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DIFFERENTIABLE PINCHING THEOREMS FOR SUBMANIFOLDS VIA RICCI FLOW

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Abstract. Two differentiable pinching theorems are verified via the Ricci flow and stable currents. We first prove a differentiable sphere theorem for positively pinched submanifolds in a space form. Moreover, we obtain a differentiable sphere theorem for submanifolds in the sphere $\mathbb{S}^{n+p}$ under extrinsic restriction.

1. Introduction. The sphere theorem is an important topic in the study of curvature and topology of Riemannian manifolds. Recently, S. Brendle and R. Schoen [6] proved a remarkable differentiable sphere theorem for Riemannian manifolds whose sectional curvatures lie in the interval (1/4, 1] by developing the theory and techniques of Ricci flow introduced by R. Hamilton [9]. This improves the topological sphere theorem for quarter-pinched Riemannian manifolds, which was firstly taken up by H. E. Rauch [16] in 1951, and solved by M. Berger [2] and W. Klingenberg [10] around 1960. In 1977, K. Grove and K. Shiohama [8] proved a topological sphere theorem for complete and connected Riemannian manifolds whose sectional curvature K and diameter $d(M)$ satisfy $K \geq 1$ and $d(M) > \pi/2$. Some other sphere theorems for Riemannian manifolds have also been obtained, see [1, 14, 15, 17, 25], etc.

In 1973, H. B. Lawson and J. Simons [11] proved a topological sphere theorem for closed submanifolds in a unit sphere by using the nonexistence for stable currents on compact submanifolds in a sphere. Let $\mathbb{F}^{n+p}(c)$ denote the complete simply connected space form with constant sectional curvature c. K. Shiohama and H. W. Xu [18] improved Lawson-Simons' result and proved a topological sphere theorem for complete submanifolds in $\mathbb{F}^{n+p}(c)$ with $c \geq 0$. Recently, H. W. Xu and J. R. Gu [20] obtained an optimal differentiable sphere theorem for complete submanifolds with pinched scalar curvature in $\mathbb{F}^{n+p}(c)$ with $c \geq 0$. Let K_M denote the sectional curvature of a Riemannian manifold M. Under the pinching condition of sectional curvature, H. W. Xu and G. X. Yang [23] proved a topological sphere theorem for compact submanifolds in the spherical space form.

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THEOREM 1.1. *Let M be a compact, oriented $n(\geq 4)$-dimensional submanifold in the space form $\mathbb{F}^{n+p}(c)$ with $c > 0$. If the sectional curvature of M satisfies*

$$K_M > \frac{c}{2} + \frac{n}{8}H^2 ,$$

then M is homeomorphic to the standard n-sphere $\mathbb{S}^n$.

Let h be the second fundamental form of a compact submanifold in a Riemannian manifold. In 1986, H. Gauchman [7] proved that if M is an n-dimensional closed minimal submanifold in $\mathbb{S}^{n+d}$, and if $||h(u,u)||^2 < \frac{1}{3}$ for any unit vector $u \in UM$, where UM is the unit tangent bundle over M, then M is a totally geodesic sphere. H. W. Xu, W. Fang and F. Xiang [21] generalized this rigidity result to the case where M is an n-dimensional closed submanifold with parallel mean curvature in $\mathbb{S}^{n+d}$. On the other hand, P. F. Leung [12] proved that if $\|h(u,u)\|^2 < \frac{1}{3}$ holds for any unit tangent vector u at any point on the submanifold, then it is a homotopy sphere. The result was improved to be a topological sphere theorem for complete submanifolds in a sphere by H. W. Xu and W. Fang [19], and further improved to be a differentiable sphere theorem by H. W. Xu and E. T. Zhao [24].

THEOREM 1.2. *Let M be an n-dimensional complete submanifold in the unit sphere $\mathbb{S}^{n+p}$. If*

$$\|h(u,u)\|^2 < \frac{1}{3}, \quad \textit{for all } u \in UM ,$$

then M is diffeomorphic to the standard n-sphere $\mathbb{S}^n$.

Let $\mathring{h}$ be the trace free second fundamental form of a submanifold in a Riemannian manifold. In [22], H. W. Xu, F. Huang and F. Xiang investigated the pinching theorem of $\mathring{h}$ for the submanifolds with parallel mean curvature in a sphere, which extends Gauchman's theorem in [7].

