Tohoku Math. J.
69 (2017), 183–194

SOME NEW PROPERTIES CONCERNING BLO MARTINGALES

EIICHI NAKAI* AND GAKU SADASUE**

(Received February 10, 2015, revised March 25, 2015)

Abstract. Some new properties concerning BLO martingales are given. The BMO-BLO boundedness of martingale maximal functions and Bennett type characterization of BLO martingales are shown. Also, a non-negative BMO martingale that is not in BLO is constructed.

1. Introduction. Coifman and Rochberg [2] gave a characterization of BMO functions on $\mathbb{R}^n$. To prove it, they introduced the notion of BLO functions and gave a characterization of BLO functions in relation with A_1-weights. The relation between BMO functions and BLO functions are studied by other several authors [1, 3, 8].

In martingale theory, Varopoulos [11, 12] and Shiota [9, 10] introduced the notion of the class BLO for continuous parameter martingales. They gave several basic properties of BLO martingales. In particular, Shiota showed Coifman-Rochberg type characterization for BLO martingales. For the discrete parameter case, BLO martingales was introduced and studied extensively in Long's book [4].

In this paper, we show several new properties concerning BLO martingales in the discrete parameter case. We first show the BMO-BLO boundedness of the martingale maximal functions (Theorem 2.5). Using this boundedness, we show Bennett type characterization of BLO martingales (Theorem 2.7).

Also, we give an example that is a non-negative BMO martingale but it is not in BLO (Proposition 4.3). In case of dyadic martingales on the interval (0, 1], for a non-positive function $f(\omega) = \log(\omega)$, $\omega \in (0, 1]$, it is easy to show that the corresponding martingale is in BMO but not in BLO. However, even in case of dyadic martingales, it is not so clear whether there exists a non-negative BMO martingale which is not in BLO, see Example 4.1 and Remark 4.2. To construct the example, we use a suitable pointwise multiplier on martingale BMO spaces. This example shows that the BMO-BLO boundedness (Theorem 2.5) is not derived from known BMO-BMO boundedness and non-negativity of maximal functions.

The organization of this paper is as follows. We state notation and results in Section 2. We give proofs of Theorems 2.5 and 2.7 in Section 3. The proof of Proposition 4.3 is given in Section 4.

2010 *Mathematics Subject Classification.* Primary 60G46; Secondary 60G42, 42B35.
Key words and phrases. Martingale, BMO, BLO, pointwise multiplier, Campanato space.
*Partly supported by Grant-in-Aid for Scientific Research (C), No. 24540159, Japan Society for the Promotion of Science. **Corresponding author. Partly supported by Grant-in-Aid for Scientific Research (C), No. 24540171, Japan Society for the Promotion of Science.

At the end of this section, we make some conventions. Throughout this paper, we always use C to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts, such as C_p, is dependent on the subscripts. If $f \le Cg$, we then write $f \lesssim g$ or $g \gtrsim f$; and if $f \lesssim g \lesssim f$, we then write $f \sim g$.

2. Notation and results. We consider a probability space $(\Omega, \mathcal{F}, P)$ such that $\mathcal{F} = \sigma(\bigcup_n \mathcal{F}_n)$, where $\{\mathcal{F}_n\}_{n\ge 0}$ is a nondecreasing sequence of sub-σ-algebras of $\mathcal{F}$. For the sake of simplicity, let $\mathcal{F}_{-1} = \mathcal{F}_0$. The expectation operator and the conditional expectation operators relative to $\mathcal{F}_n$ are denoted by E and E_n, respectively.

For a real valued integrable function f, we consider a martingale $(E_n f)_{n\ge 0}$. The class of all such martingales coincides with the class of all uniformly integrable martingales (see for example [4]). In this paper, we only treat uniformly integrable martingales. Hence we often identify $f \in L_1$ with the corresponding martingale $(E_n f)_{n\ge 0}$.

We say $\{\mathcal{F}_n\}_{n\ge 0}$ is regular if there exists $R \ge 2$ such that

$$E_n f \le R E_{n-1} f \tag{1}$$

for all non-negative integrable function f.

For $f \in L_1$, we define the maximal functions $M_n f$ and Mf by

$$M_n f = \sup_{0\le m\le n} |E_m f| \quad \text{and} \quad Mf = \sup_{n\ge 0} |E_n f|\,,$$

respectively with convention $M_{-1} f = M_0 f$. For $f \in L_1$, we also define the natural maximal functions $\mathcal{M}_n f$ and $\mathcal{M} f$ by

$$\mathcal{M}_n f = \sup_{0\le m\le n} E_m f \quad \text{and} \quad \mathcal{M} f = \sup_{n\ge 0} E_n f\,,$$

respectively with convention $\mathcal{M}_{-1} f = \mathcal{M}_0 f$. Note that we always treat real valued functions in this paper.

We now recall the definition of martingale BMO spaces.

