+ \cap W^+ \cap \tilde{\Pi} \neq \emptyset$. □

LEMMA 1.21. (1) *For $\alpha \in W$ and $\lambda \in \Sigma - W$,*

$$2\frac{\langle \alpha, \lambda \rangle}{\|\alpha\|^2} \text{ is odd if and only if } s_\alpha \lambda \in W - \Sigma\,.$$

(2) *For $\alpha \in W$ and $\lambda \in W - \Sigma$,*

$$2\frac{\langle \alpha, \lambda \rangle}{\|\alpha\|^2} \text{ is odd if and only if } s_\alpha \lambda \in \Sigma - W\,.$$

PROOF. Let $\alpha$ be in $W$ and $\lambda$ in $(\Sigma - W) \cup (W - \Sigma)$. Set $m = -2\frac{\langle \alpha, \lambda \rangle}{\|\alpha\|^2} \in \mathbb{Z}$. We may assume that $\langle \alpha, \lambda \rangle \neq 0$. Since $W$ is invariant under the multiplication by $-1$, we may assume that $\langle \alpha, \lambda \rangle < 0$. Let $X$ be in $\mathfrak{g}(\mathfrak{a}, \alpha, -1) - \{0\}$. By Lemma 1.9 the mapping

$$(\mathrm{ad}X)^m : \mathfrak{g}(\mathfrak{a}, \lambda) \to \mathfrak{g}(\mathfrak{a}, s_\alpha \lambda)$$

is a linear isomorphism. Since $\lambda$ is in $(\Sigma - W) \cup (W - \Sigma)$, we have

$$\mathfrak{g}(\mathfrak{a}, \lambda) = \mathfrak{g}(\mathfrak{a}, \lambda, \varepsilon_\lambda) \quad \text{where} \quad \varepsilon_\lambda = \begin{cases} 1 & (\lambda \in \Sigma - W)\,, \\ -1 & (\lambda \in W - \Sigma)\,. \end{cases}$$

Since $(\mathrm{ad}X)^m \mathfrak{g}(\mathfrak{a}, \lambda, \varepsilon_\lambda) \subset \mathfrak{g}(\mathfrak{a}, s_\alpha\lambda, (-1)^m \varepsilon_\lambda)$ by Lemma 1.6, (3), we have

$$\mathfrak{g}(\mathfrak{a}, s_\alpha\lambda) = \mathfrak{g}(\mathfrak{a}, s_\alpha\lambda, (-1)^m \varepsilon_\lambda)\,.$$

Hence we get the assertion. $\square$

LEMMA 1.22. *Let $\sigma$ be an outer type involution of $\mathfrak{g}$. Set $l = \max\{\|\alpha\| \mid \alpha \in \Sigma \cap W\}$. Then $\Sigma \cap W \neq \emptyset$ and $\Sigma \cap W = \{\alpha \in \tilde{\Sigma} \mid \|\alpha\| \leq l\}$.*

PROOF. By Lemma 1.20 $\Sigma \cap W \neq \emptyset$. First we show that $s_\gamma \alpha$ is in $\Sigma \cap W$ for any $\alpha \in \Sigma \cap W$ and $\gamma \in \tilde{\Sigma}$. There exists $\varepsilon \in \{\pm 1\}$ such that $\mathfrak{g}(\mathfrak{a}, \gamma, \varepsilon) \neq \{0\}$. We may assume that $\gamma$ is not proportional to $\alpha$ and that $\langle \alpha, \gamma \rangle \neq 0$. Since $s_\gamma \alpha = s_{-\gamma}\alpha$ we may assume that $\langle \alpha, \gamma \rangle < 0$. If we set $m = -\frac{2\langle \alpha, \gamma \rangle}{\|\gamma\|^2}$ then for $X \in \mathfrak{g}(\mathfrak{a}, \gamma, \varepsilon) - \{0\}$ the mapping

$$(\mathrm{ad}X)^m : \mathfrak{g}(\mathfrak{a}, \alpha) \to \mathfrak{g}(\mathfrak{a}, s_\gamma \alpha)$$

is a linear isomorphism by Lemma 1.9. Since $(\mathrm{ad}X)^m \mathfrak{g}(\mathfrak{a}, \alpha, \pm 1) \subset \mathfrak{g}(\mathfrak{a}, s_\gamma\alpha, \pm\varepsilon)$ the mapping

$$(\mathrm{ad}X)^m : \mathfrak{g}(\mathfrak{a}, \alpha, \pm 1) \to \mathfrak{g}(\mathfrak{a}, s_\gamma \alpha, \pm\varepsilon)$$

