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MODULES OF BILINEAR DIFFERENTIAL OPERATORS OVER THE ORTHOSYMPLECTIC SUPERALGEBRA $\mathfrak{osp}(1|2)$

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Abstract. Let $\mathfrak{F}_\lambda$, $\lambda \in \mathbb{C}$, be the space of tensor densities of degree λ on the supercircle $S^{1|1}$. We consider the superspace $\mathfrak{D}_{\lambda_1,\lambda_2,\mu}$ of bilinear differential operators from $\mathfrak{F}_{\lambda_1} \otimes \mathfrak{F}_{\lambda_2}$ to $\mathfrak{F}_\mu$ as a module over the orthosymplectic superalgebra $\mathfrak{osp}(1|2)$. We prove the existence and the uniqueness of a canonical conformally equivariant symbol map from $\mathfrak{D}^k_{\lambda_1,\lambda_2,\mu}$ to the corresponding space of symbols. An explicit expression of the associated quantization map is also given.

1. Introduction. Let M be an n-dimensional manifold and T^*M be the cotangent bundle over the manifold M. Usual quantization procedure consists of building a map $\mathcal{Q}$ from the space $\mathrm{Pol}(T^*M)$ of polynomials on T^*M to the space $\mathcal{D}(M)$ of linear differential operators on M called a *quatization map*. The inverse $\sigma = \mathcal{Q}^{-1}$ is thus called a *symbol map*. Generally, there is no quantization and symbol map equivariant with respect to the action of the Lie algebra $\mathrm{Vect}(M)$ of vector fields on M (or the group $\mathrm{Diff}(M)$ of diffeomorphisms of M) on the two spaces $\mathcal{D}(M)$ and $\mathrm{Pol}(T^*M)$. Thus, we restrict ourselves to equivariant symbols and quantization maps with respect to the action of a given subalgebra of $\mathrm{Vect}(M)$.

Let, for every $\lambda \in \mathbb{C}$, $\mathcal{F}_\lambda(M)$ be the space of tensor densities of degree λ on M:

$$\varphi = f(x^1, \dots, x^n)|dx^1 \wedge \cdots \wedge dx^n|^\lambda ,$$

that is, of sections of the line bundle $\Delta_\lambda(M) = |\Lambda^n(T^*M)|^{\otimes\lambda}$ over M. This provides a one-parameter family of representations of $\mathrm{Vect}(M)$. Any differential operator on M can be viewed as a linear mapping from $\mathcal{F}_\lambda(M)$ to $\mathcal{F}_\mu(M)$ ($\lambda, \mu \in \mathbb{C}$). Thus, the space of differential operators is a $\mathrm{Vect}(M)$-module, denoted $\mathcal{D}_{\lambda,\mu}(M) := \mathrm{Hom}_{\mathrm{diff}}(\mathcal{F}_\lambda(M), \mathcal{F}_\mu(M))$, i.e., it is in turn a representation of the Lie algebra $\mathrm{Vect}(M)$, we thus have a two-parameter family of representations of $\mathrm{Vect}(M)$. On the other hand, the space $\mathrm{Pol}(T^*M)$ is isomorphic as a $\mathrm{Vect}(M)$-module to the space $\mathcal{S}(M)$ of symmetric contravariant tensor fields on M (i.e., $\mathcal{S}(M) = \Gamma(STM)$) which is a Poisson algebra with a natural graduation given by the decomposition

$$\mathcal{S}(M) = \bigoplus_{k=0}^{\infty} \mathcal{S}^k(M) ,$$

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where $\mathcal{S}^k(M)$ is the space of k-th order tensor fields. The algebra $\mathcal{S}(M)$ is naturally identified with the associated graded algebra $\mathrm{gr}(\mathcal{D}_{\lambda,\mu}(M))$, that is

$$\mathcal{D}^k_{\lambda,\mu}(M)/\mathcal{D}^{k-1}_{\lambda,\mu}(M) \cong \mathcal{S}^k(M)\,.$$

The concept of equivariant quantization over $\mathbb{R}^n$ was introduced by P. Lecomte and V. Ovsienko in [15]. In this seminal work, they considered spaces of differential operators acting between densities and the Lie algebra of projective vector fields over $\mathbb{R}^n$, $\mathrm{sl}(n+1)$. In this situation, they showed the existence and uniqueness of an equivariant quantization. These results were generalized in many references (see for instance [6], [13]). In [14], P. Lecomte globalized the problem of equivariant quantization by defining the problem of natural invariant quantization on arbitrary manifolds. In [4], [5], [9], [11], [12], [18], [19], [20], [21], the authors proved the existence of such quantizations by using different methods in more and more general contexts. Finally, explicit expressions of equivariant symbol and quatization maps have been used in the study of classical modules of differential operators on tensor densities in different situations (see for examples [2], [7], [8] and [13]).

Recently, several papers dealt with the problem of equivariant quantizations in the context of supergeometry: the papers [16] and [22] exposed and solved respectively the problems of the $\mathfrak{pgl}(p+1|q)$-equivariant quantization over the superspace $\mathbb{R}^{p|q}$ and of the $\mathfrak{osp}(p+1;q+1|2r)$-equivariant quantization over $\mathbb{R}^{p+q|2r}$, whereas in [17], the authors define the problem of the natural and projectively invariant quantization on arbitrary supermanifolds and show the existence of such a map. In [10], [23], [24] the problem of equivariant quantizations over the supercircles $S^{1|1}$ and $S^{1|2}$ endowed with canonical contact structures was considered, these quantizations are equivariant with respect to Lie superalgebras $\mathfrak{osp}(1|2)$ and $\mathfrak{osp}(2|2)$ of contact projective vector fields respectively. The results stated in [10] have been used in [1] to give a full classification of the $\mathcal{K}(1)$-modules $\mathfrak{D}^k_{\lambda,\mu}$ of linear differential operators of order k acting on the superspaces of weighted densities, where $\mathcal{K}(1)$ is the Lie superalgebra of contact vector fields on $S^{1|1}$.

Our motivation in this work is the extension of the results proved in [10] and [1] to the binary case. Namely we consider the superspace $\mathfrak{D}_{\lambda_1\lambda_2,\mu}$ of bilinear differential operators $A : \mathfrak{F}_{\lambda_1} \otimes \mathfrak{F}_{\lambda_1} \to \mathfrak{F}_\mu$, where $\mathfrak{F}_\lambda$, $\lambda \in \mathbb{C}$, is the space of tensor densities on the supercircle $S^{1|1}$ of degree λ. The analogue, in the super setting, of the projective algebra $\mathrm{sl}(2)$ is the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2)$, which is the smallest simple Lie superalgebra, can be realized as a subalgebra of $\mathrm{Vect}_{\mathbb{C}}(S^{1|1})$. Naturally, the Lie superalgebra $\mathrm{Vect}_{\mathbb{C}}(S^{1|1})$, and therefore $\mathfrak{osp}(1|2)$, acts on $\mathfrak{D}_{\lambda_1\lambda_2,\mu}$, the $\mathfrak{osp}(1|2)$-module $\mathfrak{D}_{\lambda_1\lambda_2,\mu}$ is filtered as:

$$\mathfrak{D}^0_{\lambda_1\lambda_2,\mu} \subset \mathfrak{D}^{\frac{1}{2}}_{\lambda_1\lambda_2,\mu} \subset \mathfrak{D}^1_{\lambda_1\lambda_2,\mu} \subset \mathfrak{D}^{\frac{3}{2}}_{\lambda_1\lambda_2,\mu} \subset \cdots \subset \mathfrak{D}^{k-\frac{1}{2}}_{\lambda_1\lambda_2,\mu} \subset \mathfrak{D}^k_{\lambda_1\lambda_2,\mu} \subset \cdots\,.$$

The graded module $\mathrm{gr}(\mathfrak{D}_{\lambda_1,\lambda_2,\mu})$, also called the space of symbols and denoted by $\mathcal{S}_{\lambda_1,\lambda_2,\mu}$, depends only on the shift, $\delta = \mu - \lambda_1 - \lambda_2$, of the weights. Moreover, as a $\mathrm{Vect}_{\mathbb{C}}(S^{1|1})$-

module, $\mathcal{S}_{\lambda_1,\lambda_2,\mu}$ is decomposed as $\bigoplus_{k\in\frac{1}{2}\mathbb{N}} \mathcal{S}^k_{\lambda_1,\lambda_2,\mu}$ where

$$\mathcal{S}^k_{\lambda_1,\lambda_2,\mu} = \bigoplus_{\ell=0}^{2k} \mathfrak{D}^{\ell}_{\lambda_1,\lambda_2,\mu}/\mathfrak{D}^{\ell-\frac{1}{2}}_{\lambda_1,\lambda_2,\mu} = \bigoplus_{\ell=0}^{2k} \mathfrak{F}^{(\ell)}_{\delta-\frac{\ell}{2}}\,,$$

$\mathfrak{F}^{(\ell)}_{\delta-\frac{\ell}{2}}$ stands for the sum $\bigoplus \mathfrak{F}_{\delta-\frac{\ell}{2}}$ where $\mathfrak{F}_{\delta-\frac{\ell}{2}}$ is counted $2\ell+1$ times.
Moreover, in the main theorem of the paper (Theorem 4.1), we prove that, if $\delta = \mu-\lambda_1-\lambda_2 \neq \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \dots, k$, then $\mathfrak{D}^k_{\lambda_1,\lambda_2,\mu}$ is isomorphic to $\mathcal{S}^k_{\lambda_1,\lambda_2,\mu}$ as an $\mathfrak{osp}(1|2)$-module. This isomorphism, called a *a conformally equivariant symbol map*, is unique (once we fix a principal symbol). Explicit expressions of the normalized symbol and its inverse, the *conformally equivariant quatization map*, are given. To confirm the importance and the usefulness of our results, we use them to give a classification of the modules $\mathfrak{D}^k_{\lambda_1,\lambda_2,\mu}$, $k = \frac{1}{2}, 1, \frac{3}{2}, 2$. The case where $k > 2$ seems to be more intricate.

2. Basic definitions and tools.

2.1. Geometry of the supercircle $S^{1|1}$. The supercircle $S^{1|1}$ is the simplest supermanifold of dimension $1|1$ generalizing S^1. It can be defined in terms of its superalgebra of functions, denoted by $C^\infty_{\mathbb{C}}(S^{1|1})$ and consisting of elements of the form:

$$F : (x,\theta) \longmapsto f_0(x) + f_1(x)\theta\,, \tag{1}$$

where x is an arbitrary parameter on S^1 (the even variable), θ is the odd variable ($\theta^2 = 0$) and f_0, f_1 are C^∞ complex valued functions. We denote by F' the derivative of F with respect to x, i.e., $F' : (x,\theta) \mapsto f_0'(x) + f_1'(x)\theta$. Let $\mathrm{Vect}(S^{1|1})$ be the superspace of vector fields on $S^{1|1}$:

$$\mathrm{Vect}_{\mathbb{C}}(S^{1|1}) = \left\{ F_0\partial_x + F_1\partial_\theta \mid F_i \in C^\infty_{\mathbb{C}}(S^{1|1}) \right\}, \tag{2}$$

where ∂_θ (resp. ∂_x) means the partial derivative $\frac{\partial}{\partial\theta}$ (resp. $\frac{\partial}{\partial x}$).