DEFINITION 2.1. Let $p \in [1,\infty)$. For $f \in L_1$, let

$$\|f\|_{\mathrm{BMO}_p} = \sup_{n\ge 0} \|E_n[|f - E_n f|^p]^{1/p}\|_\infty\,, \quad \|f\|_{\mathrm{BMO}_p^-} = \sup_{n\ge 0} \|E_n[|f - E_{n-1} f|^p]^{1/p}\|_\infty\,.$$

Then define

$$\mathrm{BMO}_p = \{f \in L_p : \|f\|_{\mathrm{BMO}_p} < \infty\}\,, \quad \mathrm{BMO}_p^- = \{f \in L_p : \|f\|_{\mathrm{BMO}_p^-} < \infty\}\,.$$

For $p = 1$, we denote BMO_1 and BMO_1^- by BMO and BMO^- respectively. We also consider the space BD_∞ which is defined by

$$\|f\|_{\mathrm{BD}_\infty} = \sup_{n\ge 0} \|E_n f - E_{n-1} f\|_\infty\,, \quad \mathrm{BD}_\infty = \{f \in L_1 : \|f\|_{\mathrm{BD}_\infty} < \infty\}\,.$$

It is known that the space BMO_2 is the dual space of Hardy space H_1^s of all martingales having integrable conditional square functions while the space BMO_2^- is the dual space of

Hardy space H_1^* of all martingales having integrable maximal functions. The space BMO_2 is important to study interpolation of martingale Hardy spaces, while the space BMO_2^- is important in the theory of A_p-weights for martingales, see [13, Chapter 5] and [4, Chapter 6].

REMARK 2.2. It is known that, for $p \in [1, \infty)$,

$$\|f\|_{\mathrm{BMO}} \le \|f\|_{\mathrm{BMO}_p}, \quad \|f\|_{\mathrm{BMO}^-} \sim \|f\|_{\mathrm{BMO}_p^-}, \quad \|f\|_{\mathrm{BMO}_p} + \|f\|_{\mathrm{BD}_\infty} \sim \|f\|_{\mathrm{BMO}_p^-}.$$

If $\{\mathcal{F}_n\}_{n\ge 0}$ is regular, then $\|f\|_{\mathrm{BMO}} \sim \|f\|_{\mathrm{BMO}_p^-}$. See [4, 13].

We next define BLO class of martingales.

DEFINITION 2.3. For $f \in L_1$, let

$$\|f\|_{\mathrm{BLO}} = \sup_{n\ge 0} \operatorname{ess\,sup}(E_n f - f), \quad \|f\|_{\mathrm{BLO}^-} = \max(\|f\|_{\mathrm{BLO}}, \|f\|_{\mathrm{BD}_\infty}).$$

Then define

$$\mathrm{BLO} = \{f \in L_1 : \|f\|_{\mathrm{BLO}} < \infty\}, \quad \mathrm{BLO}^- = \{f \in L_1 : \|f\|_{\mathrm{BLO}^-} < \infty\}.$$

REMARK 2.4. It is known that $\|f\|_{\mathrm{BMO}} \le 2\|f\|_{\mathrm{BLO}}$ and $\|f\|_{\mathrm{BMO}^-} \le 3\|f\|_{\mathrm{BLO}^-}$. See [4, Theorem 4.3.2]. If $\{\mathcal{F}_n\}_{n\ge 0}$ is regular, then $\|f\|_{\mathrm{BLO}} \sim \|f\|_{\mathrm{BLO}^-}$.

Our first result is the BMO-BLO boundedness of the maximal functions.

THEOREM 2.5. *Let $p \in (1, \infty)$. Let C_p be the smallest constant that satisfies $\|f\|_{\mathrm{BMO}_p^-} \le C_p \|f\|_{\mathrm{BMO}^-}$ for all $f \in \mathrm{BMO}^-$. Define $C^- = \inf_{p\in(1,\infty)} C_p(3p-1)/(p-1)$. Then,*

$$\|Mf\|_{\mathrm{BLO}} \le \frac{p}{p-1}\|f\|_{\mathrm{BMO}_p} \quad (f \in \mathrm{BMO}_p), \tag{2}$$

$$\|\mathcal{M}f\|_{\mathrm{BLO}} \le \frac{p}{p-1}\|f\|_{\mathrm{BMO}_p} \quad (f \in \mathrm{BMO}_p), \tag{3}$$

$$\|Mf\|_{\mathrm{BLO}^-} \le C^-\|f\|_{\mathrm{BMO}^-} \quad (f \in \mathrm{BMO}^-), \tag{4}$$

$$\|\mathcal{M}f\|_{\mathrm{BLO}^-} \le C^-\|f\|_{\mathrm{BMO}^-} \quad (f \in \mathrm{BMO}^-). \tag{5}$$

COROLLARY 2.6. *If $\{\mathcal{F}_n\}_{n\ge 0}$ is regular, then there exists $C > 0$ independent of $f \in$ BMO such that*

$$\|Mf\|_{\mathrm{BLO}} \le C\|f\|_{\mathrm{BMO}}, \quad \|\mathcal{M}f\|_{\mathrm{BLO}} \le C\|f\|_{\mathrm{BMO}} \quad (f \in \mathrm{BMO}).$$

The BMO^--BMO^- boundedness of the maximal functions is well-known. See [4, Theorem 4.1.5]. We note that, by Proposition 4.3 below, (4) is not derived from known BMO^--BMO^- boundedness and non-negativity of maximal functions.