is a linear isomorphism. Thus $s_\gamma\alpha \in \Sigma \cap W$ since $\alpha$ is in $\Sigma \cap W$. Since $\tilde{\Sigma}$ is an irreducible root system, the Weyl group $W(\tilde{\Sigma})$ of $\tilde{\Sigma}$ acts on $\{\beta \in \tilde{\Sigma} \mid \|\beta\| = l\}$ transitively. Hence $\{\alpha \in \tilde{\Sigma} \mid \|\alpha\| = l\} \subset \Sigma \cap W$. When $l = \min\{\|\alpha\| \mid \alpha \in \tilde{\Sigma}\}$ then $\Sigma \cap W = \{\alpha \in \tilde{\Sigma} \mid \|\alpha\| = l\} = \{\alpha \in \tilde{\Sigma} \mid \|\alpha\| \leq l\}$, which implies the assertion. Hence we may assume that $l > \min\{\|\alpha\| \mid \alpha \in \tilde{\Sigma}\}$. In the case when $\tilde{\Sigma} \neq BC_r$ we have $l = \max\{\|\alpha\| \mid \alpha \in \tilde{\alpha}\}$ since any root in $\tilde{\Sigma}$ is shortest or longest. Take $\gamma \in \tilde{\Sigma}$ such that $\|\gamma\| < l$. We show that $\gamma$ is in $\Sigma \cap W$. By [4, Lemma 4.35] there exists $\beta \in \tilde{\Sigma}$ with $\|\beta\| = l$ such that $-\frac{2\langle \beta, \gamma \rangle}{\|\beta\|^2} = 1$. Then $\beta + \gamma \in \tilde{\Sigma}$. Since $\|\beta\| = l$, $\beta$ is in $\Sigma \cap W$. A simple calculation implies that

$$-2\frac{\langle -\beta, \beta + \gamma \rangle}{\| -\beta\|^2} = 2\frac{\langle \beta, \beta + \gamma \rangle}{\|\beta\|^2} = 2 + 2\frac{\langle \beta, \gamma \rangle}{\|\beta\|^2} = 2 - 1 = 1\,.$$

Let $Y \in \mathfrak{g}(\mathfrak{a}, \beta, 1) - \{0\}$ and $Y' \in \mathfrak{g}(\mathfrak{a}, -\beta, -1) - \{0\}$. The mappings

$$\mathrm{ad}Y : \mathfrak{g}(\mathfrak{a}, \gamma) \to \mathfrak{g}(\mathfrak{a}, \beta + \gamma)\,, \quad \mathrm{ad}Y' : \mathfrak{g}(\mathfrak{a}, \beta + \gamma) \to \mathfrak{g}(\mathfrak{a}, \gamma)$$

are linear isomorphisms by Lemma 1.9. Since

$$\mathrm{ad}Y : \mathfrak{g}(\mathfrak{a}, \gamma, \pm 1) \to \mathfrak{g}(\mathfrak{a}, \beta + \gamma, \pm 1)\,, \quad \mathrm{ad}Y' : \mathfrak{g}(\mathfrak{a}, \beta + \gamma, \pm 1) \to \mathfrak{g}(\mathfrak{a}, \gamma, \mp 1)\,,$$

we have $\dim\mathfrak{g}(\mathfrak{a},\gamma,1)=\dim\mathfrak{g}(\mathfrak{a},\gamma,-1)$. Thus $\gamma$ is in $\Sigma\cap W$. When $\tilde{\Sigma}$ is of type $BC_r$ then the assertion reduces to the case when $\tilde{\Sigma}$ is of type $B_r$. □

PROOF OF THEOREM 1.14. The condition (1) of Definition 1.12 was proved in Lemma 1.10. The condition (2) of Definition 1.12 was proved in Subsection 1.1. The condition (3) of Definition 1.12 was already proved in this subsection. The condition (4) of Definition 1.12 was proved in Lemma 1.22. The conditions (5) and (6) of Definition 1.12 were proved in Lemma 1.21. Thus $(\tilde{\Sigma},\Sigma,W)$ is a symmetric triad of $\mathfrak{a}$. In Subsection 1.1 we showed that $\Sigma$ is reduced and $m(\lambda)=2$ for any $\lambda$ in $\Sigma$. In Lemma 1.19 we showed that $n(\alpha)=2$ for any $\alpha$ in $W$. □

Take two maximal abelian subalgebras $\mathfrak{a}$ and $\mathfrak{a}'$ of $\mathfrak{k}_\sigma$. We obtain a symmetric triad $(\tilde{\Sigma},\Sigma,W)$ of $\mathfrak{a}$ and a symmetric triad $(\tilde{\Sigma}',\Sigma',W')$ of $\mathfrak{a}'$ in the sense of Theorem 1.14. We study a relation between them. Since $\mathfrak{a}$ and $\mathfrak{a}'$ are maximal in $\mathfrak{k}_\sigma$ there exists an element $k\in(K_\sigma)_0$, the identity component of $K_\sigma$, such that $\mathfrak{a}'=\mathrm{Ad}(k)\mathfrak{a}$. For $\beta\in\mathfrak{a}'$ and $\varepsilon=\pm1$ we have

$$\mathfrak{g}(\mathfrak{a}',\beta)=\mathrm{Ad}(k)\mathfrak{g}(\mathfrak{a},\mathrm{Ad}(k^{-1})\beta)\,,\quad \mathfrak{g}(\mathfrak{a}',\beta,\varepsilon)=\mathrm{Ad}(k)\mathfrak{g}(\mathfrak{a},\mathrm{Ad}(k^{-1})\beta,\varepsilon)\,,$$

which implies that

$$\tilde{\Sigma}'=\mathrm{Ad}(k)\tilde{\Sigma}'\,,\quad \Sigma'=\mathrm{Ad}(k)\Sigma\,,\quad W'=\mathrm{Ad}(k)W\,.$$