In this paper, using the convergence result of Ricci flow by S. Brendle [5], we obtain several differentiable sphere theorems for complete submanifolds which generalize Theorems 1.1 and 1.2. We first prove the following

THEOREM 1.3. *Let M be an $n(\geq 4)$-dimensional oriented complete submanifold in $\mathbb{F}^{n+p}(c)$ with $c > 0$. Assume that the sectional curvature of M satisfies*

$$K_M > \frac{(n-2)c}{n} + \frac{nH^2}{8} , \tag{1}$$

then M is diffeomorphic to the standard n-sphere $\mathbb{S}^n$.

Next, we investigate the differentiable pinching problem of $\mathring{h}$ and prove the following

THEOREM 1.4. *Let M be an n-dimensional complete submanifold in the unit sphere $\mathbb{S}^{n+p}$. If*

$$\|\mathring{h}(u,u)\|^2 < \frac{1}{3}, \quad \textit{for all } u \in UM ,$$

then M is diffeomorphic to the standard n-sphere $\mathbb{S}^n$.

In fact, Theorem 1.4 is a consequence of the following theorem and Lemma 3.2 in Section 3.

THEOREM 1.5. *Let M be an n-dimensional complete submanifold in an $(n+p)$-dimensional point-wise $\delta(>1/4)$-pinched Riemannian manifold N^{n+p}. Set $\overline{K}_{\max}(x) := \max_{\pi\subset T_xN}\overline{K}(x,\pi)$, where $\overline{K}(x,\pi)$ is the sectional curvature of N^{n+p} for $x\in N$ and 2-plane $\pi\in T_xN$. If*

$$||\mathring{h}(u,u)||^2(x) < \frac{4}{9}\overline{K}_{\max}(x)\left(\delta-\frac{1}{4}\right), \quad \text{for all } u\in U_xM\,, \quad x\in M\,,$$

where $\inf_{x\in N}\overline{K}_{\max}(x) > 0$, then M is diffeomorphic to a spherical space form. In particular, if M is simply connected, then M is diffeomorphic to the standard n-sphere $\mathbb{S}^n$.

2. Preliminaries. Let M^n be an n-dimensional complete submanifold isometrically immersed into an $(n+p)$-dimensional Riemannian manifold N^{n+p}. The following conventions of indices are used throughout.

$$1\le A,B,C,\ldots\le n+p\,,$$
$$1\le i,j,k,\ldots\le n\,,$$
$$n+1\le \alpha,\beta,\gamma,\ldots\le n+p\,.$$

Choose a local orthonormal frame field $\{e_A\}$ in N^{n+p} such that e_i's are tangent to M. Let $\{\omega_A\}$ be the dual frame field of $\{e_A\}$ and $\{\omega_{AB}\}$ the connection 1-forms of N^{n+p}. Restricting these forms to M, we have

$$\omega_{\alpha i} = \sum_j h^\alpha_{ij}\omega_j\,, h^\alpha_{ij} = h^\alpha_{ji}\,.$$

The curvature tensor of N and M are denoted by $\overline{R}_{ABCD}$ and R_{ijkl}, respectively. The second fundamental form of M is denoted by h and the mean curvature normal field by ξ. Set $H=\|\xi\|$ and $c_\alpha = \frac{1}{n}\sum_i h^\alpha_{ii}$. Then we have

$$h = \sum_{\alpha,i,j} h^\alpha_{ij}\omega_i\otimes\omega_j\otimes e_\alpha\,,$$

$$\xi = \sum_\alpha c_\alpha e_\alpha\,,$$

$$H = \sqrt{\sum_\alpha c_\alpha^2}\,,$$

$$R_{ijkl} = \overline{R}_{ijkl} + \sum_\alpha (h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk})\,. \tag{2}$$

The trace free second fundamental form of M is defined by

$$\mathring{h} = \sum_{i,j,\alpha} \mathring{h}^\alpha_{ij}\omega_i\otimes\omega_j\otimes e_\alpha\,,$$

where $\mathring{h}_{ij}^{\alpha} = h_{ij}^{\alpha} - c_\alpha \delta_{ij}$. Let UM denote the unit tangent bundle on M and U_xM its fiber over $x \in M$. Then $UM = \bigcup_{x\in M} U_xM$, where $U_xM = \{u \in T_xM : \|u\| = 1\}$.