Our next result is to give a characterization of BLO martingales, which is a martingale version of the result in Bennett [1]. Let L_p^0 denote the space of L_p function f which satisfies $E_0 f = 0$.

THEOREM 2.7. (i) *Let* $f \in L_1$. *Then,* f *belongs to* BLO^- *if and only if there exist* g *in* BMO^- *and* h *in* L_∞ *such that*

$$f = \mathcal{M}g + h\,. \tag{6}$$

In this case,

$$\|f\|_{\mathrm{BLO}^-} \sim \inf(\|g\|_{\mathrm{BMO}^-} + \|h\|_\infty)\,,$$

where the infimum is taken over all decompositions of the form (6)*.*
(ii) *Let* $f \in L_1^0$. *Then,* f *belongs to* BLO^- *if and only if there exist* g *in* BMO^- *and* h *in* L_∞ *such that*

$$f = Mg + h. \tag{7}$$

In this case,

$$\|f\|_{\mathrm{BLO}^-} \sim \inf(\|g\|_{\mathrm{BMO}^-} + \|h\|_\infty)\,,$$

where the infimum is taken over all decompositions of the form (7)*.*

3. Proofs of Theorems 2.5 and 2.7. In this section, we give proofs of Theorems 2.5 and 2.7.

PROOF OF THEOREM 2.5. Let $f \in \mathrm{BMO}_p$. Let N be a non-negative integer. Then,

$$Mf = \max\left(\sup_{n\le N}|E_n f|, \sup_{n\ge N}|E_n f|\right) \le M_N f + \sup_{n\ge N}|E_n f - E_N f|\,.$$

Applying conditional Doob's inequality to the martingale $(E_n f - E_N f)_{n\ge N}$, we have

$$\begin{aligned} E_N[Mf] &\le M_N f + E_N\left[\sup_{n\ge N}|E_n f - E_N f|\right] \\ &\le M_N f + E_N\left[\sup_{n\ge N}|E_n f - E_N f|^p\right]^{1/p} \\ &\le M_N f + \frac{p}{p-1}E_N[|f - E_N f|^p]^{1/p} \\ &\le M_N f + \frac{p}{p-1}\|f\|_{\mathrm{BMO}_p}\,. \end{aligned} \tag{8}$$

Combining (8) and $M_N f \le Mf$, we have

$$\operatorname{ess\,sup}(E_N[Mf] - Mf) \le \frac{p}{p-1}\|f\|_{\mathrm{BMO}_p}\,.$$

We have obtained (2).

Let $f \in \mathrm{BMO}^-$. To show (4), we note that

$$M_N f = E_N[M_N f] \le E_N[Mf]\,. \tag{9}$$

Combining (8) and (9), we have

$$0 \le E_N[Mf] - M_N f \le \frac{p}{p-1}\|f\|_{\mathrm{BMO}_p}\,. \tag{10}$$

From (10), we deduce

$$|(E_N[Mf] - E_{N-1}[Mf]) - (M_N f - M_{N-1} f)| \le \frac{p}{p-1}\|f\|_{\mathrm{BMO}_p}\,.$$

Since $0 \le M_N f - M_{N-1} f \le |E_N f - E_{N-1} f| \le \|f\|_{\mathrm{BD}_\infty}$, we obtain

$$|E_N[Mf] - E_{N-1}[Mf]| \le \frac{p}{p-1}\|f\|_{\mathrm{BMO}_p} + \|f\|_{\mathrm{BD}_\infty}\,.$$

Therefore, we have

$$\begin{aligned}\|Mf\|_{\mathrm{BLO}^-} &= \max(\|Mf\|_{\mathrm{BLO}}, \|Mf\|_{\mathrm{BD}_\infty})\\ &\le \frac{p}{p-1}\|f\|_{\mathrm{BMO}_p} + \|f\|_{\mathrm{BD}_\infty}\\ &\le \frac{2p}{p-1}\|f\|_{\mathrm{BMO}_p^-} + \|f\|_{\mathrm{BMO}_p^-}\\ &\le \frac{C_p(3p-1)}{p-1}\|f\|_{\mathrm{BMO}^-}\,.\end{aligned}$$

We have obtained (4).

For the maximal function $\mathcal{M}f$, we have the inequality

$$\mathcal{M}f = \max\left(\sup_{n\le N} E_n f, \sup_{n\ge N} E_n f\right) \le \mathcal{M}_N f + \sup_{n\ge N}|E_n f - E_N f|$$

for any non-negative integer N. Therefore, we can prove (3) and (5) by the same way. □

To prove Theorem 2.7, we show the following lemma.