For an involution $\sigma$ on $G$ and $g\in G$ we define an involution $\sigma'$ by $\sigma'=\tau_g\sigma\tau_g^{-1}$. The symmetric subgroup $K_{\sigma'}$ with respect to $\sigma'$ is given by $K_{\sigma'}=\tau_gK_\sigma=gK_\sigma g^{-1}$. For a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{k}_\sigma$ define a maximal abelian subalgebra $\mathfrak{a}'$ of $\mathfrak{k}_{\sigma'}$, the Lie algebra of $K_{\sigma'}$, by $\mathfrak{a}'=\mathrm{Ad}(g)\mathfrak{a}$. We obtain a symmetric triad $(\tilde{\Sigma},\Sigma,W)$ of $\mathfrak{a}$ and a symmetric triad $(\tilde{\Sigma}',\Sigma',W')$ of $\mathfrak{a}'$ in the sense of Theorem 1.14. We study a relation between them. For $\beta\in\mathfrak{a}'$ and $\varepsilon=\pm1$

$$\mathfrak{g}(\mathfrak{a}',\beta)=\mathrm{Ad}(g)\mathfrak{g}(\mathfrak{a},\mathrm{Ad}(g^{-1})\beta)\,,\quad \mathfrak{g}(\mathfrak{a}',\beta,\varepsilon)=\mathrm{Ad}(g)\mathfrak{g}(\mathfrak{a},\mathrm{Ad}(g^{-1})\beta,\varepsilon)\,,$$

which implies that

$$\tilde{\Sigma}'=\mathrm{Ad}(g)\tilde{\Sigma}',\quad \Sigma'=\mathrm{Ad}(g)\Sigma,\quad W'=\mathrm{Ad}(g)W\,.$$

In the rest of this subsection we determine $(\tilde{\Sigma},\Sigma,W)$ for any given $(\mathfrak{g},\mathfrak{k}_\sigma)$. The results are as follows:

| $(\mathfrak{g},\mathfrak{k}_\sigma)$ | $(\tilde{\Sigma},\Sigma,W)$ |
|---|---|
| $(\mathfrak{su}(2m),\mathfrak{so}(2m))$ $(m\geq2)$ | (I'-$C_m$) |
| $(\mathfrak{su}(2m+1),\mathfrak{so}(2m+1))$ $(m\geq1)$ | (II-$BC_m$) |
| $(\mathfrak{su}(2m),\mathfrak{sp}(m))$ | (I-$C_m$) |
| $(\mathfrak{so}(2m+2n+2),\mathfrak{so}(2m+1)\times\mathfrak{so}(2n+1))$ | (I'-$B_{m+n}$) |
| $(\mathfrak{e}_6,\mathfrak{sp}(4))$ | (I'-$F_4$) |
| $(\mathfrak{e}_6,\mathfrak{f}_4)$ | (I-$F_4$) |

We can see the set of $(\mathfrak{g}, \mathfrak{k}_\sigma)$'s by the classification of symmetric spaces of compact type. In the table above, when $\mathfrak{g}$ is of classical type, that is $\mathfrak{g} = \mathfrak{su}(N), \mathfrak{so}(N)$, then we can verify the type of $(\tilde{\Sigma}, \Sigma, W)$ by a matrix calculation ([5]). Here we used the following notation.

| type | $\tilde{\Sigma}$ | $\Sigma$ | $W$ |
|---|---|---|---|
| (I'-$C_m$) | $C_m$ | $D_m$ | $C_m$ |
| (II-$BC_m$) | $BC_m$ | $B_m$ | $BC_m$ |
| (I-$C_m$) | $C_m$ | $C_m$ | $D_m$ |
| (I'-$B_{m+n}$) | $B_{m+n}$ | $B_m \cup B_n$ | $(\tilde{\Sigma} - \Sigma) \cup \{\pm e_i\}$ |
| (I-$F_4$) | $F_4$ | $F_4$ | $\{$shortest roots in $F_4\} \cong D_4$ |

$(\tilde{\Sigma}, \Sigma, W) =$ (I'-$F_4$) means that $\tilde{\Sigma} = F_4$ and

$$\Sigma = \{\text{shortest roots in } F_4\} \cup \{\pm(e_1 \pm e_2), \pm(e_3 \pm e_4)\} \cong C_4\,,$$

$$W = \{\text{shortest roots in } F_4\} \cup \{\pm(e_1 \pm e_3), \pm(e_1 \pm e_4), \pm(e_2 \pm e_3), \pm(e_2 \pm e_4)\}\,.$$

Here we followed the same notations of positive roots in [1].