Consider the vector fields D and $\overline{D}$ defined by (see [25] for the interpretation of these fields):

$$D = \partial_\theta + \theta\partial_x \quad \text{and} \quad \overline{D} = \partial_\theta - \theta\partial_x\,, \tag{3}$$

these vector fields satisfy the condition

$$\overline{D}^{2j} = -D^{2j} = (-1)^j\partial_x^j, \forall j \in \mathbb{N}\,. \tag{4}$$

One can easily check the *super Leibniz formula*:

$$\overline{D}^j \circ F = \sum_{i=0}^{j} \binom{j}{i}_s (-1)^{|F|(j-i)}\overline{D}^i(F)\overline{D}^{j-i}\,, \tag{5}$$

where the notions $\binom{j}{i}_s$ and $|\ |$ stand respectively for the *super combination* defined by

$$\binom{j}{i}_s = \begin{cases} \binom{[\frac{j}{2}]}{[\frac{i}{2}]} & \text{if i is even or j is odd} \\ 0 & \text{otherwise}, \end{cases} \tag{6}$$

and for the parity function ($[x]$ denotes the integer part of a real number x).

The distribution generated by $\overline{D}$ defines a codimension 1 non-integrable distribution on $S^{1|1}$ called the *standard contact structure* on $S^{1|1}$ which is equivalently the kernel of differential 1-form

$$\alpha = dx + \theta d\theta\,.$$

A vector field X is said to be contact if it preserves the contact distribution, i.e.,

$$[X, \overline{D}] = F_X \overline{D}\,, \tag{7}$$

where $F_X \in C^\infty_{\mathbb{C}}(S^{1|1})$ is a function depending on X. We denote by $\mathcal{K}(1)$ the *Lie superalgebra of contact vector fields* on $S^{1|1}$. An element in $\mathcal{K}(1)$ can be expressed for any $f \in C^\infty_{\mathbb{C}}(S^{1|1})$ as (see [10]):

$$X_f = -f\overline{D}^2 + \frac{1}{2}D(f)\overline{D}\,. \tag{8}$$

The contact bracket is defined by

$$[X_f, X_g] = X_{\{f,g\}} \tag{9}$$

and the space $C^\infty_{\mathbb{C}}(S^{1|1})$ is thus equipped with a Lie superalgebra structure (isomorphic to $\mathcal{K}(1)$) thanks to the bracket:

$$\{f, g\} = fg' - f'g + \frac{1}{2}(-1)^{|f|(|g|+1)}D(f)D(g)\,. \tag{10}$$

The action of $\mathcal{K}(1)$ on $C^\infty_{\mathbb{C}}(S^{1|1})$ is defined by:

$$\mathfrak{L}_{X_f}(g) = fg' + \frac{1}{2}D(f)\overline{D}(g) = fg' + \frac{1}{2}(-1)^{|f|+1}\overline{D}(f)\overline{D}(g)\,. \tag{11}$$

2.2. The orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2)$. We consider the *orthosymplectic Lie superalgebra* $\mathfrak{osp}(1|2)$, which is the smallest simple Lie superalgebra. It can be defined as the real algebra with basis (H, X, Y, A, B), the elements H, X and Y are even (with parity 0) and the elements A, B are odd (with parity 1), the bracket is graded skewsymmetric, it satisfies the graded Jacobi identity

$$(-1)^{|U||W|}[[U, V], W] + (-1)^{|V||U|}[[V, W], U] + (-1)^{|W||V|}[[W, U], V] = 0\,. \tag{12}$$

The commutation relations are:

$$
\begin{array}{llll}
 & [H,X]=X\,, & [H,Y]=-Y\,, & [X,Y]=2H\,,\\
(13) & [H,A]=\frac{1}{2}A\,, & [X,A]=0\,, & [Y,A]=-B\,,\\
 & [H,B]=-\frac{1}{2}B\,, & [X,B]=-A\,, & [Y,B]=0\,,\\
 & [A,A]=2X\,, & [A,B]=2H\,, & [B,B]=-2Y\,.
\end{array}
$$

The even subalgebra $(\mathfrak{osp}(1|2))_0$ of $\mathfrak{osp}(1|2)$ is of course the simple Lie algebra $\mathfrak{sl}(2)$, with basis $\{X,\ Y,\ H\}$. From the relations, it is clear that, as a Lie superalgebra, $\mathfrak{osp}(1|2)$ is generated by its odd part $(\mathfrak{osp}(1|2))_1 = \mathrm{Span}(A, B)$.
We can realize the superalgebra $\mathfrak{osp}(1|2)$ as a subalgebra of $\mathcal{K}(1)$ (and evidently of $\mathrm{Vect}_{\mathbb{C}}(S^{1|1})$) by setting

$$(-X_x, X_1, -X_{x^2}, 2X_\theta, X_{x\theta}) = (H, X, Y, A, B)\,. \tag{14}$$

It is well known that if we identify S^1 with $\mathbb{RP}^1$ with homogeneous coordinates $(x_1 : x_2)$ and choose the affine coordinate $x = x_1/x_2$, the vector fields $\frac{d}{dx},\ x\frac{d}{dx},\ x^2\frac{d}{dx}$ are globally defined and correspond to the standard projective structure on $\mathbb{RP}^1$. In this adapted coordinates the action of the algebra $\mathfrak{sl}(2)$ viewed as the subalgebra $\mathrm{Span}\left(\frac{d}{dx},\ x\frac{d}{dx},\ x^2\frac{d}{dx}\right)$ of the Lie algebra $\mathrm{Vect}(S^1)$ is well defined.

As in the S^1-case, there exist adapted coordinates (x, θ) for which the $\mathfrak{osp}(1|2)$-action is well defined (see [10] for more details).

2.3. The space of weighted densities on $S^{1|1}$. In the super setting, by replacing dx by the 1-form α, we get an analogous definition for weighted densities, i.e., we define the space of λ-densities as

$$\mathfrak{F}_\lambda = \left\{F\alpha^\lambda \mid F \in C^\infty_{\mathbb{C}}(S^{1|1})\right\}. \tag{15}$$

As a vector space, $\mathfrak{F}_\lambda$ is isomorphic to $C^\infty_{\mathbb{C}}(S^{1|1})$.

Let X_F a contact vector field , we define a one-parameter family of first order differential operators on $C^\infty_{\mathbb{C}}(S^{1|1})$

$$\mathfrak{L}^\lambda_{X_F} = \mathfrak{L}_{X_F} + \lambda F', \lambda \in \mathbb{C}\,. \tag{16}$$

One easily checks that the map $X_F \mapsto \mathfrak{L}^\lambda_{X_F}$ is a homomorphism of Lie superalgebra , that is, $[\mathfrak{L}^\lambda_{X_F}, \mathfrak{L}^\lambda_{X_G}] = \mathfrak{L}^\lambda_{[X_F,\,X_G]}$, for every λ. Thus $\mathfrak{F}_\lambda$ becomes a $\mathcal{K}(1)$-module on $C^\infty_{\mathbb{C}}(S^{1|1})$. Evidently, the Lie derivative of the density $G\alpha^\lambda$ along the vector field X_F in $\mathcal{K}(1)$ is given by:

$$\mathfrak{L}^\lambda_{X_F}(G\alpha^\lambda) := \mathfrak{L}^\lambda_{X_F}(G)\alpha^\lambda = \left(FG' + \frac{1}{2}D(F)\overline{D}(G) + \lambda F'G\right)\alpha^\lambda. \tag{17}$$

One can easily see that:

(1) The adjoint $\mathcal{K}(1)$-module, is isomorphic to $\mathfrak{F}_{-1}$.
(2) As a $\mathrm{Vect}(S^1)$-module, $\mathfrak{F}_\lambda \simeq \mathcal{F}_\lambda \oplus \Pi(\mathcal{F}_{\lambda+\frac{1}{2}})$.

2.4. Bilinear differential operators on weighted densities. Obviously, $\forall \lambda_1, \lambda_2 \in \mathbb{R}$, $\mathfrak{F}_{\lambda_1} \otimes \mathfrak{F}_{\lambda_2}$ also a $\mathcal{K}(1)$-module with the action

$$\mathfrak{L}^{\lambda_1,\lambda_2}_{X_F}(\Phi_1 \otimes \Phi_2) = \mathfrak{L}^{\lambda_1}_{X_F}(\Phi_1) \otimes \Phi_2 + (-1)^{|F||\Phi_1|}\Phi_1 \otimes \mathfrak{L}^{\lambda_2}_{X_F}(\Phi_2)\,. \tag{18}$$

Thus, we consider a family of $\mathcal{K}(1)$-actions on the superspace of bilinear differential operators $\mathfrak{D}_{\lambda_1,\lambda_2,\mu} := \mathrm{Hom}_{\mathrm{diff}}(\mathfrak{F}_{\lambda_1} \otimes \mathfrak{F}_{\lambda_2}, \mathfrak{F}_\mu)$:

$$\mathfrak{L}^{\lambda_1,\lambda_2,\mu}_{X_F}(A) = \mathfrak{L}^{\mu}_{X_F} \circ A - (-1)^{|A||F|} A \circ \mathfrak{L}^{\lambda_1,\lambda_2}_{X_F}\,. \tag{19}$$

Since $\overline{D}^2 = -\partial_x$, every differential operator $A \in \mathfrak{D}_{\lambda_1,\lambda_2,\mu}$ can be expressed in the form (see [10])

$$A = \sum_{\ell=0}^{2k} \sum_{i+j=\ell} a_{i,j}(x,\theta)\overline{D}^i \otimes \overline{D}^j \tag{20}$$

where the coefficients $a_{i,j}$ are arbitrary functions and $\ell \in \mathbb{N}$. That is, forall $F = f\alpha_1^\lambda \in \mathfrak{F}_{\lambda_1}$, $G = g\alpha_2^\lambda \in \mathfrak{F}_{\lambda_2}$,

$$A(F \otimes G) = \Bigg(\sum_{\ell=0}^{2k} \sum_{i+j=\ell} a_{i,j}(x,\theta)(-1)^{j|f|}\overline{D}^i(f)\overline{D}^j(g)\Bigg)\alpha^\mu\,. \tag{21}$$