LEMMA 2.1. *Let M^n be a submanifold in a Riemannian manifold N^{n+p}. Then*

$$\|\mathring{h}(u, v)\| \leq \mathring{h}_x\,,$$

for all unit vectors $u, v \in U_xM$ at each point $x \in M$, where $\mathring{h}_x := \max_{w\in U_xM} \|\mathring{h}(w, w)\|$.

PROOF. If $u \neq \pm v$, set $y = \frac{u+v}{\|u+v\|}$ and $z = \frac{u-v}{\|u-v\|}$. Then

$$\begin{aligned}\mathring{h}(u, v) &= \frac{1}{4}[\mathring{h}(u+v, u+v) - \mathring{h}(u-v, u-v)]\\ &= \frac{1}{4}[\|u+v\|^2\mathring{h}(y, y) - \|u-v\|^2\mathring{h}(z, z)]\,.\end{aligned}$$

Using the triangle inequality, we get

$$\begin{aligned}\|\mathring{h}(u, v)\| &\leq \frac{1}{4}\mathring{h}_x(\|u+v\|^2 + \|u-v\|^2)\\ &\leq \mathring{h}_x\,. \qquad (3)\end{aligned}$$

If $u = \pm v$, then $\|\mathring{h}(u, v)\| = \|\mathring{h}(u, u)\| \leq \mathring{h}_x$. This completes the proof. □

LEMMA 2.2. *Let M^n be a submanifold in a Riemannian manifold N^{n+p}. Then*

$$\langle h(e_i, e_i), h(e_j, e_j)\rangle \geq -\mathring{h}_x^2\,,$$

for any orthonormal frame field $\{e_i\}$ at each point $x \in M$.

PROOF.

$$\begin{aligned}&\|h(e_i, e_i)\|^2 + \|h(e_j, e_j)\|^2 - 2\langle h(e_i, e_i), h(e_j, e_j)\rangle\\ &\quad= \|\mathring{h}(e_i, e_i)\|^2 + \|\mathring{h}(e_j, e_j)\|^2 + 2\langle\mathring{h}(e_i, e_i), \xi\rangle + 2\langle\mathring{h}(e_j, e_j), \xi\rangle + 2H^2\\ &\qquad - 2(\langle\mathring{h}(e_i, e_i), \mathring{h}(e_j, e_j)\rangle + \langle\mathring{h}(e_i, e_i), \xi\rangle + \langle\mathring{h}(e_j, e_j), \xi\rangle + H^2)\\ &\quad= \|\mathring{h}(e_i, e_i)\|^2 + \|\mathring{h}(e_j, e_j)\|^2 - 2\langle\mathring{h}(e_i, e_i), \mathring{h}(e_j, e_j)\rangle\\ &\quad\leq 4\mathring{h}_x^2\,,\end{aligned}$$

where Lemma 2.1 is used in the last inequality. Hence we have

$$\langle h(e_i, e_i), h(e_j, e_j)\rangle \geq \frac{1}{2}(\|h(e_i, e_i)\|^2 + \|h(e_j, e_j)\|^2) - 2\mathring{h}_x^2\,. \qquad (4)$$

Thus,

$$\begin{aligned}\langle h(e_i, e_i), h(e_j, e_j)\rangle &= \frac{1}{2}\langle h(e_i, e_i), h(e_j, e_j)\rangle + \frac{1}{2}\langle h(e_i, e_i), h(e_j, e_j)\rangle\\ &\geq \frac{1}{2}\left[\frac{1}{2}(\|h(e_i, e_i)\|^2 + \|h(e_j, e_j)\|^2) - 2\mathring{h}_x^2\right]\\ &\quad -\frac{1}{2}\|h(e_i, e_i)\|\cdot\|h(e_j, e_j)\|\\ &= \frac{1}{4}\Big[\|h(e_i, e_i)\|^2 + \|h(e_j, e_j)\|^2\end{aligned}$$

$$
\begin{aligned}
&-2\|h(e_i, e_i)\| \cdot \|h(e_j, e_j)\|\big] - \mathring{h}_x^2 \\
&\geq -\mathring{h}_x^2 .
\end{aligned}
$$

□

S. Brendle [5] obtained the following useful result.

LEMMA 2.3. *Let (M, g_0) be a compact Riemannian manifold of dimension $n \geq 4$. Assume that*

$$
R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} > 0\,, \tag{5}
$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$ and all $\lambda \in [-1, 1]$. Then the normalized Ricci flow with initial metric g_0

$$
\frac{\partial}{\partial t} g(t) = -2\operatorname{Ric}_{g(t)} + \frac{2}{n} r_{g(t)} g(t)\,,
$$

exists for all time and converges to a positive constant sectional curvature metric as $t \to \infty$. Here $r_{g(t)}$ denotes the mean value of the scalar curvature of $g(t)$.