When $\mathfrak{g} = \mathfrak{e}_6$ we can verify the type of $(\tilde{\Sigma}, \Sigma, W)$ using Vogan diagrams. In order to explain this, first let $\mathfrak{g}$ be a compact simple Lie algebra, and $\sigma$ an automorphism of $\mathfrak{g}$ of outer type. Take a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{k}_\sigma = F(\sigma, \mathfrak{g})$. Then $\mathfrak{t} = \mathfrak{a} \oplus V(\mathfrak{m}_\sigma)$ is a maximal compact Cartan subalgebra of $\mathfrak{g}$ (Lemma 1.11). Take an invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$. Denote by $\bar{\ }: \mathfrak{t} \to \mathfrak{a}$ the orthogonal projection. For $\alpha \in \mathfrak{t}$ we define a subspace $\mathfrak{g}_\alpha^{\mathbb{C}}$ of the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$ by

$$\mathfrak{g}_\alpha^{\mathbb{C}} = \{X \in \mathfrak{g}^{\mathbb{C}} \mid [Z, X] = \sqrt{-1}\langle\alpha, Z\rangle X \quad (Z \in \mathfrak{t})\}\,.$$

Then $\alpha \in \mathfrak{t}$ is called a *root* of $\mathfrak{g}$ with respect to $\mathfrak{t}$ if $\mathfrak{g}_\alpha^{\mathbb{C}} \neq \{0\}$. A root $\alpha$ is *compact* if $\mathfrak{g}_\alpha^{\mathbb{C}} \subset \mathfrak{k}_\sigma^{\mathbb{C}}$, and $\alpha$ is *noncompact* if $\mathfrak{g}_\alpha^{\mathbb{C}} \subset \mathfrak{m}_\sigma^{\mathbb{C}}$. A root $\alpha$ is compact or noncompact if and only if $\alpha$ is in $\mathfrak{a}$. A root $\alpha$ is *complex* if it is neither compact nor noncompact. Then

$$\Sigma \cap W = \{\overline{\text{complex roots}}\}\,, \quad \Sigma = \{\text{compact roots}, \overline{\text{complex roots}}\}\,,$$

$$W = \{\text{noncompact roots}, \overline{\text{complex roots}}\}\,.$$

The set of compact roots, noncompact roots and complex roots can be readable from the Vogan diagram of (the dual of) $(\mathfrak{g}, \mathfrak{k}_\sigma)$. And the orthogonal projection can be also readable from the Vogan diagram (see [6] for the detail).

In the sequel we assume that $(\mathfrak{g}, \mathfrak{k}_\sigma) = (\mathfrak{e}_6, \mathfrak{f}_4)$ or $(\mathfrak{g}, \mathfrak{k}_\sigma) = (\mathfrak{e}_6, \mathfrak{sp}(4))$. In both cases we have

$$\mathfrak{t} = \sum_{i=1}^{6} \mathbb{R}\alpha_i\,, \quad \mathfrak{a} = \mathbb{R}\alpha_2 \oplus \mathbb{R}\alpha_4 \oplus \mathbb{R}(\alpha_3 + \alpha_5) \oplus \mathbb{R}(\alpha_1 + \alpha_6)$$

and

$$\overline{\alpha_1} = \overline{\alpha_6} = \frac{1}{2}(\alpha_1 + \alpha_6)\,, \quad \overline{\alpha_3} = \overline{\alpha_5} = \frac{1}{2}(\alpha_3 + \alpha_5)\,, \quad \overline{\alpha_2} = \alpha_2, \quad \overline{\alpha_4} = \alpha_4\,.$$

Hence $\tilde{\Sigma} = F_4$.

In the case when $(\mathfrak{g}, \mathfrak{k}_\sigma) = (\mathfrak{e}_6, \mathfrak{f}_4)$ we get $W \subset \Sigma = \tilde{\Sigma}$ since there does not exist a noncompact root. The set of compact roots coincide with the set of long roots in $F_4$. The set of the projections of complex roots coincide with the set of short roots in $F_4$. Hence $(\tilde{\Sigma}, \Sigma, W) = (\text{I-}F_4)$.

In the case when $(\mathfrak{g}, \mathfrak{k}_\sigma) = (\mathfrak{e}_6, \mathfrak{sp}(4))$, we get $\Sigma \cap W = \{\text{short roots in } F_4\}$. Hence $(\tilde{\Sigma}, \Sigma, W) = (\text{I'-}F_4)$.

**2. The orbit spaces of $\sigma$-actions.** In this section let $G$ be a compact connected simple Lie group and $\sigma$ an involution of $G$ of outer type. Take an adjoint invariant inner product $\langle\ ,\ \rangle$ on the Lie algebra $\mathfrak{g}$ of $G$. We use the same notation in the previous section. Take a maximal torus $A$ of $K_\sigma$ and denote by $\mathfrak{a}$ its Lie algebra. We denote by $(\tilde{\Sigma}, \Sigma, W)$ the symmetric triad of $\mathfrak{a}$ obtained in the previous section. Since $\Sigma$ is a root system of $\mathfrak{k}_\sigma$ with respect to $\mathfrak{a}$, we have the following root space decomposition of $\mathfrak{k}_\sigma$:

$$\mathfrak{k}_\sigma = \mathfrak{a} \oplus \sum_{\lambda \in \Sigma^+} (\mathbb{R}F_\lambda \oplus \mathbb{R}G_\lambda)\,,$$

where, for any $H \in \mathfrak{a}$,

$$[H, F_\lambda] = \langle\lambda, H\rangle G_\lambda\,, \quad [H, G_\lambda] = -\langle\lambda, H\rangle F_\lambda\,.$$