Moreover, if $A \in \mathfrak{D}^k_{\lambda_1,\lambda_2,\mu}$ then $\ell = 2k$. For short, we will write the operator A as:

$$A = \sum_{\ell=0}^{2k} \sum_{i+j=\ell} a_{i,j}\overline{D}^i \otimes \overline{D}^j\,. \tag{22}$$

Thus, we have a $\mathcal{K}(1)$-invariant *finer filtration*:

$$\mathfrak{D}^0_{\lambda_1,\lambda_2,\mu} \subset \mathfrak{D}^{\frac{1}{2}}_{\lambda_1,\lambda_2,\mu} \subset \mathfrak{D}^1_{\lambda_1,\lambda 2,\mu} \subset \mathfrak{D}^{\frac{3}{2}}_{\lambda_1,\lambda_2,\mu} \subset \cdots \mathfrak{D}^{k-\frac{1}{2}}_{\lambda_1,\lambda_2,\mu} \subset \mathfrak{D}^k_{\lambda_1,\lambda_2,\mu} \subset \cdots\,. \tag{23}$$

2.5. Principal symbol map. The quotient module $\mathfrak{D}^k_{\lambda_1,\lambda_2,\mu}/\mathfrak{D}^{k-\frac{1}{2}}_{\lambda_1,\lambda_2,\mu}$, $k \in \frac{1}{2}\mathbb{N}$, can be decomposed into $2k+1$ components that transform under coordinates change as $\delta - \frac{k}{2}$ densities, where $\delta = \mu - \lambda_1 - \lambda_2$. Therefore, the multiplication of these components by any non-singular matrix, say ω, gives rise to an isomorphism

$$\sigma^\omega : \mathfrak{D}^k_{\lambda_1,\lambda_2,\mu}/\mathfrak{D}^{k-\frac{1}{2}}_{\lambda_1,\lambda_2,\mu} \xrightarrow{\simeq} \mathfrak{F}_{\delta-\frac{k}{2}} \oplus \mathfrak{F}_{\delta-\frac{k}{2}} \oplus \cdots \oplus \mathfrak{F}_{\delta-\frac{k}{2}}\ (2k+1 \text{ copies})\,. \tag{24}$$

The map σ^ω is what we call *the principal symbol map*. By the very definition, a principal symbol map is $\mathcal{K}(1)$-invariant but, unlike the unary case, is not unique. Let us consider *the space of symbols*, i.e., the graded space

$$\mathcal{S}_\delta = \mathcal{S}_{\mu-\lambda_1-\lambda_2} = \bigoplus_{k=0}^{\infty} \mathfrak{D}^k_{\lambda_1,\lambda_2,\mu}/\mathfrak{D}^{k-\frac{1}{2}}_{\lambda_1,\lambda_2,\mu}\,, \tag{25}$$

associated to the filtration (23) of $\mathfrak{D}_{\lambda_1,\lambda_2,\mu}$. The space of symbols of order $\leq k$, $k \in \frac{1}{2}\mathbb{N}$, is

$$\mathcal{S}_\delta^k = \bigoplus_{\ell=0}^{2k} \mathfrak{D}^{\ell}_{\lambda_1,\lambda_2,\mu}/\mathfrak{D}^{\ell-\frac{1}{2}}_{\lambda_1,\lambda_2,\mu} = \bigoplus_{\ell=0}^{2k} \mathfrak{F}^{(\ell)}_{\delta-\frac{\ell}{2}}\,, \tag{26}$$

here the notation $\mathfrak{F}_\lambda^{(i)}$, $i \in \mathbb{N}$ and $\lambda \in \mathbb{C}$, stands for the sum $\bigoplus \mathfrak{F}_\lambda$ where $\mathfrak{F}_\lambda$ is counted $2i+1$ times. Thanks to the isomorphism (24), an element P of $\mathcal{S}_\delta^k$ can be written in a unique way in the form

$$P = \alpha^\delta \sum_{\ell=0}^{2k} \sum_{i+j=\ell} \bar{a}_{i,j}(x,\theta)\, \alpha^{-\frac{i+j}{2}} \tag{27}$$

where $\bar{a}_{i,j}$ are arbitrary functions in $C^\infty_{\mathbb{C}}(S^{1|1})$. Obviously, the space of symbols $\mathcal{S}_\delta$ is a module over the orthosymplectic superalgebra $\mathfrak{osp}(1|2)$.

3. Modules of bilinear differential operators.

3.1. Left and right conjugations. Let us denote by $\mathcal{B}$ the Berezin integral $\mathcal{B} : \mathfrak{F}_{\frac{1}{2}} \to \mathbb{C}$ given, for any $f = f_0 + \theta f_1$, by the formula [3]

$$\mathcal{B}(f\alpha^{\frac{1}{2}}) = \int_{S^1} f_1(x)dx\,. \tag{28}$$

It is well known that the Berezin integral $\mathcal{B}$ is $\mathcal{K}(1)$-invariant, that is

$$\mathcal{B}\left(\mathfrak{L}^{\frac{1}{2}}_{X_F}(f\alpha^{\frac{1}{2}})\right) = 0,\ \forall F, f \in C^\infty_{\mathbb{C}}(S^{1|1})\,. \tag{29}$$

So, the product of densities composed with $\mathcal{B}$ yields a bilinear $\mathcal{K}(1)$-invariant form:

$$\langle .\,,\ .\rangle : \mathfrak{F}_\lambda \otimes \mathfrak{F}_{\frac{1}{2}-\lambda} \to \mathbb{C}\,, \quad \lambda \in \mathbb{C}\,, \tag{30}$$

given by

$$\langle f\alpha^\lambda, g\alpha^{\frac{1}{2}-\lambda}\rangle = \int_{S^1} (f_1 g_0 + f_0 g_1)(x)dx, \tag{31}$$

where $f = f_0 + \theta f_1 \in \mathfrak{F}_\lambda$ and $g = g_0 + \theta g_1 \in \mathfrak{F}_{\frac{1}{2}-\lambda}$.

THEOREM 3.1. *For each value of $k \in \frac{1}{2}\mathbb{N}$, there exist the following isomorphisms of $\mathcal{K}(1)$-modules:*

$$\begin{aligned}\mathfrak{D}^k_{\lambda_1,\lambda_2,\mu} &\cong \mathfrak{D}^k_{\frac{1}{2}-\mu,\lambda_2,\frac{1}{2}-\lambda_1} \cong \mathfrak{D}^k_{\lambda_1,\frac{1}{2}-\mu,\frac{1}{2}-\lambda_2} \cong \mathfrak{D}^k_{\frac{1}{2}-\mu,\lambda_1,\frac{1}{2}-\lambda_2}\\ &\cong \mathfrak{D}^k_{\lambda_2,\frac{1}{2}-\mu,\frac{1}{2}-\lambda_1} \cong \mathfrak{D}^k_{\lambda_2,\lambda_1,\mu}\end{aligned} \tag{32}$$

given respectively by the $\mathfrak{osp}(1|2)$-invariant maps $C_1, C_2, C_1 \circ C_2, C_2 \circ C_1$ and $C_2 \circ C_1 \circ C_2$(or $C_1 \circ C_2 \circ C_1$) where C_1 (respectively C_2) is the left conjugation (respectively right *conjugation) given for a bilinear operator* $A = \sum_{\ell=0}^{2k} \sum_{i+j=\ell} a_{i,j} \overline{D}^i \otimes \overline{D}^j$ *by the rule*

$$C_1(A) = \sum_{\ell=0}^{2k} \sum_{i+j=\ell} (-1)^{[\frac{i+1}{2}]+i|a_{i,j}|} \sum_{\ell=0}^{i} (-1)^{\ell|a_{i,j}|} \binom{i}{\ell}_s \overline{D}^{i-\ell} \circ a_{i,j} \otimes \overline{D}^{j+\ell}, \tag{33}$$

(*respectively by* $C_2(A) = \sum_{\ell=0}^{2k} \sum_{i+j=\ell} (-1)^{[\frac{j+1}{2}]+j|a_{i,j}|} \sum_{\ell=0}^{j} (-1)^{\ell|a_{i,j}|} \binom{j}{\ell}_s \overline{D}^{i+\ell} \otimes \overline{D}^{j-\ell} \circ a_{i,j}$).

PROOF. Let $A \in \mathfrak{D}^k_{\lambda_1,\lambda_2,\mu}$. Then there exists a unique bilinear differential operator $C_1(A) \in \mathfrak{D}^k_{\frac{1}{2}-\mu,\lambda_2,\frac{1}{2}-\lambda_1}$ such that

$$\langle A(F \otimes G), \Psi\rangle = (-1)^{|A||f|+|g||\psi|}\langle F, C_1(A)(\Psi \otimes G)\rangle \tag{34}$$

where $F = f\alpha^{\lambda_1} \in \mathfrak{F}_{\lambda_1}$, $G = g\alpha^{\lambda_2} \in \mathfrak{F}_{\lambda_2}$ and $\Psi = \psi\alpha^{\frac{1}{2}-\mu} \in \mathfrak{F}_{\frac{1}{2}-\mu}$. Thus we can easily show that we get a $\mathcal{K}(1)$-invariant linear bijective map $C_1 : \mathfrak{D}^k_{\lambda_1,\lambda_2,\mu} \to \mathfrak{D}^k_{\frac{1}{2}-\mu,\lambda_2,\frac{1}{2}-\lambda_1}$.