LEMMA 3.1. (i) *Let $f \in L_1$. Then, f belongs to* BLO *if and only if $\mathcal{M}f - f$ belongs to L_∞. In this case,*

$$\|\mathcal{M}f - f\|_\infty = \|f\|_{\mathrm{BLO}}\,. \tag{11}$$

(ii) *Let $f \in L_1^0$. Then, f belongs to* BLO *if and only if $Mf - f$ belongs to L_∞. In this case,*

$$\|f\|_{\mathrm{BLO}} \le \|Mf - f\|_\infty \le 3\|f\|_{\mathrm{BLO}}\,. \tag{12}$$

PROOF. We first show (i). If $f \in$ BLO, then we have $0 \le \mathcal{M}f - f \le \|f\|_{\mathrm{BLO}}$. Therefore, we have $\|\mathcal{M}f - f\|_\infty \le \|f\|_{\mathrm{BLO}}$.

Conversely, if $f \in L_1$ such that $\mathcal{M}f - f$ belongs to L_∞, then we have $E_n f - f \le \mathcal{M}f - f \le \|\mathcal{M}f - f\|_\infty$. Therefore we obtain that f is in BLO and $\|f\|_{\mathrm{BLO}} \le \|\mathcal{M}f - f\|_\infty$. We have shown (i).

We now show (ii). Let $f \in L_1^0 \cap \mathrm{BLO}$. Since $E_0 f = 0$, we have

$$-\operatorname{ess\,inf} f = \operatorname{ess\,sup}(E_0 f - f) \le \|f\|_{\mathrm{BLO}}\,. \tag{13}$$

Therefore, we have

$$\begin{aligned}0 \le Mf - f &\le M(f + \|f\|_{\mathrm{BLO}}) - (f + \|f\|_{\mathrm{BLO}}) + 2\|f\|_{\mathrm{BLO}}\\ &= \{\mathcal{M}(f + \|f\|_{\mathrm{BLO}}) - (f + \|f\|_{\mathrm{BLO}})\} + 2\|f\|_{\mathrm{BLO}}\end{aligned}$$

$$= \mathcal{M}f - f + 2\|f\|_{\mathrm{BLO}}\,.$$

Hence, using (11), we have

$$\|Mf - f\|_\infty \le 3\|f\|_{\mathrm{BLO}}.$$

Conversely, let $f \in L_1^0$ such that $Mf - f$ is in L_∞. Then we have $E_n f - f \le |E_n f| - f \le Mf - f \le \|Mf - f\|_\infty$.

The proof of Lemma 3.1 is completed.

PROOF OF THEOREM 2.7. We show only (i) because (ii) is proved by the same method. Let $f = \mathcal{M}g + h$ where $g \in \mathrm{BMO}^-$ and $h \in L_\infty$. By Theorem 2.5, we have

$$E_n[\mathcal{M}g + h] - (\mathcal{M}g + h) \le \|\mathcal{M}g\|_{\mathrm{BLO}^-} + 2\|h\|_\infty \le C^-\|g\|_{\mathrm{BMO}^-} + 2\|h\|_\infty$$

and

$$|E_n[\mathcal{M}g + h] - E_{n-1}[\mathcal{M}g + h]| \le \|\mathcal{M}g\|_{\mathrm{BLO}^-} + 2\|h\|_\infty \le C^-\|g\|_{\mathrm{BMO}^-} + 2\|h\|_\infty.$$

Therefore, f is in BLO^- with

$$\|f\|_{\mathrm{BLO}^-} \le \inf(C^-\|g\|_{\mathrm{BMO}^-} + 2\|h\|_\infty) \sim \inf(\|g\|_{\mathrm{BMO}^-} + \|h\|_\infty)\,. \tag{14}$$

Conversely, let $f \in \mathrm{BLO}^-$. Then, f is in BMO^- by Remark 2.4. Moreover, $f - \mathcal{M}f$ is bounded and $\|\mathcal{M}f - f\|_\infty = \|f\|_{\mathrm{BLO}}$ by Lemma 3.1. Hence, letting $g = f$ and $h = f - \mathcal{M}f$, we have the decomposition $f = \mathcal{M}g + h$ where g is in BMO^- and h is in L_∞ with $\|h\|_\infty = \|f\|_{\mathrm{BLO}}$. In this case, we have

$$\|g\|_{\mathrm{BMO}^-} + \|h\|_\infty \le 3\|f\|_{\mathrm{BLO}^-} + \|f\|_{\mathrm{BLO}} \le 4\|f\|_{\mathrm{BLO}^-}\,. \tag{15}$$

Combining (14) and (15), we have

$$\|f\|_{\mathrm{BLO}^-} \sim \inf(\|g\|_{\mathrm{BMO}^-} + \|h\|_\infty)\,.$$

The proof of Theorem 2.7 is completed. □

4. Difference between BMO and BLO.

In this section, we give an example that is a non-negative BMO martingale but it is not in BLO. To show this, we begin by recalling the notion of atoms.