Inequality (5) is closely related to the positivity of the isotropic curvature. We refer the readers to [13] for isotropic curvature. As a matter of fact, $R_{1313}+\lambda^2 R_{1414}+R_{2323}+\lambda^2 R_{2424}-2\lambda R_{1234} > 0$ holds for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$ and all $\lambda \in [-1, 1]$ if and only if $M \times \mathbb{R}$ has positive isotropic curvature (see [5, 6]). It follows from Berger's inequality that every manifold with positively pointwise 1/4-pinched sectional curvatures satisfies the curvature condition (5) in Lemma 2.3. In fact, we have the following Berger's inequality [3]

$$
|R_{ijkl}| \leq \frac{2}{3}\big(K_{\max} - K_{\min}\big)\,,
$$

for all distinct indices i, j, k, l. Here $K_{\max}$ and $K_{\min}$ are the maximum and minimum of the sectional curvatures at a point of M. The curvature assumption implies that $K_{\max} < 4K_{\min}$. So we have

$$
\begin{aligned}
&R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} \\
&\geq 2(1+\lambda^2)K_{\min} - \frac{4}{3}|\lambda|\big(K_{\max} - K_{\min}\big) \\
&\geq 2(1+\lambda^2)K_{\min} - 4|\lambda| K_{\min} \\
&= 2(1-|\lambda|)^2 K_{\min} \\
&\geq 0\,.
\end{aligned}
$$

Observe that at least one of the second and last inequalities is strict, which implies $R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} > 0$. In fact, the 1/4-pinched differentiable sphere theorem is an immediate consequence of Lemma 2.3.

We also need the following topological lemma due to H. B. Lawson and J. Simons [11].

LEMMA 2.4. *Let M^n be a compact submanifold in a unit sphere $\mathbb{S}^{n+p}$. Let k and l be positive integers with $k+l=n$. If the following inequality*

$$\sum_{j=k+1}^{n}\sum_{i=1}^{k}(2\|h(e_i,e_j)\|^2-\langle h(e_i,e_i),h(e_j,e_j)\rangle)<k\cdot l\,,$$

holds for any orthonormal basis $\{e_1,e_2,\dots,e_n\}$ of tangent space T_xM at any point $x\in M$, then there does not exist any stable k-current, and

$$H_k(M,\mathbb{Z})=H_l(M,\mathbb{Z})=0\,,$$

where $H_i(M,\mathbb{Z})$ is the i-th homology group of M with integer coefficients.

3. Proof of theorems. We first give the proof of Theorem 1.3.

PROOF OF THEOREM 1.3. Since $K_M>\frac{(n-2)c}{n}+\frac{nH^2}{8}>0$, we obtain that M is a compact submanifold by Myers' Theorem.

Suppose that $\{e_1,e_2,e_3,e_4\}$ is an orthonormal four-frame and $\lambda\in\mathbb{R}$. We extend $\{e_1,e_2,e_3,e_4\}$ to an orthonormal frame $\{e_1,\dots,e_n\}$. From the Gauss equation (2) we get

$$R_{ijij}=c+\sum_\alpha h_{ii}^\alpha h_{jj}^\alpha-\sum_\alpha(h_{ij}^\alpha)^2\,.$$

From the assumption we obtain

$$\sum_\alpha h_{ii}^\alpha h_{jj}^\alpha-\sum_\alpha(h_{ij}^\alpha)^2>\frac{nH^2}{8}-\frac{2c}{n}\,.$$