Now, we shall prove that the operator given in (33) satisfies the condition (34). Indeed, let $A = a\overline{D}^i \otimes \overline{D}^j \in \mathfrak{D}_{\lambda_1,\lambda_2,\mu}$ and suppose that $i = 2p+1$ is odd. Then $a\overline{D}^i(f)\overline{D}^j(g) = (-1)^p a_0 f_1^{(p)}\overline{D}^j(g)_0 + (-1)^p\theta\Big(a_0 f_1^{(p)}\overline{D}^j(g)_1 - a_0 f_0^{(p+1)}\overline{D}^j(g)_0 + a_1 f_1^{(p)}\overline{D}^j(g)_0\Big)$. Thus

$$\begin{aligned}
\langle A(F \otimes G), \Psi\rangle &= (-1)^{|f|j}(-1)^p \int_{S^1} \Big(a_0 f_1^{(p)}\overline{D}^j(g)_0\psi_1\Big)(x)dx \\
&+(-1)^{|f|j}(-1)^p \int_{S^1} \Big(a_0 f_1^{(p)}\overline{D}^j(g)_1 - a_0 f_0^{(p+1)}\overline{D}^j(g)_0 + a_1 f_1^{(p)}\overline{D}^j(g)_0\Big)(x)\psi_0(x)dx \\
&= \int_{S^1} f_0(x)\Big(a_0\overline{D}^j(g)_0\psi_0\Big)^{(p+1)}(x)dx \\
&+ (-1)^j \int_{S^1} f_1(x)\Big(a_0\overline{D}^j(g)_0\psi_1 + a_0\overline{D}^j(g)_1\psi_0 + a_1\overline{D}^j(g)_0\psi_0\Big)^{(p)}(x)dx\,.
\end{aligned}$$

On the other hand, using (5) and (6), we have

$$\begin{aligned}
C_1(A) &= (-1)^{p+1+(2p+1)|a|} \sum_{\ell=0}^{2p+1} (-1)^{\ell|a|} \binom{2p+1}{\ell}_s \overline{D}^{2p+1-\ell} \circ a \otimes \overline{D}^{j+\ell} \\
&= (-1)^{p+1+|a|}\Big[\sum_{\ell=0}^{p} \binom{p}{\ell} \overline{D}^{2p-2\ell+1} \circ a \otimes \overline{D}^{j+2\ell} \\
&\qquad + \sum_{\ell=0}^{p} (-1)^{|a|} \binom{p}{\ell} \overline{D}^{2p-2\ell} \circ a \otimes \overline{D}^{j+2\ell+1}\Big],
\end{aligned}$$

that is

$$\begin{aligned}
C_1(A)(\Psi \otimes G) &= \sum_{\ell=0}^{p} \binom{p}{\ell} \Big[(a_1\psi_0 - (-1)^j a_0\psi_1)^{(p-\ell)}\overline{D}^j(g)_0^{(\ell)} \\
&+ \theta\Big((a_1\psi_0 + (-1)^j a_0\psi_1)^{(p-\ell)}\overline{D}^j(g)_1^{(\ell)} + (a_0\psi_0)^{(p-\ell+1)}\overline{D}^j(g)_0^{(\ell)}\Big)\Big]
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{\ell=0}^{p} (-1)^{(j+1)|\psi|+p} \binom{p}{\ell} \Big[(a_0\psi_0)^{(p-\ell)} \overline{D}^j (g)_1^{(\ell)} \\
&+ \theta\Big(- (a_0\psi_0)^{(p-\ell)} \overline{D}^j (g)_0^{(\ell+1)} + (a_1\psi_0 + (-1)^{j+1} a_0\psi_1)^{(p-\ell)} \overline{D}^j (g)_1^{(\ell)} \Big) \Big].
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\langle F, C_1(A)(\Psi \otimes G)\rangle &= \sum_{\ell=0}^{p} \binom{p}{\ell} \int_{S^1} f_1(x)(a_1\psi_0 - (-1)^j a_0\psi_1)^{(p-\ell)}(x) \overline{D}^j (g)_0^{(\ell)}(x)dx \\
&+ \sum_{\ell=0}^{p} \binom{p}{\ell} \int_{S^1} f_0(x)\Big((a_1\psi_0 - (-1)^j a_0\psi_1)^{(p-\ell)}(x) \overline{D}^j (g)_1^{(\ell)}(x) \\
&+ (a_0\psi_0)^{(p-\ell+1)}(x) \overline{D}^j (g)_0^{(\ell)}(x) \Big) dx \\
&- \sum_{\ell=0}^{p} \binom{p}{\ell} \int_{S^1} f_1 (a_0\psi_0)^{(p-\ell)}(x) \overline{D}^j (g)_1^{(\ell)}(x)dx \\
&- \sum_{\ell=0}^{p} \binom{p}{\ell} \int_{S^1} f_0 \Big(- (a_0\psi_0)^{(p-\ell)}(x) \overline{D}^j (g)_0^{(\ell+1)}(x) \\
&+ (a_1\psi_0 + (-1)^{j+1} a_0\psi_1)^{(p-\ell)}(x) \overline{D}^j (g)_1^{(\ell)}(x) \Big) dx\,.
\end{aligned}
$$

Now, given that

$$
\begin{aligned}
(-1)^{|g||\psi|} \psi_1^{(p-\ell)} \overline{D}^j (g)_0^{(\ell)} &= (-1)^j \psi_1^{(p-\ell)} \overline{D}^j (g)_0^{(\ell)} \\
(-1)^{|g||\psi|} \psi_1^{(p-\ell)} \overline{D}^j (g)_1^{(\ell)} &= (-1)^{j+1} \psi_1^{(p-\ell)} \overline{D}^j (g)_0^{(\ell)}\,,
\end{aligned}
$$

one has

$$
\begin{aligned}
&(-1)^{|A||f|+|g||\psi|} \langle F, C_1(A)(\Psi \otimes G)\rangle = (-1)^{(|a|+j+1)|f|+|g||\psi|} \langle F, C_1(A)(\Psi \otimes G)\rangle \\
&\quad = (-1)^j \sum_{\ell=0}^{p} \binom{p}{\ell} \int_{S^1} f_1(x) \Big[(a_1\psi_0 - a_0\psi_1)^{(p-\ell)} \overline{D}^j (g)_0^{(\ell)} + (a_0\psi_0)^{(p-\ell)} \overline{D}^j (g)_1^{(\ell)} \Big] dx \\
&\quad + \sum_{\ell=0}^{p} \binom{p}{\ell} \int_{S^1} f_0(x) \Big[(a_0\psi_0)^{(p-\ell+1)}(x) \overline{D}^j (g)_0^{(\ell)}(x) \\
&\quad + (a_0\psi_0)^{(p-\ell)}(x) \overline{D}^j (g)_0^{(\ell+1)}(x) \Big] dx\,.
\end{aligned}
$$

Since

$$
\sum_{\ell=0}^{p} \binom{p}{\ell} \Big((a_0\psi_0)^{(p-\ell+1)} \overline{D}^j (g)_0^{(\ell)} + (a_0\psi_0)^{(p-\ell)} \overline{D}^j (g)_0^{(\ell+1)} \Big) = \Big(a_0 \overline{D}^j (g)_0 \psi_0 \Big)^{(p+1)}
$$

and

$$\sum_{\ell=0}^{p}\binom{p}{\ell}\Big[(a_1\psi_0-a_0\psi_1)^{(p-\ell)}\overline{D}^j(g)_0^{(\ell)}+(a_0\psi_0)^{(p-\ell)}\overline{D}^j(g)_1^{(\ell)}\Big]$$
$$=\Big(a_0\overline{D}^j(g)_0\psi_1+a_0\overline{D}^j(g)_1\psi_0+a_1\overline{D}^j(g)_0\psi_0\Big)^{(p)},$$

we clearly see that $\langle A(F\otimes G),\Psi\rangle=(-1)^{|A||f|+|g||\psi|}\langle F, C_1(A)(\Psi\otimes G)\rangle$.

Finally, a more easy calculation can be made when $i=2p$ is even. Formula (33) is thus proved. □

DEFINITION 3.2.

1) We call the modules $\mathfrak{D}^k_{\frac{1}{2}-\mu,\lambda_2,\frac{1}{2}-\lambda_1}$, $\mathfrak{D}^k_{\lambda_1,\frac{1}{2}-\mu,\frac{1}{2}-\lambda_2}$, $\mathfrak{D}^k_{\frac{1}{2}-\mu,\lambda_1,\frac{1}{2}-\lambda_2}$, $\mathfrak{D}^k_{\lambda_2,\frac{1}{2}-\mu,\frac{1}{2}-\lambda_1}$, $\mathfrak{D}^k_{\lambda_2,\lambda_1,\mu}$ the adjoint modules of $\mathfrak{D}^k_{\lambda_1,\lambda_2,\mu}$. Especially, $\mathfrak{D}^k_{\frac{1}{2}-\mu,\lambda_2,\frac{1}{2}-\lambda_1}$ (respectively $\mathfrak{D}^k_{\lambda_1,\frac{1}{2}-\mu,\frac{1}{2}-\lambda_2}$, $\mathfrak{D}^k_{\lambda_2,\lambda_1,\mu}$) is called the left-adjoint (respectively the right-adjoint, the symmetric) module of $\mathfrak{D}^k_{\lambda_1,\lambda_2,\mu}$.

2) A module of the form $\mathfrak{D}^k_{\lambda,\lambda,\frac{1}{2}-\lambda}$, $\lambda\in\mathbb{C}$ will be said a self-adjoint module.

3) A module will be said singular if it is not isomorphic to any module other than itself and previously mentioned modules.

3.2. Explicit formulas for the action of $\mathcal{K}(1)$ on $\mathfrak{D}^k_{\lambda_1,\lambda_2,\mu}$. Let X_F, $F\in C^\infty_{\mathbb{C}}(S^{1|1})$, an arbitrary contact vector field and $A=\sum_{\ell=0}^{2k}\sum_{i+j=\ell}a_{i,j}\overline{D}^i\otimes\overline{D}^j\in\mathfrak{D}^k_{\lambda_1,\lambda_2,\mu}$. The following result gives the expression of the action of X_F on A.

PROPOSITION 3.3. *The action of the vector field X_F on the operator A is given by*

$$\mathfrak{L}^{\lambda_1,\lambda_2,\mu}_{X_F}(A)=\sum_{\ell=0}^{2k}\sum_{i+j=\ell}a^X_{i,j}\overline{D}^i\otimes\overline{D}^j \tag{35}$$

where:

$$\begin{aligned}a^X_{0,0}=\mathfrak{L}^{\delta}_{X_F}(a_{0,0})&-\sum_{n=1}^{2k}(-1)^{n(|F|+|a_{n,0}|)}\lambda_1\overline{D}^n(F')a_{n,0}\\&-\sum_{n=1}^{2k}(-1)^{n(|F|+|a_{0,n}|)}\lambda_2\overline{D}^n(F')a_{0,n}\end{aligned} \tag{36}$$

and

$$
\begin{aligned}
a^X_{i,j} &= \mathfrak{L}^{\delta-\frac{i+j}{2}}_{X_F}(a_{i,j}) \\
&- \sum_{n=1}^{2k-(i+j)} (-1)^{n(|F|+|a_{n+i,j}|)} \Bigg[\binom{n+i}{n+2}_s - \frac{1}{2}(-1)^i \binom{n+i}{n+1}_s \\
&\qquad\qquad + \lambda_1 \binom{n+i}{n}_s \Bigg] \overline{D}^n (F') a_{n+i,j} \\
&- \sum_{n=1}^{2k-(i+j)} (-1)^{n(|F|+|a_{i,n+j}|+i)} \Bigg[\binom{n+j}{n+2}_s - \frac{1}{2}(-1)^j \binom{n+j}{n+1}_s \\
&\qquad\qquad + \lambda_2 \binom{n+j}{n}_s \Bigg] \overline{D}^n (F') a_{i,n+j}
\end{aligned}
\tag{37}
$$