A measurable set $B \in \mathcal{F}_n$ such that $P(B) > 0$ is called an atom (more precisely a $(\mathcal{F}_n, P)$-atom), if any $A \subset B$ with $A \in \mathcal{F}_n$ satisfies $P(A) = P(B)$ or $P(A) = 0$. We denote by $A(\mathcal{F}_n)$ the set of all atoms in $\mathcal{F}_n$.

For the rest of this paper, we postulate the following conditions on $\{\mathcal{F}_n\}_{n\ge0}$:

(F1) Every σ-algebra $\mathcal{F}_n$ is generated by countable atoms.
(F2) $\{\mathcal{F}_n\}_{n\ge0}$ is regular.
(F3) $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

For such $\{\mathcal{F}_n\}_{n\geq 0}$, BMO $=$ BMO$^-$ and BLO $=$ BLO$^-$ with equivalent norms as is mentioned in Remark 2.2 and Remark 2.4 respectively. Moreover, we have the following expression of norms:

$$\|f\|_{\mathrm{BMO}} = \sup_{n\geq 0} \sup_{B\in A(\mathcal{F}_n)} \frac{1}{P(B)} \int_B |f - E_n f| \, dP \,,$$
$$\|f\|_{\mathrm{BLO}} = \sup_{n\geq 0} \sup_{B\in A(\mathcal{F}_n)} \frac{1}{P(B)} \int_B \Big(f - \operatorname*{ess\,inf}_B f\Big) \, dP \,.$$

By (F3), if $\operatorname{ess\,inf} f = -\infty$, then $f \notin$ BLO. Hence, if BMO $\setminus L_\infty \neq \emptyset$, then BMO $\setminus$ BLO $\neq \emptyset$. Therefore, to obtain a BMO martingale which is not in BLO, we only have to find an unbounded BMO martingale. We recall the following well-known example.

EXAMPLE 4.1. Let $((0,1], \mathcal{B}, P)$ be the Lebesgue's probability space. Let

$$\mathcal{F}_n = \sigma[\{I_{n,k} : k = 1, 2, \dots, 2^n\}] \,, \quad I_{n,k} = ((k-1)/2^n, k/2^n] \,.$$

Let $f(\omega) = \log(1/\omega)$ ($\omega \in (0,1]$). It is well-known that

$$f \in \mathrm{BLO} \subset \mathrm{BMO} \quad \text{and} \quad -f \in \mathrm{BMO} \setminus \mathrm{BLO} \,.$$

Actually, for $\omega \in ((k-1)/2^n, k/2^n]$,

$$\begin{aligned} E_n[f](\omega) - f(\omega) &= 2^n \int_{(k-1)/2^n}^{k/2^n} \log(1/x)\, dx - \log(1/\omega) \\ &\leq 2^n \Big[-(x\log x - x) \Big]_{(k-1)/2^n}^{k/2^n} + \log \frac{k}{2^n} \\ &\leq 1 \,, \end{aligned} \tag{16}$$

and $\operatorname{ess\,inf}(-f) = -\infty$.

REMARK 4.2. In Example 4.1, $\log(1/\omega)$ provides a simple example of a non-negative BLO martingale and $\log(\omega)$ provides a simple example of a non-positive BMO martingale which is not in BLO. Such examples are also known in general case. See [4, Lemma 4.3.8]. However, even in case of dyadic martingales, it is not known that there exists a simple example of a non-negative BMO martingale which is not in BLO. Thus, it is not so clear whether there exists a non-negative BMO martingale which is not in BLO.

The following is the main result in this section.

PROPOSITION 4.3. *Let $\{\mathcal{F}_n\}_{n\geq 0}$ satisfy the conditions* (F1), (F2) *and* (F3). *Suppose that there exists a sequence of measurable sets $(B_n)_{n\geq 0}$ that satisfies*

$$B_n \in A(\mathcal{F}_n), \quad B_0 \supset B_1 \supset \cdots \supset B_n \supset \cdots \quad \text{and} \quad \lim_{n\to\infty} P(B_n) = 0 \,. \tag{17}$$

Then, there exists a non-negative BMO *martingale which is not in* BLO.

To show Proposition 4.3, we use pointwise multipliers on generalized martingale Campanato spaces in [7]. Generalized martingale Campanato spaces $\mathcal{L}_{p,\phi}$ and $\mathcal{L}^{\natural}_{p,\phi}$ are defined by the following:

DEFINITION 4.4. Let $p \in [1, \infty)$ and ϕ be a function from $(0, 1]$ to $(0, \infty)$. For $f \in L_1$, let

$$\|f\|_{\mathcal{L}_{p,\phi}} = \sup_{n \geq 0} \sup_{B \in A(\mathcal{F}_n)} \frac{1}{\phi(P(B))} \left(\frac{1}{P(B)} \int_B |f - E_n f|^p \, dP \right)^{1/p}, \tag{18}$$

and

$$\|f\|_{\mathcal{L}^\natural_{p,\phi}} = \|f\|_{\mathcal{L}_{p,\phi}} + |Ef| . \tag{19}$$