Then we have

$$\begin{aligned} \mathrm{Ad}(\exp H)F_\lambda &= \cos(\langle\lambda, H\rangle)F_\lambda + \sin(\langle\lambda, H\rangle)G_\lambda\,,\\ \mathrm{Ad}(\exp H)G_\lambda &= -\sin(\langle\lambda, H\rangle)F_\lambda + \cos(\langle\lambda, H\rangle)G_\lambda\,. \end{aligned}$$

Since $W$ is the set of nonzero weights of $\mathfrak{m}_\sigma$ with respect to $\mathfrak{a}$ and $n(\alpha) = 2$ for any $\alpha \in W$, we have the following weight space decomposition of $\mathfrak{m}_\sigma$:

$$\mathfrak{m}_\sigma = V(\mathfrak{m}_\sigma) \oplus V^\perp(\mathfrak{m}_\sigma)\,, \quad V^\perp(\mathfrak{m}_\sigma) = \sum_{\alpha \in W^+} (\mathbb{R}X_\alpha \oplus \mathbb{R}Y_\alpha)\,,$$

where, for any $H \in \mathfrak{a}$,

$$[H, X_\alpha] = \langle\alpha, H\rangle Y_\alpha\,, \quad [H, Y_\alpha] = -\langle\alpha, H\rangle X_\alpha\,.$$

Then we have

$$\begin{aligned} \mathrm{Ad}(\exp H)X_\alpha &= \cos(\langle\alpha, H\rangle)X_\alpha + \sin(\langle\alpha, H\rangle)Y_\alpha,\\ \mathrm{Ad}(\exp H)Y_\alpha &= -\sin(\langle\alpha, H\rangle)X_\alpha + \cos(\langle\alpha, H\rangle)Y_\alpha\,. \end{aligned}$$

For $H \in \mathfrak{g}$ we denote by $O_H = \bigcup_{g \in G} g\exp(2H)\sigma(g)^{-1}$ the orbit of $\sigma$-action through $\exp 2H$. Since $A$ is a section of $\sigma$-action, we may assume that $H$ is in $\mathfrak{a}$. The tangent space of $O_H$ at $\exp 2H$ is given by

$$\begin{aligned} &dL_{\exp(-2H)}T_{\exp 2H}(O_H)\\ &\quad = \{\mathrm{Ad}(\exp(-2H))X - \sigma(X) \mid X \in \mathfrak{g}\}\\ &\quad = \sum_{\substack{\lambda \in \Sigma^+\\ \langle\lambda, H\rangle \notin \pi\mathbb{Z}}} (\mathbb{R}F_\lambda \oplus \mathbb{R}G_\lambda) \oplus \sum_{\substack{\alpha \in W^+\\ \langle\alpha, H\rangle \notin \frac{\pi}{2} + \pi\mathbb{Z}}} (\mathbb{R}X_\alpha \oplus \mathbb{R}Y_\alpha) \oplus V(\mathfrak{m}_\sigma)\,, \end{aligned}$$

where $L_g : G \to G; x \mapsto gx$ is a left translation. The normal space of $O_H$ at $\exp 2H$ is given by

$$dL_{\exp(-2H)} T^{\perp}_{\exp 2H}(O_H) = \mathfrak{a} \oplus \sum_{\substack{\lambda \in \Sigma^+ \\ \langle \lambda, H\rangle \in \pi\mathbb{Z}}} (\mathbb{R}F_\lambda \oplus \mathbb{R}G_\lambda) \oplus \sum_{\substack{\alpha \in W^+ \\ \langle \alpha, H\rangle \in \frac{\pi}{2}+\pi\mathbb{Z}}} (\mathbb{R}X_\alpha \oplus \mathbb{R}Y_\alpha)\,.$$

In [4] we defined that $H$ is a *regular* point if

$$\langle \lambda, H\rangle \notin \pi\mathbb{Z} \quad (\lambda \in \Sigma)\,, \quad \langle \alpha, H\rangle \notin \frac{\pi}{2} + \pi\mathbb{Z} \quad (\alpha \in W)\,.$$

Thus we get the following Proposition:

PROPOSITION 2.1. *The orbit $O_H$ is regular if and only if $H$ is a regular point.*

The orbit $O_0 = \bigcup_{g \in G} g\sigma(g)^{-1}$ through the identity element is the image of $G/K_\sigma$ by a Cartan embedding $F_\sigma : G/K_\sigma \to G; gK_\sigma \mapsto g\sigma(g)^{-1}$.

We will review the definition of a reflective submanifold, which was introduced by Leung.

DEFINITION 2.2 ([8]). Let $\tilde{M}$ be a complete Riemannian manifold. A connected component of the fixed point set of an involutive isometry $F$ of $\tilde{M}$ is called a *reflective submanifold*. $F$ is called a *reflection*.

REMARK 2.3. $O_0$ is a reflective submanifold of $G$.