PROOF. Let $\Phi_1 = \varphi_1(x)\alpha^{\lambda_1} \in \mathfrak{F}_{\lambda_1}$, $\Phi_2 = \varphi_2(x)\alpha^{\lambda_2} \in \mathfrak{F}_{\lambda_2}$. Upon using (17), (18) and (19), we get

$$
\begin{aligned}
\mathfrak{L}^{\lambda_1,\lambda_2,\mu}_{X_F}(A)\Big(\Phi_1 \otimes \Phi_2\Big) &= \mathfrak{L}^{\mu}_{X_F}\Big(A(\Phi_1 \otimes \Phi_2)\Big) - (-1)^{|A||F|} A\Big(\mathfrak{L}^{\lambda_1}_{X_F}(\Phi_1) \otimes \Phi_2\Big) \\
&- (-1)^{|A|(|F|+|\Phi_1|)} A\Big(\Phi_1 \otimes \mathfrak{L}^{\lambda_2}_{X_F}(\Phi_2)\Big) \\
&= \Bigg[\sum_{\ell=0}^{2k} \sum_{i+j=\ell} F\Big(a_{i,j}(-1)^{j|\varphi_1|}\overline{D}^i(\varphi_1)\overline{D}^j(\varphi_2)\Big)' \\
&+ \frac{1}{2} D(F)\overline{D}\Big(a_{i,j}(-1)^{j|\varphi_1|}\overline{D}^i(\varphi_1)\overline{D}^j(\varphi_2)\Big) + \mu F'(-1)^{j|\varphi_1|}\overline{D}^i(\varphi_1)\overline{D}^j(\varphi_2) \\
&- (-1)^{|A||F|}(-1)^{j|\mathfrak{L}^{\lambda_1}_{X_F}(\Phi_1)|} a_{i,j}\overline{D}^i\Big(F\varphi_1' + \frac{1}{2}D(F)\overline{D}(\varphi_1) + \lambda_1 F'\varphi_1\Big)\overline{D}^j(\varphi_2) \\
&- (-1)^{|A|(|F|+|\Phi_1|)}(-1)^{j|\varphi_1|} a_{i,j}\overline{D}^i(\varphi_1)\overline{D}^j\Big(F\varphi_2' + \frac{1}{2}D(F)\overline{D}(\varphi_2) + \lambda_2 F'\varphi_2\Big)\Bigg]\alpha^{\mu}.
\end{aligned}
$$

Using the super Leibniz formula (5) and by writing (35) in the form

$$
\mathfrak{L}^{\lambda_1,\lambda_2,\mu}_{X_F}(A)\big(\Phi_1 \otimes \Phi_2\big) = \Bigg[\sum_{\ell=0}^{2k} \sum_{i+j=\ell} (-1)^{j|\varphi_1|} a^X_{i,j} \overline{D}^i(\varphi_1) \otimes \overline{D}^j(\varphi_2) \Bigg]\alpha^{\mu},
$$

formulas (37) are easily obtained by identification. □

4. Conformally equivariant symbol and quantization maps. We fix a principal symbol map σ^{ω} as in (24), where ω is a non singular matrix. A map $\sigma^{\omega}_{\lambda_1,\lambda_2,\mu}$ is called a *symbol map* if it is a linear bijection

$$
\sigma^{\omega}_{\lambda_1,\lambda_2,\mu} : \mathfrak{D}_{\lambda_1,\lambda_2,\mu} \to \mathcal{S}_{\delta}
\tag{38}
$$

such that the highest-order term of $\sigma^{\omega}_{\lambda_1,\lambda_2,\mu}(A)$, where $A \in \mathfrak{D}_{\lambda_1,\lambda_2,\mu}$ coincides with the principal symbol $\sigma^{\omega}(A)$. Hence, the inverse map, $Q = (\sigma^{\omega}_{\lambda_1,\lambda_2,\mu})^{-1}$, will be called a *quantization* *map*. Unlike the unary case, the problem of existence and uniqueness of $\mathfrak{osp}(1|2)$-equivariant symbol (and so quantization) map can be tackled once the symbol map σ^{ω} is fixed.

The following statement is the main result of this paper, it shows that for generic values of δ, the $\mathfrak{osp}(1|2)$-module $\mathfrak{D}^k_{\lambda_1,\lambda_2,\mu}$ is isomorphic to $\mathcal{S}^k_\delta$. The $\mathfrak{osp}(1|2)$-equivariant map is called a *conformally equivariant symbol mapping*. The main theorem of the paper is the following

THEOREM 4.1. *If δ is non-resonant, i.e., $\delta = \mu - \lambda_1 - \lambda_2 \neq \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \dots, k$ then, there exits a family of $\mathfrak{osp}(1|2)$-equivariant maps $\sigma^{\omega}_{\lambda_1,\lambda_2,\mu} : \mathfrak{D}^k_{\lambda_1,\lambda_2,\mu} \to \mathcal{S}^k_\delta$*

$$A = \sum_{p=0}^{2k}\sum_{i+j=p} a_{i,j}\overline{D}^i \otimes \overline{D}^j \longmapsto \alpha^\delta \sum_{p=0}^{2k}\sum_{i+j=p}\sum_{\ell=p}^{2k}\sum_{s+t=\ell} \omega^{s,t}_{i,j} D^{\ell-p}(a_{s,t})\alpha^{-\frac{i+j}{2}} \tag{39}$$

where $\omega^{s,t}_{i,j}$ are constants given by the induction formula

$$\begin{aligned}&(-1)^{\ell-p}\Big([\tfrac{\ell-p}{2}] + (1-(-1)^{\ell-p})(\delta - \tfrac{\ell}{2})\Big)\omega^{s,t}_{i,j} - \Big([\tfrac{s}{2}] + (1-(-1)^s)\lambda_1\Big)\omega^{s-1,t}_{i,j}\\ &-(-1)^s\Big([\tfrac{t}{2}] + (1-(-1)^t)\lambda_2\Big)\omega^{s,t-1}_{i,j} = 0.\end{aligned} \tag{40}$$

The "normalized" symbol map $\sigma^0_{\lambda_1,\lambda_2,\mu} := \sigma^{\mathrm{Id}}_{\lambda_1,\lambda_2,\mu}$ is given by the rule

$$\sigma^0_{\lambda_1,\lambda_2,\mu}(A) = \alpha^\delta \sum_{p=0}^{2k}\sum_{i+j=p}\sum_{\ell=p}^{2k}\sum_{\substack{s+t=\ell\\ s\geq i, t\geq j}} \gamma^{s,t}_{i,j} D^{\ell-p}(a_{s,t})\alpha^{-\frac{i+j}{2}} \tag{41}$$

where

$$\gamma^{s,t}_{i,j} = (-1)^{[\frac{\ell-p+1}{2}]+(t-j)s} \frac{\binom{\ell-p}{t-j}_s \binom{[\frac{s}{2}]}{[\frac{i}{2}]}\binom{[\frac{t}{2}]}{[\frac{j}{2}]}\binom{2\lambda_1+[\frac{s-1}{2}]}{[\frac{2(s-i)+1+(-1)^i}{4}]}\binom{2\lambda_2+[\frac{t-1}{2}]}{[\frac{2(t-j)+1+(-1)^j}{4}]}}{\binom{[\frac{\ell-p}{2}]}{[\frac{t-j}{2}]}\binom{[\frac{\ell-p+1}{2}]}{[\frac{t-j+1}{2}]}\binom{2\delta-p-1}{[\frac{\ell-p+1}{2}]}} \tag{42}$$

and the binomial coefficients in (42) *are defined by $\binom{\nu}{q} = \frac{\nu(\nu-1)\cdots(\nu-q+1)}{q!}$, this expression makes sense for arbitrary $\nu \in \mathbb{C}$. Moreover, once the principal symbol is fixed, the symbol map $\sigma^{\omega}_{\lambda_1,\lambda_2,\mu}$ is unique.*

PROOF. We begin the proof by proving the $\mathfrak{osp}(1|2)$-equivariance of the map $\sigma^0_{\lambda_1,\lambda_2,\mu}$. Indeed, Let $X = X_F \in \mathcal{K}(1)$. We have

$$\sigma^0_{\lambda_1,\lambda_2,\mu}\big(\mathfrak{L}^{\lambda_1,\lambda_2,\mu}_X(A)\big) = \alpha^\delta \sum_{p=0}^{2k}\sum_{i+j=p} \overline{a}^X_{i,j}\alpha^{-\frac{i+j}{2}}\,.$$

Then, we readily see that

$$\overline{a}^X_{i,j} = \sum_{\ell=p}^{2k}\sum_{s+t=\ell} \gamma^{s,t}_{i,j} D^{(\ell-p)}(a^X_{i,j}),\ \ p = i+j\,.$$

Thanks to Proposition 3.3, forall $0 \le p = i + j \le k$, we get

$$\begin{aligned}
\overline{a}^X_{i,j} &= \sum_{\ell=p}^{2k} \sum_{s+t=\ell} (-1)^{|a^X_{s,t}|(\ell-p)+\frac{(\ell-p)(\ell-p+1)}{2}} \gamma^{s,t}_{i,j} \overline{D}^{(\ell-p)}(a^X_{s,t}) \\
&= \sum_{\ell=p}^{2k} \sum_{s+t=\ell} (-1)^{|a^X_{s,t}|(\ell-p)+\frac{(\ell-p)(\ell-p+1)}{2}} \gamma^{s,t}_{i,j} \overline{D}^{(\ell-p)} \Bigg[\mathfrak{L}^{\delta-\frac{s+t}{2}}_{X_F}(a_{s,t}) \\
&\quad - \sum_{n=1}^{2k-(s+t)} (-1)^{n(|F|+|a_{n+s,t}|)} \left(\binom{n+s}{n+2}_s - \tfrac{1}{2}(-1)^s \binom{n+s}{n+1}_s + \lambda_1 \binom{n+s}{n}_s \right) \overline{D}^n(F') a_{n+s,t} \\
&\quad - \sum_{n=1}^{2k-(s+t)} (-1)^{n(|F|+|a_{s,n+t}|+s)} \left(\binom{n+t}{n+2}_s - \tfrac{1}{2}(-1)^t \binom{n+j}{n+1}_s + \lambda_2 \binom{n+t}{n}_s \right) \overline{D}^n(F') a_{s,n+t} \Bigg].
\end{aligned}$$

Thus

$$\begin{aligned}
\overline{a}^X_{i,j} - \mathfrak{L}^{(\delta-\frac{p}{2})}_X(\overline{a}_{i,j}) &= \sum_{\ell=p}^{2k} \sum_{s+t=\ell} (-1)^{|a_{s,t}|+|F|+(\ell-p)} \gamma^{s,t}_{i,j} \Big[(\delta - \tfrac{l}{2}) \binom{\ell-p}{1}_s \\
&\quad + \tfrac{1}{2}(-1)^{\ell-p} \binom{\ell-p}{2}_s + \binom{\ell-p}{3}_s \Big] \overline{D}(F') D^{\ell-p-1}(a_{i,j}) \\
&\quad - \sum_{\ell=p}^{2k} \sum_{s+t=\ell} (-1)^{|a_{s,t}|+|F|} \Big[\gamma^{s-1,t}_{i,j} \Big(\tbinom{s}{3}_s + \tfrac{1}{2}(-1)^s \tbinom{s}{2}_s + \lambda_1 \big(\tbinom{s}{3}_s \Big) \\
&\quad + (-1)^s \gamma^{s,t-1}_{i,j} \Big(\tbinom{t}{3}_s + \tfrac{1}{2}(-1)^t \tbinom{t}{2}_s + \lambda_2 \tbinom{t}{3}_s \Big) \overline{D}(F') D^{\ell-p-1}(a_{i,j}) \Big] \\
&\quad + \text{ higher terms in } \overline{D}^n(F').
\end{aligned}$$

Now, through a simple calculation, one can check out that the scalars $\gamma^{s,t}_{i,j}$ satisfie the relationship

$$(-1)^{\ell-p} \Upsilon\left(\delta - \frac{\ell}{2}, \ell - p \right) \gamma^{s,t}_{i,j} - \Upsilon(\lambda_1, s) \gamma^{s-1,t}_{i,j} - (-1)^s \Upsilon(\lambda_2, t) \gamma^{s,t-1}_{i,j} = 0$$

where for $\lambda \in \mathbb{C}$ and $m \in \mathbb{N}$ we put

$$\Upsilon(\lambda, m) = \frac{1}{2} \left(\left[\frac{m}{2} \right] + (1 - (-1)^m) \lambda \right).$$

Therefore, the term in $\overline{D}(F')$ vanishes, the map $\sigma^0_{\lambda_1,\lambda_2,\mu}$ is clearly $\mathfrak{osp}(1|2)$-equivariant.