Define

$$\mathcal{L}_{p,\phi} = \{f \in L_1 : \|f\|_{\mathcal{L}_{p,\phi}} < \infty\} \quad \text{and} \quad \mathcal{L}^\natural_{p,\phi} = \{f \in L_1 : \|f\|_{\mathcal{L}^\natural_{p,\phi}} < \infty\} .$$

Note that $\mathcal{L}_{p,\phi}$ and $\mathcal{L}^\natural_{p,\phi}$ coincide as sets of measurable functions. We regard $\mathcal{L}_{p,\phi} = (\mathcal{L}_{p,\phi}, \|\cdot\|_{\mathcal{L}_{p,\phi}})$ is a seminormed space and $\mathcal{L}^\natural_{p,\phi} = (\mathcal{L}^\natural_{p,\phi}, \|\cdot\|_{\mathcal{L}^\natural_{p,\phi}})$ is a normed space. Moreover, if $\phi \equiv 1$, then $\mathcal{L}_{1,\phi}$ coincides with BMO. So we denote $\mathcal{L}^\natural_{1,\phi}$ by BMO$^\natural$ for $\phi \equiv 1$.

Let $\mathcal{X}$ be a normed space of $\mathcal{F}$-measurable functions. We say that an $\mathcal{F}$-measurable function g is a pointwise multiplier on $\mathcal{X}$, if the pointwise multiplication fg is in $\mathcal{X}$ for any $f \in \mathcal{X}$. We denote by PWM($\mathcal{X}$) the set of all pointwise multipliers on $\mathcal{X}$.

If $\mathcal{X}$ is a Banach space and has the property

$$f_n \to f \text{ in } \mathcal{X}\ (n \to \infty) \implies \exists \{n(j)\} \text{ s.t. } f_{n(j)} \to f \text{ a.s. } (j \to \infty), \tag{20}$$

then every $g \in$ PWM($\mathcal{X}$) is a bounded operator on $\mathcal{X}$ by the pointwise multiplication. We denote by $\|g\|_{\mathrm{Op}}$ the operator norm.

Then the following is known.

LEMMA 4.5 ([7, Corollary 1.5]). *The space* PWM(BMO$^\natural$) *is characterized as*

$$\mathrm{PWM}(\mathrm{BMO}^\natural) = \mathcal{L}_{1,\phi} \cap L_\infty$$

where $\phi(r) = 1/(1 - \log r)$, $r \in (0, 1]$. *Moreover, for* $g \in$ PWM(BMO$^\natural$), $\|g\|_{\mathrm{Op}}$ *is equivalent to* $\|g\|_{\mathcal{L}_{1,\phi}} + \|g\|_{L_\infty}$.

To show Proposition 4.3, we give some more lemmas. The following lemma is a direct consequence of [6, Lemma 3.3].

LEMMA 4.6. *Let* $(B_n)_{n \geq 0}$ *be the same as in Proposition* 4.3. *Let* $(k_j)_{j \geq 0}$ *be a sequence of integers defined inductively by* $k_0 = 0$ *and*

$$k_j = \min\{n > k_{j-1} : B_n \neq B_{k_{j-1}}\} \quad (j \geq 1) . \tag{21}$$

Then, for each $j \geq 1$, *we have*

$$\left(1 + \frac{1}{R}\right) P(B_{k_j}) \leq P(B_{k_{j-1}}) \leq R P(B_{k_j}),$$

where R *is the constant in* (1).

REMARK 4.7. In (21), the set $\{n > k_{j-1} : B_n \neq B_{k_{j-1}}\}$ is not empty by the condition $\lim_{n\to\infty} P(B_n) = 0$ in (17).

To describe the next lemma, we recall the doubling condition for functions. A function $\theta : (0, 1] \to (0, \infty)$ is said to satisfy the doubling condition if there exists a constant $C > 0$ such that

$$\frac{1}{C} \le \frac{\theta(r)}{\theta(s)} \le C \quad \text{for} \quad r, s \in (0, 1]\,, \ \ \frac{1}{2} \le \frac{r}{s} \le 2\,.$$

The next lemma is essentially the same as [7, Lemma 2.4]. But for the later use, we give a proof.

LEMMA 4.8. *Let $p \in [1, \infty)$ and $\phi : (0, 1] \to (0, \infty)$. Let $(B_n)_{n\ge 0}$ and $(k_j)_{j\ge 0}$ be the same as in Lemma 4.6. Define a function h by*

$$h = \sum_{j=0}^{\infty} \phi(P(B_{k_j}))\chi_{B_{k_j}}\,, \tag{22}$$

where $\chi_{B_{k_j}}$ stands for the characteristic function of B_{k_j}. Assume that ϕ satisfies the doubling condition and

$$\int_0^r \phi(t)^p\,dt \le Cr\phi(r)^p \quad \textit{for all } r \in (0, 1]\,. \tag{23}$$

Then, h belongs to $\mathcal{L}_{p,\phi}$.