PROOF. The tangent and normal spaces of $O_0$ at the identity element $e$ are given by

$$T_e(O_0) = \{X - \sigma(X) \mid X \in \mathfrak{g}\} = \mathfrak{m}_\sigma\,, \quad T_e^{\perp}(O_0) = \mathfrak{k}_\sigma\,.$$

The mapping $F : G \to G; x \mapsto \sigma(x^{-1})$ is the identity on $O_0$. The differential of $F$ is $-1$ on $T_e^{\perp}(O_0)$. Hence $F$ is a reflection. $\square$

DEFINITION 2.4 ([4]). Let $(\tilde{\Sigma}, \Sigma, W)$ be a symmetric triad of $\mathfrak{a}$. Then $H \in \mathfrak{a}$ is a *totally geodesic point* if $\langle \alpha, H\rangle \in \frac{\pi}{2}\mathbb{Z}$ for any $\alpha \in \tilde{\Sigma}$.

PROPOSITION 2.5. *If $H$ is a totally geodesic point, then the orbit $O_H$ is a reflective submanifold.*

PROOF. By the assumption, $\langle \alpha, 4H\rangle \in 2\pi\mathbb{Z}$ for any $\alpha \in \tilde{\Sigma}$. Hence $\mathrm{Ad}(\exp(4H)) = 1$, which implies that $\exp 4H$ is in the center of $G$. If we set $\sigma'(g) = \exp(2H)\sigma(g)\exp(-2H)$ then $\sigma'$ is an involution of $G$. Since $O_H \exp(-2H) = \bigcup_{g \in G} g\sigma'(g^{-1})$, by the remark above $O_H \exp(-2H)$ is a reflective submanifold. Hence $O_H$ is also reflective. $\square$

The isotropy subgroup $G_H$ of the $\sigma$-action at $\exp 2H$ is given by

$$\begin{aligned} G_H &= \{g \in G \mid g \exp 2H = (\exp 2H)\sigma(g)\} \\ &= \{g \in G \mid (\exp(-2H))g\exp(2H) = \sigma(g)\}\,. \end{aligned}$$

The group $G_H$ is $\sigma$-invariant, and its Lie algebra $\mathfrak{g}_H$ is given by

$$\begin{aligned}\mathfrak{g}_H &= \{X \in \mathfrak{g} \mid \mathrm{Ad}(\exp(-2H))X = \sigma(X)\} \\ &= \mathfrak{a} \oplus \sum_{\substack{\lambda\in\Sigma^+ \\ \langle\lambda,H\rangle\in\pi\mathbb{Z}}} (\mathbb{R}F_\lambda \oplus \mathbb{R}G_\lambda) \oplus \sum_{\substack{\alpha\in W^+ \\ \langle\alpha,H\rangle\in\frac{\pi}{2}+\pi\mathbb{Z}}} (\mathbb{R}X_\alpha \oplus \mathbb{R}Y_\alpha) \\ &= dL_{\exp 2H}^{-1} T^{\perp}_{\exp 2H}(O_H)\,.\end{aligned}$$

The subspace $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{g}_H$. Hence

$$\mathfrak{g}_H = \bigcup_{g\in G_H} \mathrm{Ad}(g)\mathfrak{a} = \bigcup_{g\in G_H} \mathrm{Ad}(\sigma(g))\mathfrak{a}\,.$$

The action of $G_H$ on the normal space $T^{\perp}_{\exp 2H}(O_H)$ is given as follows: For $g \in G_H$, and $X \in \mathfrak{g}_H$,

$$g_* dL_{\exp 2H} X = \frac{d}{dt} g \exp 2H \exp tX \sigma(g)^{-1}_{|t=0} = dL_{\exp 2X} \mathrm{Ad}(\sigma(g))X\,.$$

Hence as representation spaces we have the following isomorphism:

$$[G_H \curvearrowright T^{\perp}_{\exp 2H}(O_H)] \cong [G_H \overset{\mathrm{Ad}\circ\sigma}{\curvearrowright} \mathfrak{g}_H]\,.$$

Thus

$$T^{\perp}_{\exp 2H}(O_H) = \bigcup_{g\in G_H} g_* dL_{\exp 2H}\mathfrak{a}\,. \tag{2.1}$$

The notion of austere submanifold was first given by Harvey-Lawson.

DEFINITION 2.6 ([2]). Let $M$ be a submanifold of a Riemannian manifold $\tilde{M}$. We denote the shape operator of $M$ by $A$. Then $M$ is called an *austere submanifold*, if for each $x \in M$ and for each normal vector $\xi \in T_x^{\perp}(M)$, the set of eigenvalues with their multiplicities of $A^{\xi}$ is invariant under the multiplication by $-1$.

It is obvious that an austere submanifold is a minimal submanifold.

DEFINITION 2.7 ([4]). Let $(\tilde{\Sigma}, \Sigma, W)$ be a symmetric triad of $\mathfrak{a}$. For each $\lambda \in \Sigma$ and $\alpha \in W$ set $m(\lambda) = n(\alpha) = 2$. For $H \in \mathfrak{a}$ we define a *mean curvature vector* $m_H \in \mathfrak{a}$ of $H$ by

$$\begin{aligned} m_H &= -\sum_{\substack{\lambda\in\Sigma^+ \\ \langle\lambda,H\rangle\notin\frac{\pi}{2}\mathbb{Z}}} m(\lambda)\cot(\langle\lambda, H\rangle)\lambda + \sum_{\substack{\alpha\in W^+ \\ \langle\alpha,H\rangle\notin\frac{\pi}{2}\mathbb{Z}}} n(\alpha)\tan(\langle\alpha, H\rangle)\alpha \\ &= -2\sum_{\substack{\lambda\in\Sigma^+ \\ \langle\lambda,H\rangle\notin\frac{\pi}{2}\mathbb{Z}}} \cot(\langle\lambda, H\rangle)\lambda + 2\sum_{\substack{\alpha\in W^+ \\ \langle\alpha,H\rangle\notin\frac{\pi}{2}\mathbb{Z}}} \tan(\langle\alpha, H\rangle)\alpha\,.\end{aligned}$$