Now, the main ingredient of the proof of the first point of our theorem is the locality property of an $\mathfrak{osp}(1|2)$-equivariant symbol map, that is, such map is given by differential operators. Indeed we can easily adapt the proof of locality given in [10] for the unary case to our case. Thus, in addition, from the expression of "normalized" symbol map $\sigma^0_{\lambda_1,\lambda_2,\mu}$ we can suppose that a general symbol map $\sigma^\omega_{\lambda_1,\lambda_2,\mu}$ have the form

$$A = \sum_{p=0}^{2k} \sum_{i+j=p} a_{i,j} \overline{D}^i \otimes \overline{D}^j \longmapsto \alpha^\delta \sum_{p=0}^{2k} \sum_{i+j=p} \sum_{\ell=p}^{2k} \sum_{s+t=\ell} \omega^{s,t}_{i,j}(x,\theta) D^{\ell-p}(a_{s,t}) \alpha^{-\frac{i+j}{2}}. \tag{43}$$

Clearly, to calculate the condition of $\mathfrak{osp}(1|2)$-equivariance, it suffices to impose invariance with respect to the vector fields $D = 2X_\theta$ and $xD = 2X_{x\theta}$. Effortlessly, we can get that, firstly, a symbol map (43) commutes with the action of D if and only if the coefficients $\omega_{i,j}^{s,t}$ are constants (i.e., do not depend on x, θ), secondly that it commutes with the action of xD if and only if the coefficients $\omega_{i,j}^{s,t}$ satisfy the induction formula (40).

If $\delta = \mu - \lambda_1 - \lambda_2$ is non-resonant, i.e., $\delta = \mu - \lambda_1 - \lambda_2 \neq \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \dots, k$, then, it is easy to see that the solution of the equation (40) and once the principal symbol σ^ω where $\omega = (w_{i,j}^{i,j})_{i+j=2k}$ is fixed, the symbol map $\sigma^\omega_{\lambda_1,\lambda_2,\mu}$ is unique. $\square$

REMARK 4.2. By setting $\zeta_{s,t}^{i,j} = \gamma_{i-s,j-t}^{i,j}$, we can write the the symbol map $\sigma^0_{\lambda_1,\lambda_2,\mu}$ as in [10] (Theorem 6.1). That is, for $A = a_{i,j}\overline{D}^i \otimes \overline{D}^j \in \mathfrak{D}^k_{\lambda_1,\lambda_2,\mu}$ and $i + j = 2k$,

$$\sigma^\omega_{\lambda_1,\lambda_2,\mu}(A) = \alpha^\delta \sum_{\ell=0}^{2k} \sum_{s+t=\ell} \zeta_{i,j}^{s,t} D^\ell(a_{i,j})\alpha^{\frac{s+t-i-j}{2}} \tag{44}$$

where

$$\zeta_{s,t}^{i,j} = \begin{cases} (-1)^{[\frac{\ell+1}{2}]+it} \dfrac{\binom{\ell}{t}_s \binom{[\frac{i}{2}]}{[\frac{i-s}{2}]} \binom{[\frac{j}{2}]}{[\frac{j-t}{2}]} \binom{2\lambda_1+[\frac{i-1}{2}]}{[\frac{2s+1+(-1)^{i+s}}{4}]} \binom{2\lambda_2+[\frac{j-1}{2}]}{[\frac{2t+1+(-1)^{j+t}}{4}]}}{\binom{[\frac{\ell}{2}]}{[\frac{t}{2}]} \binom{[\frac{\ell+1}{2}]}{[\frac{t+1}{2}]} \binom{2\delta+\ell-(i+j)-1}{[\frac{\ell+1}{2}]}} \\ \text{if } i \geq s\,, \quad j \geq t\,, \\ 0 \text{ otherwhise.} \end{cases} \tag{45}$$

Let us now give the explicit formula for the quantization map $Q^0_{\lambda_1,\lambda_2,\mu}$.

PROPOSITION 4.3. *The quantization map $Q^0_{\lambda_1,\lambda_2,\mu}$, i.e., the inverse of the symbol map $\sigma^0_{\lambda_1,\lambda_2,\mu}$ given in Theorem 4.1 associates to a polynomial $P = \alpha^\delta \sum_{\ell=0}^{2k} \sum_{i+j=\ell} \bar{b}_{i,j}\alpha^{-\frac{i+j}{2}} \in \mathcal{S}^k_\delta$ the differential operator $Q^0_{\lambda_1,\lambda_2,\mu}(P) = \sum_{\ell=0}^{2k} \sum_{i+j=p} \tilde{b}_{i,j}\overline{D}^i \otimes \overline{D}^j \in \mathfrak{D}^k_{\lambda_2,\lambda_1,\mu}$ such that $\tilde{b}_{i,j} = \sum_{\ell=p}^{2k} \sum_{s+t=\ell} \beta_{i,j}^{s,t} D^{\ell-p}(\bar{b}_{s,t})$, where*

$$\begin{cases} \beta_{i,j}^{s,t} = (-1)^{[\frac{\ell-p-1}{2}]+(t-j)s} \dfrac{\binom{\ell-p}{t-j}_s \binom{[\frac{s}{2}]}{[\frac{i}{2}]} \binom{[\frac{t}{2}]}{[\frac{j}{2}]} \binom{2\lambda_1+[\frac{s-1}{2}]}{[\frac{2(s-i)+1+(-1)^i}{4}]} \binom{2\lambda_2+[\frac{t-1}{2}]}{[\frac{2(t-j)+1+(-1)^j}{4}]}}{\binom{[\frac{\ell-p}{2}]}{[\frac{t-j}{2}]} \binom{[\frac{\ell-p+1}{2}]}{[\frac{t-j+1}{2}]} \binom{2\delta-l}{[\frac{\ell-p+1}{2}]}} \\ \textit{if } \ell = s+t > p = i+j \\ \beta_{i,j}^{s,t} = \gamma_{i,j}^{s,t} \textit{ if } s+t = i+j\,. \end{cases} \tag{46}$$

REMARK 4.4. Following [10] (Section 6.2), we can see that, if $\delta = \mu - \lambda_1 - \lambda_2$ is resonant, then the equation (40) can also be easily solved for special values of $(\lambda_1, \lambda_2, \mu)$. Namely

$$\left(\frac{1-s}{4}, \frac{1-t}{4}, \frac{s+t}{4}\right) \tag{47}$$

where s, t are odd. The solution in this case is no longer unique. However, if $(\lambda_1, \lambda_2, \mu)$ are not the same as above, there is no $\mathfrak{osp}(1|2)$-isomorphism between $\mathfrak{D}^k_{\lambda_1,\lambda_2,\mu}$ and $\mathcal{S}^k_\delta$.

5. $\mathcal{K}(1)$-isomorphisms and intertwining operators. The following result is adapted from the unary case.

PROPOSITION 5.1. *For the $\mathcal{K}(1)$-module $\mathfrak{D}^k_{\lambda_1,\lambda_2,\mu}$, the difference $\delta = \mu - \lambda_1 - \lambda_2$ of weights is an invariant. That is:*

$$\mathcal{D}^k_{\lambda_1,\lambda_2,\mu} \cong \mathcal{D}^k_{\rho_1,\rho_2,\nu} \Rightarrow \mu - \lambda_1 - \lambda_2 = \nu - \rho_1 - \rho_2 \,. \tag{48}$$

PROOF. This is an immediate consequence of the equivariance with respect to the vector field X_x. □

Let now $\delta \neq 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots, k$ and let denote by $\sigma^0_{\lambda_1,\lambda_2,\mu} := \sigma^{\mathrm{Id}}_{\lambda_1,\lambda_2,\mu}$ the $\mathfrak{osp}(1|2)$-equivariant symbol map (associated with $\omega = \mathrm{Id}$). Consider $T : \mathfrak{D}^k_{\lambda_1,\lambda_2,\mu} \to \mathfrak{D}^k_{\rho_1,\rho_2,\nu}$ an isomorphism of $\mathcal{K}(1)$ modules . As T is $\mathcal{K}(1)$-equivariant, it follows that the composition

$$\mathfrak{D}^k_{\lambda_1,\lambda_2,\mu} \to \mathfrak{D}^k_{\rho_1,\rho_2,\nu} \xrightarrow{\sigma^0_{\rho_1,\rho_2,\nu}} \mathcal{S}^k_\delta = \bigoplus_{\ell=0}^{2k} \mathfrak{F}^{(\ell)}_{\delta-\frac{\ell}{2}}\,,$$

is $\mathfrak{osp}(1|2)$-equivariant. Therefore, it coincides with the symbol map $\sigma^\omega_{\lambda_1,\lambda_2,\mu}$ for some ω. Namely, $\sigma^0_{\lambda_1,\lambda_2,\mu} \circ T = \sigma^\omega_{\lambda_1,\lambda_2,\mu}$. It follows that

$$\sigma^0_{\lambda_1,\lambda_2,\mu} \circ T \circ Q^0_{\lambda_1,\lambda_2,\mu} = \sigma^\omega_{\lambda_1,\lambda_2,\mu} \circ Q^0_{\lambda_1,\lambda_2,\mu}\,.$$