PROOF. By [5, Lemma 7.1], the doubling condition and (23) implies

$$\int_0^r \phi(t)t^{1/p-1}\,dt \le C_p\phi(r)r^{1/p} \quad \text{for all } \ r \in (0, 1]\,. \tag{24}$$

From (24), we can deduce

$$\sum_{j:k_j>n} \phi(P(B_{k_j}))\|\chi_{B_{k_j}}\|_p \le CC_p\phi(P(B_n))P(B_n)^{1/p}\,. \tag{25}$$

Indeed, we have

$$\begin{aligned}
&\sum_{j:k_j>n} \phi(P(B_{k_j}))\|\chi_{B_{k_j}}\|_p \\
&\qquad = \sum_{j:k_j>n} \frac{1}{\log(P(B_{k_{j-1}})/P(B_{k_j}))} \int_{P(B_{k_j})}^{P(B_{k_{j-1}})} \phi(P(B_{k_j}))P(B_{k_j})^{1/p}\,\frac{dt}{t} \\
&\qquad \le C \sum_{j:k_j>n} \int_{P(B_{k_j})}^{P(B_{k_{j-1}})} \phi(t)t^{1/p-1}\,dt \\
&\qquad = C \int_0^{P(B_n)} \phi(t)t^{1/p-1}\,dt \\
&\qquad \le CC_p\phi(P(B_n))P(B_n)^{1/p}\,.
\end{aligned}$$

From (25), it follows that h is in L_p. Using the equality

$$E_n h = \sum_{j:k_j\le n} \phi(P(B_{k_j}))\chi_{B_{k_j}} + E_n\Bigg[\sum_{j:k_j>n} \phi(P(B_{k_j}))\chi_{B_{k_j}}\Bigg],$$

we have

$$\begin{aligned} (26) \qquad \|h - E_n h\|_p &\le \Bigg\|\sum_{j:k_j>n} \phi(P(B_{k_j}))\chi_{B_{k_j}}\Bigg\|_p + \Bigg\|E_n\Bigg[\sum_{j:k_j>n} \phi(P(B_{k_j}))\chi_{B_{k_j}}\Bigg]\Bigg\|_p \\ &\le 2CC_p\phi(P(B_n))P(B_n)^{1/p}. \end{aligned}$$

If $B \in A(\mathcal{F}_n)$ and $B \neq B_n$, then

$$(27) \qquad \int_B |h - E_n h|^p \, dP = 0 .$$

From (26) and (27), we conclude that h is in $\mathcal{L}_{p,\phi}$. □

We need the following lemma on a stability of martingale Campanato space $\mathcal{L}_{p,\phi}$. For the proof, see [7, Remark 2.7].

LEMMA 4.9. *Let $p \in [1,\infty)$ and $\phi : (0,1] \to (0,\infty)$. Let F be a Lipschitz continuous function. Then, for any $h \in \mathcal{L}_{p,\phi}$, $F(h)$ belongs to $\mathcal{L}_{p,\phi}$ with $\|F(h)\|_{\mathcal{L}_{p,\phi}} \le 2C\|h\|_{\mathcal{L}_{p,\phi}}$ where C is the Lipschitz constant of F.*

PROOF OF PROPOSITION 4.3. Let $\phi(r) = 1/(1-\log r)$ $(0 < r \le 1)$. Let $(B_n)_{n\ge 0}$ be a sequence of measurable sets that satisfies (17). Let $(k_j)_{j\ge 0}$ be the same as in Lemma 4.6. Then, we have

$$R^{-j} \le P(B_{k_j}) \le (1 + 1/R)^{-j}.$$

Therefore, we have

$$\frac{1}{1 + j\log R} \le \phi(P(B_{k_j})) \le \frac{1}{1 + j\log(1 + 1/R)} .$$

Choose j_0 such that $j_0\log(1 + 1/R) \ge 3$. Then $\phi(P(B_{k_j})) \le 1/4$ for $j \ge j_0$. Let

$$b_0 = 0\,, \quad b_\ell = \sum_{j=j_0+1}^{j_0+\ell} \phi(P(B_{k_j})) \quad (\ell \ge 1)\,.$$

Then $b_\ell \to \infty$ as $\ell \to \infty$. For the sequence $\{b_\ell\}$, take a subsequence $\{b_{\ell(m)}\}$ such that

$$b_{\ell(1)} = b_0 = 0, \quad b_{\ell(m)} + 1 \le b_{\ell(m+1)} < b_{\ell(m)} + 2, \quad m = 1, 2, \dots .$$

Take also another subsequence $\{b_{\ell(m)'}\}$ such that

$$b_{\ell(m)} + \frac{1}{4} \le b_{\ell(m)'} \le b_{\ell(m+1)} - \frac{1}{4}, \quad m = 1, 2, \dots .$$

We define a bounded Lipschitz continuous function $F : [0, \infty) \to [0, \infty)$ as

$$F(x) = \begin{cases} x - b_{\ell(m)} & (b_{\ell(m)} \le x < (b_{\ell(m)} + b_{\ell(m+1)})/2)\,, \\ b_{\ell(m+1)} - x & ((b_{\ell(m)} + b_{\ell(m+1)})/2 \le x \le b_{\ell(m+1)})\,. \end{cases}$$