This together with (1) implies

$$\begin{aligned}
&R_{1313}+\lambda^2R_{1414}+R_{2323}+\lambda^2R_{2424}-2\lambda R_{1234}\\
&\quad>(1+\lambda^2)\Big(\frac{2(n-2)}{n}c+\frac{nH^2}{4}\Big)-2\lambda\sum_\alpha(h_{13}^\alpha h_{24}^\alpha-h_{14}^\alpha h_{23}^\alpha)\\
&\quad\ge(1+\lambda^2)\Big(\frac{2(n-2)}{n}c+\frac{nH^2}{4}\Big)-|\lambda|\sum_\alpha\sum_{i=3}^{n}[(h_{1i}^\alpha)^2+(h_{2i}^\alpha)^2]\\
&\quad\ge(1+\lambda^2)\Big(\frac{2(n-2)}{n}c+\frac{nH^2}{4}\Big)-|\lambda|\Big[\sum_\alpha(h_{11}^\alpha+h_{22}^\alpha)\Big(\sum_{i=3}^{n}h_{ii}^\alpha\Big)\\
&\qquad-\Big(\frac{n(n-2)H^2}{4}-\frac{4(n-2)c}{n}\Big)\Big]\\
&\quad\ge(1+\lambda^2)\Big(\frac{2(n-2)}{n}c+\frac{nH^2}{4}\Big)-|\lambda|\frac{n^2}{4}H^2+|\lambda|\Big(\frac{n(n-2)H^2}{4}-\frac{4(n-2)c}{n}\Big)\\
&\quad=\frac{2(n-2)}{n}(1-|\lambda|)^2c+\frac{n}{4}(1-|\lambda|)^2H^2\\
&\quad\ge 0\,.
\end{aligned}$$

Hence, the inequality (5) holds for all $\lambda \in [-1, 1]$, which implies that M is diffeomorphic to a spherical space form by Lemma 2.3. On the other hand, since $K_M > \frac{(n-2)c}{n} + \frac{nH^2}{8} \geq \frac{c}{2} + \frac{nH^2}{8}$, from Theorem 1.1 we know that M is homeomorphic to $\mathbb{S}^n$. Combining the results above, we get that M is diffeomorphic to $\mathbb{S}^n$. □

In the following we consider the submanifolds with restriction on $\mathring{h}$.

LEMMA 3.1. *Let M be an $n(\geq 4)$-dimensional submanifold in an $(n+p)$-dimensional point-wise $\delta(> 1/4)$-pinched Riemannian manifold N^{n+p}. If*

$$\|\mathring{h}(u,u)\|^2(x) < \frac{4}{9}\overline{K}_{\max}(x)\left(\delta - \frac{1}{4}\right), \quad \text{for all } u \in U_xM\,, \quad x \in M\,, \tag{6}$$

where $\inf_{x\in N} \overline{K}_{\max}(x) > 0$, then inequality (5) is satisfied for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$ and $\lambda \in [-1, 1]$.

PROOF. For any point $x \in M$, let $\mathring{h}_x := \max_{w\in U_xM} \|\mathring{h}(w,w)\|$. We suppose $\{e_1, e_2, e_3, e_4\}$ to be an orthonormal four-frame and $\lambda \in [-1, 1]$. By the Gauss equation we have

$$\begin{aligned}
&R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234}\\
&\quad = \overline{R}_{1313} + \langle h(e_1,e_1), h(e_3,e_3)\rangle - \|h(e_1,e_3)\|^2\\
&\qquad + \lambda^2[\overline{R}_{1414} + \langle h(e_1,e_1), h(e_4,e_4)\rangle - \|h(e_1,e_4)\|^2]\\
&\qquad + [\overline{R}_{2323} + \langle h(e_2,e_2), h(e_3,e_3)\rangle - \|h(e_2,e_3)\|^2]\\
&\qquad + \lambda^2[\overline{R}_{2424} + \langle h(e_2,e_2), h(e_4,e_4)\rangle - \|h(e_1,e_4)\|^2]\\
&\qquad - 2|\lambda|[\overline{R}_{1234} + \langle h(e_1,e_3), h(e_2,e_4)\rangle - \langle h(e_1,e_4), h(e_2,e_3)\rangle].
\end{aligned}$$

Applying Lemma 2.1, Lemma 2.2 and $h(e_i,e_j) = \mathring{h}(e_i,e_j)$, $i \neq j$, we obtain

$$\begin{aligned}
&R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234}\\
&\quad \geq \overline{R}_{1313} - 2\mathring{h}_x^2 + \lambda^2[\overline{R}_{1414} - 2\mathring{h}_x^2]\\
&\qquad + \overline{R}_{2323} - 2\mathring{h}_x^2 + \lambda^2[\overline{R}_{2424} - 2\mathring{h}_x^2]\\
&\qquad - 2|\lambda|[\overline{R}_{1234} + 2\mathring{h}_x^2].
\end{aligned}$$