Now, it is a matter of a direct computation of the map

$$\sigma^\omega_{\lambda_1,\lambda_2,\mu} \circ Q^0_{\lambda_1,\lambda_2,\mu} : \bigoplus_{\ell=0}^{2k} \mathfrak{F}^{(\ell)}_{\delta-\frac{\ell}{2}} \to \bigoplus_{\ell=0}^{2k} \mathfrak{F}^{(\ell)}_{\delta-\frac{\ell}{2}}$$

to see that the isomorphism $T : \mathfrak{D}^k_{\lambda_1,\lambda_2,\mu} \to \mathfrak{D}^k_{\rho_1,\rho_2,\nu}$ is *block diagonal* in terms of the $\mathfrak{osp}(1|2)$-equivariant symbols in the following sense:

Let $A = \displaystyle\sum_{p=0}^{2k}\sum_{i=0}^{p} a_{i,p-i}\overline{D}^i \otimes \overline{D}^{p-i}$ in $\mathfrak{D}^k_{\lambda_1,\lambda_2,\mu}$, we denote by

$$\sigma^p_{\lambda_1,\lambda_2,\mu}(A) = \alpha^\delta \sum_{i=0}^{p} \overline{a}_{i,p-i}\alpha^{-\frac{p}{2}} \text{ and } \sigma^p_{\lambda_1,\lambda_2,\mu}\big(T(A)\big) = \alpha^\delta \sum_{i=0}^{p} \overline{a}^T_{i,p-i}\alpha^{-\frac{p}{2}} \tag{49}$$

the homogeneous components of order p of $\sigma^0_{\lambda_1,\lambda_2,\mu}(A)$ and $\sigma^0_{\lambda_1,\lambda_2,\mu}\big(T(A)\big)$ respectively. Then, T is $\mathfrak{osp}(1|2)$-equivariant if and only if, for all $p \in \{0, 1, \ldots, k\}$ the symbols $\sigma^p_{\lambda_1,\lambda_2,\mu}(A)$ and $\sigma^p_{\lambda_1,\lambda_2,\mu}\big(T(A)\big)$ are proportional, that is there exits a non singular matrix $\Upsilon_p = \left(\varepsilon^{j,p-j}_{i,p-i}\right)_{\substack{0\le i\le p\\ 0\le j\le p}} \in GL_{p+1}(\mathbb{R})$ such that

$$(\overline{a}^T_{0,p}, \overline{a}^T_{1,p-1}, \ldots, \overline{a}^T_{p,0})^t = \Upsilon_p(\overline{a}_{0,p}, \overline{a}_{1,p-1}, \ldots, \overline{a}_{p,0})^t \tag{50}$$

where the notation u^t means the transpose of the vector $u \in \mathbb{R}^n$. Equivalently,

$$\overline{a}^T_{i,p-i} = \sum_{j=0}^{p} \varepsilon^{j,p-j}_{i,p-i} \overline{a}_{j,p-j};\ \forall i \in \{0, 1, \dots, p\}\,. \tag{51}$$

6. Modules of bilinear differential operators of order ≤ 2. Throughout this section, using the results of the previous section, we plan the cases of the modules $\mathfrak{D}^k_{\lambda_1,\lambda_2,\mu}$, $k = \frac{1}{2}, 1, \frac{3}{2}, 2$ with $\delta = \mu - \lambda_1 - \lambda_2$ non resonant.
Let

$$A = \sum_{\ell=0}^{2k} \sum_{i+j=\ell} a_{i,j} \overline{D}^i \otimes \overline{D}^j \in \mathfrak{D}^k_{\lambda_1,\lambda_2,\mu},\ P = \alpha^\delta \sum_{\ell=0}^{2k} \sum_{i+j=\ell} b_{i,j} \alpha^{-\frac{i+j}{2}} \in \mathcal{S}^k_\delta\,, \tag{52}$$

we set

$$\sigma^0_{\lambda_1,\lambda_2,\mu}(A) = \alpha^\delta \sum_{p=0}^{2k} \sum_{i+j=p} \overline{a}_{i,j} \alpha^{-\frac{i+j}{2}},\ Q^0_{\lambda_1,\lambda_2,\mu}(P) = \sum_{\ell=0}^{2k} \sum_{i+j=\ell} \tilde{b}_{i,j} \overline{D}^i \otimes \overline{D}^j\,. \tag{53}$$

THEOREM 6.1. 1) *All the $\mathcal{K}(1)$-modules $\mathfrak{D}^{\frac{1}{2}}_{\lambda_1,\lambda_2,\mu}$ with $\delta \neq \frac{1}{2}$ are isomorphic.*
2) *All the $\mathcal{K}(1)$-modules $\mathfrak{D}^1_{\lambda_1,\lambda_2,\mu}$ with $\delta \neq \{\frac{1}{2}, 1\}$ are isomorphic.*

PROOF. 1) Let $T : \mathfrak{D}^{\frac{1}{2}}_{\lambda_1,\lambda_2,\mu} \to \mathfrak{D}^{\frac{1}{2}}_{\rho_1,\rho_2,\nu}$ be an isomorphism of $\mathcal{K}(1)$-modules then obviously $\delta = \mu - \lambda\_\_2 = \nu - \rho_1 - \rho_2$. Let denote by $\widetilde{T} := \sigma^0_{\rho_1,\rho_2,\nu} \circ T \circ Q^0_{\lambda_1,\lambda_2,\mu} : \mathcal{S}^{\frac{1}{2}}_\delta \to \mathcal{S}^{\frac{1}{2}}_\delta$. Since $\widetilde{T}$ is diagonal, there exit $\varepsilon_0 \in \mathbb{R}^*$, $\Upsilon_1 = \left(\varepsilon^{j,1-j}_{i,1-i}\right)_{\substack{0\leq i\leq 1\\ 0\leq j\leq 1}} \in GL_2(\mathbb{R})$ such that

$$\overline{a}^T_{0,0} = \varepsilon_0 \overline{a}_{0,0}\,,\ (\overline{a}^T_{0,1}, \overline{a}^T_{1,0})^t = \Upsilon_1 (\overline{a}_{0,1}, \overline{a}_{1,0})^t\,. \tag{54}$$

Following (37), (42) and (4.3) we have

$$a^X_{0,0} = \mathfrak{L}^\delta_{X_F}(a_{0,0})\,, \quad a^X_{0,1} = \mathfrak{L}^{\delta-\frac{1}{2}}_{X_F}(a_{0,1})\,, \quad a^X_{1,0} = \mathfrak{L}^{\delta-\frac{1}{2}}_{X_F}(a_{1,0})\,, \tag{55}$$

$$\begin{cases} \overline{a}_{0,0} = a_{0,0} - \frac{2\lambda_2}{2\delta-1} D(a_{0,1}) - \frac{2\lambda_1}{2\delta-1} D(a_{1,0}) \\ \overline{a}_{0,1} = a_{0,1} \\ \overline{a}_{1,0} = a_{1,0}\,, \end{cases} \tag{56}$$

$$\begin{cases} \tilde{b}_{0,0} = b_{0,0} + \frac{2\lambda_2}{2\delta-1} D(b_{0,1}) + \frac{2\lambda_1}{2\delta-1} D(b_{1,0}) \\ \tilde{b}_{0,1} = b_{0,1} \\ \tilde{b}_{1,0} = b_{1,0}\,. \end{cases}$$

Thus, the conclusion can easily be stated.

2) By similar reasoning. □

THEOREM 6.2. *Let* $A=\{(\lambda,0,\frac{1}{2}),(0,\lambda,\frac{1}{2}),(0,0,\frac{1}{2}-\lambda),\lambda\neq 0,-\frac{1}{2},-1\}$. *Then*
(i) *All the* $\mathcal{K}(1)$*-modules* $\mathfrak{D}^{\frac{3}{2}}_{\lambda_1,\lambda_2,\mu}$ *with* $\delta\neq\{\frac{1}{2},1,\ \frac{3}{2}\}$ *and* $(\lambda_1,\lambda_2,\mu)\notin A$ *are isomorphic.*
(ii) *The modules of the form*

$$\mathfrak{D}^{\frac{3}{2}}_{\lambda,0,\frac{1}{2}}\cong\mathfrak{D}^{\frac{3}{2}}_{0,\lambda,\frac{1}{2}}\cong\mathfrak{D}^{\frac{3}{2}}_{0,0,\frac{1}{2}-\lambda}\ ;\quad \lambda\neq 0\,,\ -\frac{1}{2}\,,-1 \tag{57}$$

are singular. Moreover, the module $\mathfrak{D}^{\frac{3}{2}}_{0,\lambda,\frac{1}{2}}$ (*respectively* $\mathfrak{D}^{\frac{3}{2}}_{\lambda,0,\frac{1}{2}}$) *is a self-left-adjoint* (*respectively self-right-adjoint*) *module.*

PROOF. Let $T:\mathfrak{D}^{\frac{3}{2}}_{\lambda_1,\lambda_2,\mu}\to\mathfrak{D}^{\frac{3}{2}}_{\rho_1,\rho_2,\nu}$ be an isomorphim of $\mathcal{K}(1)$-modules. The diagonally property of $\widetilde{T}$ reads

$$\begin{aligned}\overline{a}^T_{0,0}=\varepsilon_0\overline{a}_{0,0}\ ,\ \ (\overline{a}^T_{0,1},\overline{a}^T_{1,0})^t&=\Upsilon_1(\overline{a}_{0,1},\overline{a}_{1,0})^t\ ,\ \ (\overline{a}^T_{0,2},\overline{a}^T_{1,1},\overline{a}^T_{2,0})^t\\ =\Upsilon_2(\overline{a}_{0,2},\overline{a}_{1,1},\overline{a}_{2,0})^t,\text{ and }(\overline{a}^T_{0,3},\overline{a}^T_{1,2},\overline{a}^T_{2,1},\overline{a}^T_{3,0})^t&=\Upsilon_3(\overline{a}_{0,3},\overline{a}_{1,2},\overline{a}_{2,1},\overline{a}_{3,0})^t\end{aligned}$$

where $\varepsilon_0\in\mathbb{R}^*$ and $\Upsilon_i\in GL(i+1,\mathbb{R})$, $i=1,2,3$. Unlike the cases $k=\frac{1}{2}$ and $k=1$, we obtain here an additional condition that expresses a relationship between ε_0 and Υ_3, namely

$$\Upsilon_3\Gamma(\rho_1,\rho_2,\nu)^t=\varepsilon_0\Gamma(\lambda_1,\lambda_2,\mu)^t$$

were Γ stands for the function defined by

$$\Gamma(a,b,c)=\left(\frac{a(2c-2b-1)}{2(c-a-b-1)},\frac{2ab}{2(c-a-b-1)},\frac{2ab}{2(c-a-b-1)},\frac{b(2c-2a-1)}{2(c-a-b-1)}\right).$$

Since T is an isomorphism, two cases arise:

• $\Gamma(\rho_1,\rho_2,\nu)\neq 0$ and $\Gamma(\lambda_1,\lambda_2,\mu)\neq 0$, that is (ρ_1,ρ_2,ν), $(\lambda_1,\lambda_2,\mu)\notin A$. Thus we get a family of $\mathcal{K}(1)$-isomorphisms given by the conditions

$$\varepsilon_0=1\,,\ \ \Upsilon_1\in GL(2,\mathbb{R})\,,\ \ \Upsilon_2\in GL(3,\mathbb{R})\,,\qquad\text{and}\quad \Upsilon_3\Gamma(\rho_1,\rho_2,\nu)^t=\Gamma(\lambda_1,\lambda_2,\mu)^t,$$

• $\Gamma(\rho_1,\rho_2,\nu)=\Gamma(\lambda_1,\lambda_2,\mu)=0$, then (ρ_1,ρ_2,ν), $(\lambda_1,\lambda_2,\mu)\in A$ and ii) is clearly obtained. This concludes the proof of the theorem. □

THEOREM 6.3. *Let* $B=B_1\cup B_2$ *where* $B_1=\{(\lambda,0,\frac{1}{2}),(0,\lambda,\frac{1}{2}),(0,0,\frac{1}{2}-\lambda)\,;\lambda\neq 0,-\frac{1}{2},-1,-\frac{3}{2}\}$ *and* $B_2=\{(\lambda,0,\frac{1}{2}-\lambda),(0,\lambda,\frac{1}{2}-\lambda),(\lambda,\lambda,\frac{1}{2})\,;\lambda\neq 0,-\frac{1}{4},-\frac{1}{2},-\frac{3}{4}\}$. *Then*
(i) All the $\mathcal{K}(1)$-modules $\mathfrak{D}^2_{\lambda_1,\lambda_2,\mu}$ with $\delta\neq\{\frac{1}{2},1,\frac{3}{2},2\}$ and $(\lambda_1,\lambda_2,\mu)\notin B$ are isomorphic.
(ii) The modules of the form

$$\mathfrak{D}^2_{\lambda,0,\frac{1}{2}}\cong\mathfrak{D}^2_{0,\lambda,\frac{1}{2}}\cong\mathfrak{D}^2_{0,0,\frac{1}{2}-\lambda}\ ;\quad\lambda\neq 0\,,\ -\frac{1}{2}\,,\ -1,-\frac{3}{2} \tag{58}$$

and

$$\mathfrak{D}^2_{\lambda,0,\frac{1}{2}-\lambda}\cong\mathfrak{D}^2_{0,\lambda,\frac{1}{2}-\lambda}\cong\mathfrak{D}^2_{\lambda,\lambda,\frac{1}{2}}\ ;\quad\lambda\neq 0\,,\ -\frac{1}{4}\,,\ -\frac{1}{2}\,,\ -\frac{3}{4} \tag{59}$$

are singular.

PROOF. In this case, the isomorphism $T : \mathfrak{D}^2_{\lambda_1,\lambda_2,\mu} \to \mathfrak{D}^2_{\rho_1,\rho_2,\nu}$ satisfy the following equations

$$\Upsilon_3 \Gamma_1(\rho_1,\rho_2,\nu)^t = \varepsilon_0 \Gamma_1(\lambda_1,\lambda_2,\mu)^t$$

$$\Upsilon_4 \Gamma_2(\rho_1,\rho_2,\nu)^t = \varepsilon_0 \Gamma_2(\lambda_1,\lambda_2,\mu)^t$$

$$\Upsilon_4 \Gamma_3(\rho_1,\rho_2,\nu)^t = \varepsilon^{1,0}_{1,0} \Gamma_3(\lambda_1,\lambda_2,\mu)^t + \varepsilon^{0,1}_{1,0} \Gamma_4(\lambda_1,\lambda_2,\mu)^t$$

$$\Upsilon_4 \Gamma_4(\rho_1,\rho_2,\nu)^t = \varepsilon^{1,0}_{0,1} \Gamma_3(\lambda_1,\lambda_2,\mu)^t + \varepsilon^{0,1}_{0,1} \Gamma_4(\lambda_1,\lambda_2,\mu)^t$$

where $\varepsilon_0 \in \mathbb{R}^*$, $\Upsilon_1 = \left(\varepsilon^{j,1-j}_{i,1-i}\right)_{\substack{0\le i\le 1\\ 0\le j\le 1}} \in GL_2(\mathbb{R})$, $\Upsilon_i \in GL(i+1,\mathbb{R})$, $i = 2,3,4$ and the functions Γ_i, $i = 1,\dots,4$ are respectively given by

$$\Gamma_1(a,b,c) = \left(\frac{a(2c-2b-1)}{2(c-a-b-1)}, \frac{2ab}{2(c-a-b-1)}, \frac{2ab}{2(c-a-b-1)}, \frac{b(2c-2a-1)}{2(c-a-b-1)}\right),$$

$\Gamma_2(a,b,c) = (\Gamma_2^1(a,b,c),\dots,\Gamma_2^5(a,b,c))$ where

$$\Gamma_2^1(a,b,c) = \frac{3a(2c-2b-1)}{(2c-2a-2b-1)(2c-2a-2b-4)},$$
$$\Gamma_2^5(a,b,c) = \frac{3b(2c-2b-1)}{(2c-2a-2b-1)(2c-2a-2b-4)},$$

and

$$\Gamma_2^2(a,b,c) = \Gamma_2^3(a,b,c) = \Gamma_2^4(a,b,c) = \frac{6ab}{(2c-2a-2b-1)(2c-2a-2b-4)},$$

$$\Gamma_3(a,b,c) = \left(\frac{-(2c+2a-2b-1)}{2(2c-2a-2b-3)}, \frac{b(2a+1)}{2c-2a-2b-3}, \frac{-b}{2c-2a-2b-3}, \frac{b(2c-2a-2)}{2c-2a-2b-3}, 0\right)$$

and

$$\Gamma_4(a,b,c) = \left(0, \frac{-a(2c-2b-2)}{2c-2a-2b-3}, \frac{-a}{2c-2a-2b-3}, \frac{-a(2b+1)}{2c-2a-2b-3}, \frac{-(2c-2a+2b-1)}{2(2c-2a-2b-3)}\right).$$

Thus, the following cases arise

• $\Gamma_i(\rho_1,\rho_2,\nu) \neq 0$ and $\Gamma_i(\lambda_1,\lambda_2,\mu) \neq 0$, $\forall i = 1,2,3,4$, that is (ρ_1,ρ_2,ν), $(\lambda_1,\lambda_2,\mu) \notin B$, then we get a family of $\mathcal{K}(1)$-isomorphisms given by the conditions
$\varepsilon_0 = 1$, $\Upsilon_1 = I_2$, $\Upsilon_2 \in GL(3,\mathbb{R})$, and

$$\Upsilon_3 \Gamma_1(\rho_1,\rho_2,\nu)^t = \Gamma_1(\lambda_1,\lambda_2,\mu)^t,\ \Upsilon_4 \Gamma_i(\rho_1,\rho_2,\nu)^t = \Gamma_i(\lambda_1,\lambda_2,\mu)^t,\ i = 1,2,3.$$

• $\Gamma_1(\rho_1,\rho_2,\nu) = \Gamma_1(\lambda_1,\lambda_2,\mu) = \Gamma_2(\rho_1,\rho_2,\nu) = \Gamma_2(\lambda_1,\lambda_2,\mu) = 0$, then (ρ_1,ρ_2,ν), $(\lambda_1,\lambda_2,\mu) \in B_1$, which leads to (58).

• $\Gamma_3(\lambda_1,\lambda_2,\mu) = 0$, that is $(\lambda_1,\lambda_2,\mu) \in B_2$. Then $\Gamma_4(\lambda_1,\lambda_2,\mu) \neq 0$ and
$\varepsilon^{0,1}_{0,1}\Gamma_3(\rho_1,\rho_2,\nu)^t - \varepsilon^{1,0}_{1,0}\Gamma_4(\rho_1,\rho_2,\nu)^t = 0$ (since it belongs to the Kernel of Υ_4). So we get
$\varepsilon^{0,1}_{0,1}(2\nu+2\rho_1-2\rho_2-1) = 0$ and $\varepsilon^{0,1}_{1,0}(2\nu-2\rho_1+2\rho_2-1) = 0$.
Then, the following situations appear:
$(\rho_1,\rho_2,\nu) = (\lambda,\lambda,\frac{1}{2})$ if $\varepsilon^{0,1}_{0,1} \neq 0$ and $\varepsilon^{0,1}_{1,0} \neq 0$
$(\rho_1,\rho_2,\nu) = (0,\lambda,\frac{1}{2}-\lambda)$ if $\varepsilon^{0,1}_{0,1} = 0$ and
$\Upsilon_4\Gamma_4(\rho_1,\rho_2,\nu) = \varepsilon^{0,1}_{0,1}\Gamma_4(\lambda_1,\lambda_2,\mu)$ with $\varepsilon^{0,1}_{0,1} \neq 0$.

• $\Gamma_4(\lambda_1, \lambda_2, \mu) = 0$, that is $(\lambda_1, \lambda_2, \mu) \in B_2$. Then $\Gamma_3(\rho_1, \rho_2, \nu) \neq 0$, by a similar reasoning we end up with the following situations
$(\rho_1, \rho_2, \nu) = (\lambda, \lambda, \frac{1}{2})$ if $\varepsilon^{1,0}_{1,0} \neq 0$ and $\varepsilon^{1,0}_{0,1} \neq 0$
$(\rho_1, \rho_2, \nu) = (\lambda, 0, \frac{1}{2} - \lambda)$ if $\varepsilon^{1,0}_{1,0} = 0$ and
$\Upsilon_4\Gamma_3(\rho_1, \rho_2, \nu) = \varepsilon^{1,0}_{1,0}\Gamma_3(\lambda_1, \lambda_2, \mu)$ with $\varepsilon^{1,0}_{1,0} \neq 0$.
Thanks to the latter two cases, (59) is promptly proved. This achieves the proof of the theorem. □

REMARK 6.4. For the resonant case, we may conjecture that, using the notion of *normal symbol* instead of equivariant symbol, (see [1], [7] and [8]), the singular modules are the following (and their adjoint):

$(0, 0, \frac{1}{2})$ if $k = \frac{1}{2}$ or $k = 1$.
$(0, 0, \mu)$, $\mu = \frac{1}{2}, 1, \frac{3}{2}$ and $(-\frac{1}{2}, 0, 1)$ if $k = \frac{3}{2}$.
$(0, 0, \mu)$, $\mu = \frac{1}{2}, 1, \frac{3}{2}, 2$ and $(\lambda, 0, \lambda + 2)$, $\lambda \in \mathbb{C}$ if $k = 2$.

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