(1) $H \in \mathfrak{a}$ is an *austere point* if the subset

$$\left\{-\lambda \cot(\langle\lambda, H\rangle) \ \mid \lambda \in \Sigma^+, \langle\lambda, H\rangle \notin \frac{\pi}{2}\mathbb{Z}\right\}$$
$$\cup \left\{\alpha \tan(\langle\alpha, H\rangle) \ \mid \alpha \in W^+, \langle\alpha, H\rangle \notin \frac{\pi}{2}\mathbb{Z}\right\}$$

of $\mathfrak{a}$ is invariant under the multiplication by $-1$.

(2) $H \in \mathfrak{a}$ is a *minimal point* if $m_H = 0$.

After some preparations we will show the following theorem:

THEOREM 2.8. *For the orbit $O_H \subset G$ ($H \in \mathfrak{a}$), we have the following:*

(1) *$O_H \subset G$ is totally geodesic if and only if $H$ is a totally geodesic point.*
(2) *$O_H \subset G$ is austere if and only if $H$ is an austere point.*
(3) *$O_H \subset G$ is minimal if and only if $H$ is a minimal point.*

In [4] the totally geodesic points and austere points were classified. Hence we can classify the totally geodesic orbits and austere orbits. The orbit space can be identified with a simplex in $\mathfrak{a}$. The simplex is a closure of a cell. We can stratify a cell and see that each strata has a unique minimal point ([4, Theorem 2.24]).

For $X \in \mathfrak{g}$ we define a Killing vector field $X^*$ on $G$ by

$$X^*_x = \frac{d}{dt}(\exp tX)x\sigma(\exp tX)^{-1}|_{t=0} \in T_x(G)\,.$$

Then $X^*$ is tangent to $O_H$ at each point in $O_H$. We denote by $h$ the second fundamental form of $O_H \subset G$ at $g = \exp 2H$.

LEMMA 2.9. *For $X, Y \in \mathfrak{g}$,*

$$2dL_g^{-1}h(X^*, Y^*) = -[\mathrm{Ad}(g^{-1})X, \sigma(Y)]^\perp - [\mathrm{Ad}(g^{-1})Y, \sigma(X)]^\perp\,,$$

*where we denote by $Z^\perp$ the $dL_g^{-1}T_g^\perp(O_H)$-component of $Z \in \mathfrak{g}$ with respect to the decomposition $\mathfrak{g} = dL_g^{-1}T_g(O_H) \oplus dL_g^{-1}T_g^\perp(O_H)$.*

PROOF. We denote by $X^L$ and $X^R$ the left invariant and the right invariant vector field on $G$ corresponding to $X \in \mathfrak{g} = T_e(G)$ respectively. Then $X^L$ and $X^R$ are Killing vector fields, and we have

$$X^* = X^R - \sigma(X)^L\,.$$

For $X, Y \in \mathfrak{g}$ we have

$$\nabla_{X^L}Y^L = \frac{1}{2}[X^L, Y^L] = \frac{1}{2}[X, Y]^L\,,$$
$$\nabla_{X^R}Y^R = \frac{1}{2}[X^R, Y^R] = -\frac{1}{2}[X, Y]^R\,,$$

where we denote by $\nabla$ the Levi-Civita connection on $G$. Hence we have

$$\begin{aligned}\nabla_{X^*}Y^* &= \nabla_{X^R-\sigma(X)^L}(Y^R-\sigma(Y)^L)\\ &= \left(-\frac{1}{2}[X,Y]^R+\frac{1}{2}[\sigma(X),\sigma(Y)]^L\right)-\nabla_{X^R}\sigma(Y)^L-\nabla_{\sigma(X)^L}Y^R\\ &= -\frac{1}{2}[X,Y]^*-\nabla_{X^R}\sigma(Y)^L-\nabla_{Y^R}\sigma(X)^L\,.\end{aligned}$$

Here we used the relation $[\sigma(X)^L, Y^R]=0$. By a formula of Koszul,

$$2\langle X^L, \nabla_{Z^R}Y^L\rangle = \langle [Y^L, X^L], Z^R\rangle = \langle [Y,X]^L, Z^R\rangle$$

for $X, Y, Z \in \mathfrak{g}$. Evaluating at $x \in G$ and using the invariance of metric, we get

$$2\langle X^L, \nabla_{Z^R}Y^L\rangle_x = -\langle X^L_x, dL_x[Y, \mathrm{Ad}(x^{-1})Z]\rangle\,.$$