Since we have Berger's inequality $|\overline{R}_{1234}| \leq \frac{2}{3}(\overline{K}_{\max} - \overline{K}_{\min})$, the assumption (6) implies that

$$\begin{aligned}
&R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234}\\
&\quad > (2+2\lambda^2)\left[\overline{K}_{\min} - \frac{8}{9}\overline{K}_{\max}\left(\delta - \frac{1}{4}\right)\right] - 2|\lambda|\left[\frac{2}{3}(\overline{K}_{\max} - \overline{K}_{\min}) + \frac{8}{9}\overline{K}_{\max}\left(\delta - \frac{1}{4}\right)\right]\\
&\quad \geq (2+2\lambda^2)\left[\delta\overline{K}_{\max} - \frac{8}{9}\overline{K}_{\max}\left(\delta - \frac{1}{4}\right)\right] - 2|\lambda|\left[\frac{2}{3}(1-\delta)\overline{K}_{\max} + \frac{8}{9}\overline{K}_{\max}\left(\delta - \frac{1}{4}\right)\right]\\
&\quad = \frac{2}{9}(1+\lambda^2 - 2|\lambda|)(\delta+2)\overline{K}_{\max}\\
&\quad \geq 0\,.
\end{aligned}$$

This completes the proof of the Lemma 3.1. □

Now we give the proof of Theorem 1.5.

PROOF OF THEOREM 1.5. Since $\|\mathring{h}(u,u)\|^2(x) < \frac{4}{9}(\delta - \frac{1}{4})\overline{K}_{\max}$ for any $u \in U_xM$, from the Gauss equation, we know that the sectional curvature of M is bounded from bellow by $\frac{2+\delta}{9}\overline{K}_{\max} \geq \frac{2+\delta}{9}\inf_{x\in M}\overline{K}_{\max}(x) > 0$. By Myers' Theorem, M is a compact submanifold.

When $n = 2$, it's easy to see that M is diffeomorphic to $\mathbb{S}^2$ or $\mathbb{RP}^2$. When $n = 3$, Hamilton's Theorem [9] says that M is diffeomorphic to a spherical space form. When $n \geq 4$, by Lemma 2.3 and Lemma 3.1, we see that M is diffeomorphic to a spherical space form. In particular, if M is simply connected, then M must be diffeomorphic to the standard unit n-sphere $\mathbb{S}^n$. This completes the proof of Theorem 1.5. □

LEMMA 3.2. *Let M^n be an n-dimensional complete submanifold in the unit sphere $\mathbb{S}^{n+p}$. If*

$$\|\mathring{h}(u,u)\|^2 < \frac{1}{3}, \quad \text{for all } u \in UM,$$

then M is simply connected.

PROOF. It is easy to see that the sectional curvature of M is bounded by $\frac{1}{3}$ from below. Hence M is compact by Myers' Theorem.

Let $\{e_1, e_2, \ldots, e_n\}$ be any orthonormal basis of the tangent space at any point $x \in M$. Using Lemma 2.1 and Lemma 2.2, we have

$$\|h(e_i,e_j)\| = \|\mathring{h}(e_i,e_j)\| < \frac{1}{3},$$

$$\langle h(e_i,e_i), h(e_j,e_j)\rangle > -\frac{1}{3},$$

for any unit vector field e_i and e_j $(i \neq j)$. Hence for any $1 \leq k \leq n-1$,

$$\begin{aligned}
&\sum_{j=k+1}^{n}\sum_{i=1}^{k}(2\|h(e_i,e_j)\|^2 - \langle h(e_i,e_i), h(e_j,e_j)\rangle) \\
&\quad < \sum_{j=k+1}^{n}\sum_{i=1}\left(\frac{2}{3}+\frac{1}{3}\right) \\
&\quad = kl\,.
\end{aligned}$$

By Lemma 2.4, there does not exist any stable integral current in M.

Suppose that $\pi_1(M) \neq 0$. Since M is compact, it follows from a classical theorem due to Cartan and Hadamard (see [4]) that there exists a minimal closed geodesic in any nontrivial homotopy class in $\pi_1(M)$. Then we get a contradiction. Therefore, $\pi_1(M) = 0$ and M is simply connected. □

Theorem 1.4 is an immediate consequence of Lemma 3.2 and Theorem 1.5.

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