Then

$$F(b_{\ell(m)}) = 0 \quad \text{and} \quad F(b_{\ell(m)'}) \ge \frac{1}{4}\,, \quad m = 1, 2, \ldots\,. \tag{28}$$

Let

$$g = \sum_{j=j_0+1}^{\infty} \chi_{B_{k_j}}\,, \quad h = \sum_{j=j_0+1}^{\infty} \phi(P(B_{k_j}))\chi_{B_{k_j}}\,.$$

Then $g \in \mathrm{BMO}^\natural$ and $F(h) \in \mathcal{L}_{1,\phi} \cap L_\infty$ by Lemmas 4.8 and 4.9. Hence $gF(h)$ is in BMO by Lemma 4.5 and $gF(h) \ge 0$.

Next we show that $gF(h)$ is not in BLO. Since

$$g = \sum_{j=j_0+1}^{j_0+\ell} = \ell\,, \quad h = \sum_{j=j_0+1}^{j_0+\ell} \phi(P(B_{k_j})) = b_\ell \quad \text{on} \quad L_\ell \equiv B_{k_{j_0+\ell}} \setminus B_{k_{j_0+\ell+1}}\,,$$

we have $gF(h) = \ell F(b_\ell)$ on L_ℓ. Hence, by (28) we have

$$gF(h) = 0 \text{ on } L_{\ell(m)} \quad \text{and} \quad gF(h) \ge \frac{\ell(m)'}{4} \text{ on } L_{\ell(m)'}, \quad m = 1, 2, \ldots\,.$$

Therefore, for $j = j_0 + \ell(m)'$,

$$\begin{aligned} E_{k_j}[gF(h)] &\ge \frac{\ell(m)'}{4} E_{k_j}[\chi_{L_{\ell(m)'}}]\chi_{B_{k_j}} \\ &= \frac{\ell(m)'}{4} \frac{P(B_{k_j}) - P(B_{k_{j+1}})}{P(B_{k_j})} \chi_{B_{k_j}} \\ &\ge \frac{\ell(m)'}{4(R+1)} \chi_{B_{k_j}}\,, \end{aligned}$$

and

$$\operatorname{ess\,sup}(E_{k_j}[gF(h)] - gF(h)) \ge \frac{\ell(m)'}{4(R+1)}\,.$$

This shows $gF(h) \notin \mathrm{BLO}$. □

REFERENCES

[1] C. BENNETT, Another characterization of BLO, Proc. Amer. Math. Soc. 85 (1982), 552–556.

[2] R. R. COIFMAN AND R. ROCHBERG, Another characterization of BMO, Proc. Amer. Math. Soc. 79 (1980), 249–254.

[3] H. LIN, E. NAKAI AND D. YANG, Boundedness of Lusin-area and g_λ^* functions on localized BMO spaces over doubling metric measure spaces, Bull. Sci. Math. 135 (2011), 59–88.

[4] R. L. LONG, Martingale spaces and inequalities, Peking University Press, Beijing, 1993.

[5] E. NAKAI, A generalization of Hardy spaces H^p by using atoms, Acta Math. Sin. (Engl. Ser.) 24 (2008), 1243–1268.

[6] E. NAKAI AND G. SADASUE, Martingale Morrey-Campanato spaces and fractional integrals, J. Funct. Spaces Appl. 2012 (2012), Article ID 673929, 29 pages. DOI:10.1155/2012/673929

[7] E. NAKAI AND G. SADASUE, Pointwise multipliers on martingale Campanato spaces, Studia Math. 220 (2014), 87–100.

[8] A. OSĘKOWSKI, Sharp estimates for functions of bounded lower oscillation, Bull. Aust. Math. Soc. 87 (2013), 68–81.

[9] Y. SHIOTA, On a decomposition of BMO-martingales, Tohoku Math. J. (2) 33 (1981), 515–520.

[10] Y. SHIOTA, Certain decompositions of BMO-martingales, Tohoku Math. J. (2) 33 (1981), 561–565.

[11] N. TH. VAROPOULOS, A probabilistic proof of the Garnett-Jones theorem on BMO, Pacific J. Math. 90 (1980), 201–221.

[12] N. TH. VAROPOULOS, The Helson-Szegö theorem and A_p-functions for Brownian motion and several variables, J. Funct. Anal. 39 (1980), 85–121.

[13] F. WEISZ, Martingale Hardy spaces and their applications in Fourier analysis, Lecture Notes in Mathematics, 1568, Springer-Verlag, Berlin, 1994.

DEPARTMENT OF MATHEMATICS
IBARAKI UNIVERSITY
MITO, IBARAKI 310–8512
JAPAN

E-mail address: eiichi.nakai.math@vc.ibaraki.ac.jp

DEPARTMENT OF MATHEMATICS
OSAKA KYOIKU UNIVERSITY
KASHIWARA, OSAKA 582–8582
JAPAN

E-mail address: sadasue@cc.osaka-kyoiku.ac.jp