Hence

$$(\nabla_{Z^R}Y^L)_x = \frac{1}{2}dL_x[\mathrm{Ad}(x^{-1})Z, Y]\,.$$

Thus

$$(\nabla_{X^*}Y^*)_x = -\frac{1}{2}[X^*,Y^*]_x - \frac{1}{2}dL_x[\mathrm{Ad}(x^{-1})X, \sigma(Y)] - \frac{1}{2}dL_x[\mathrm{Ad}(x^{-1})Y, \sigma(X)]\,.$$

Since $[X,Y]^*$ is tangent to $O_H$, we get the assertion. □

LEMMA 2.10. *Let $\xi$ be in $\mathfrak{a}$. We denote by $A^{dL_g\xi}$ the shape operator of $O_H \subset G$ with respect to the normal vector $dL_g\xi$. The set of eigenvalues with multiplicities of $A^{dL_g\xi}$ is given by*

$$\begin{aligned}&\left\{-\frac{\langle\xi,\lambda\rangle}{2}\cot(\langle\lambda,H\rangle)\ (\textit{multiplicity}=2)\mid \lambda\in\Sigma^+,\ \langle\lambda,H\rangle\notin\pi\mathbb{Z}\right\}\\ &\cup\left\{\frac{\langle\xi,\alpha\rangle}{2}\tan(\langle\alpha,H\rangle)\ (\textit{multiplicity}=2)\mid \alpha\in W^+,\ \langle\alpha,H\rangle\notin\frac{\pi}{2}+\pi\mathbb{Z}\right\}\\ &\cup\{0\ (\textit{multiplicity}=\dim V(\mathfrak{m}_\sigma)\}\,.\end{aligned}$$

PROOF. For $\lambda \in \Sigma^+$ with $\langle\lambda, H\rangle \notin \pi\mathbb{Z}$ we have

$$A^{dL_g\xi}F^*_\lambda = -\frac{\langle\xi,\lambda\rangle\cot(\langle\lambda,H\rangle)}{2}F^*_\lambda\,,\quad A^{dL_g\xi}G^*_\lambda = -\frac{\langle\xi,\lambda\rangle\cot(\langle\lambda,H\rangle)}{2}G^*_\lambda\,,$$

where $(F^*_\lambda)_g$ and $(G^*_\lambda)_g$ are abbreviated as $F^*_\lambda$ and $G^*_\lambda$ respectively. For $\alpha \in W^+$ with $\langle\alpha, H\rangle \notin \frac{\pi}{2}+\pi\mathbb{Z}$ we have

$$A^{dL_g\xi}X^*_\alpha = \frac{\langle\xi,\alpha\rangle\tan(\langle\alpha,H\rangle)}{2}X^*_\alpha\,,\quad A^{dL_g\xi}Y^*_\alpha = \frac{\langle\xi,\alpha\rangle\tan(\langle\alpha,H\rangle)}{2}Y^*_\alpha\,.$$

And $A^{dL_g\xi}X^*=0$ for $X \in V(\mathfrak{m}_\sigma)$. Hence we get the assertion. □

PROOF OF THEOREM 2.8. (1) By (2.1) the orbit $O_H \subset G$ is totally geodesic if and only if $A^{dL_g\xi} = 0$ for each $\xi \in \mathfrak{a}$. The assertion follows from Lemma 2.10.

(2) By (2.1) the orbit $O_H \subset G$ is austere if and only if the set of eigenvalues of $A^{dL_g\xi}$ is invariant under the multiplication by $-1$ for each $\xi \in \mathfrak{a}$. Using Lemma 2.10 we get the assertion in a similar way of the proof of [4, Cor. 4.29].

(3) By Lemma 2.9 $h(X^*, X^*) = 0$ for $X \in V(\mathfrak{m}_\sigma)$. For $\lambda \in \Sigma^+$ with $\langle\lambda, H\rangle \notin \pi\mathbb{Z}$, we have

$$dL_g^{-1}h(F_\lambda^*, F_\lambda^*) = dL_g^{-1}h(G_\lambda^*, G_\lambda^*) = -\sin(2\langle\lambda, H\rangle)\lambda\,,$$
$$F_\lambda^* \perp G_\lambda^*, \quad \|F_\lambda^*\|^2 = \|G_\lambda^*\|^2 = 4\sin^2(\langle\lambda, H\rangle)\,.$$

For $\alpha \in W^+$ with $\langle\alpha, H\rangle \notin \frac{\pi}{2} + \pi\mathbb{Z}$, we have

$$dL_g^{-1}h(X_\alpha^*, X_\alpha^*) = dL_g^{-1}h(Y_\alpha^*, Y_\alpha^*) = \sin(2\langle\alpha, H\rangle)\alpha\,,$$
$$X_\alpha^* \perp Y_\alpha^*, \quad \|X_\alpha^*\|^2 = \|Y_\alpha^*\|^2 = 4\cos^2(\langle\alpha, H\rangle)\,.$$

If we denote by $\tilde{m}_H$ the mean curvature vector of $O_H \subset G$ at $\exp 2H$, then $\tilde{m}_H = \frac{1}{2}dL_g m_H$. Thus we get the assertion. $\square$

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FACULTY OF ARTS AND SCIENCES
KYOTO INSTITUTE OF TECHNOLOGY
MATSUGASAKI, SAKYOKU
KYOTO 606–8585
JAPAN

*E-mail address*: ikawa@kit.ac.jp