Tohoku Math. J.
63 (2011), 427–439

ON LOW PASS FILTERS IN A FRAME MULTIRESOLUTION ANALYSIS

ANGEL SAN ANTOLÍN

(Received July 16, 2010)

Abstract. We give necessary and sufficient conditions for a measurable function to be a low pass filter associated to a scaling function in a frame multiresolution analysis. Those conditions involve the class of real-valued bounded measurable functions such that the origin is a point of approximate continuity of such functions. The main result here is proved in a general context where the considered dilation is given by a fixed expansive linear map.

1. Introduction. We give necessary and sufficient conditions on a measurable function H to be a low pass filter associated to a scaling function in a frame multiresolution analysis.

Characterizations of low pass filters associated to scaling functions in a multiresolution analysis are known (see Papadakis, Sikić and Weiss [20], Dobrić, Gundy and Hitczenko [10] and [21]). Furthermore, necessary and sufficient conditions on the low pass filters associated to scaling functions ϕ, such that $\{\phi(\mathbf{x}-\mathbf{k}); \mathbf{k} \in \boldsymbol{Z}^n\}$ is a Riesz basis for the core subspace in a multiresolution analysis were given by Gundy [13].

Although the results presented here are new under the classical definition of FMRA, we present our conditions in a general context where, instead of the dyadic dilation, one considers the dilation given by a fixed expansive linear map $A : \boldsymbol{R}^n \to \boldsymbol{R}^n$ such that $A(\boldsymbol{Z}^n) \subset \boldsymbol{Z}^n$. Recall that A is expansive if all (complex) eigenvalues of A have modulus greater than 1.

Let us write basic notions. The theory of frames was introduced by Duffin and Schaeffer [11]. A sequence $\{\phi_n\}_{n=1}^{\infty}$ of elements in a separable Hilbert space $\boldsymbol{H}$ is a *frame* for $\boldsymbol{H}$ if there exist constants $C, D > 0$ such that

$$C\|h\|^2 \leq \sum_{n=1}^{\infty} |\langle h, \phi_n\rangle|^2 \leq D\|h\|^2 \quad \text{for all } h \in \boldsymbol{H} .$$

The constants C and D are called *frame bounds* and $\langle\cdot,\cdot\rangle$ denotes the inner product on $\boldsymbol{H}$. Recall that a frame is a complete set of elements in $\boldsymbol{H}$. A frame $\{\phi_n\}_{n=1}^{\infty}$ is *tight* if we can choose $C = D$, and if in fact $C = D = 1$, we will call the frame a *Parseval frame*. A sequence $\{h_n\}_{n=1}^{\infty}$ of elements in a Hilbert space $\boldsymbol{H}$ is a *frame sequence* if it is a frame for $\overline{\text{span}}\{h_n\}_{n=1}^{\infty}$.

2000 *Mathematics Subject Classification.* Primary 42C15; Secondary 47A75, 42B99.

Key words and phrases. Approximate continuity, Fourier transform, frame multiresolution analysis, low pass filter, scaling function.

The author was partially supported by MTM 2010-15790.

A multiresolution analysis (MRA) is a general method introduced by Mallat [17] and Meyer [18] for constructing wavelets. Afterwards, the notion of a *frame multiresolution analysis* (FMRA) was formulated by Benedetto and Li [1] as a natural extension of MRA. We will consider an FMRA in a general context: Given a linear invertible map A as above, one defines an A-FMRA as a sequence of closed subspaces V_j, $j \in \boldsymbol{Z}$, of the Hilbert space $L^2(\boldsymbol{R}^n)$ that satisfies the following conditions:

(i) $V_j \subset V_{j+1}$ for every $j \in \boldsymbol{Z}$;
(ii) $f(\mathbf{x}) \in V_j$ if and only if $f(A\mathbf{x}) \in V_{j+1}$ for every $j \in \boldsymbol{Z}$;
(iii) $\overline{\bigcup_{j\in \boldsymbol{Z}} V_j} = L^2(\boldsymbol{R}^n)$;
(iv) There exists a function $\phi \in V_0$, that is called *scaling function*, such that the system $\{\phi(\mathbf{x}-\mathbf{k})\ ;\ \mathbf{k} \in \boldsymbol{Z}^n\}$ is a frame for V_0.

If in the condition (iv), the system $\{\phi(\mathbf{x}-\mathbf{k})\ ;\ \mathbf{k} \in \boldsymbol{Z}^n\}$ is a Parseval frame for V_0 we say that an A-FMRA is an A-PFMRA.

DEFINITION 1.1. A function $\phi \in L^2(\boldsymbol{R}^n)$ *generates an A-FMRA* if $\{\phi(\mathbf{x}-\mathbf{k})\ ;\ \mathbf{k} \in \boldsymbol{Z}^n\}$ is a frame sequence and the subspaces

$$V_j = \overline{\mathrm{span}}\{\phi(A^j\mathbf{x}-\mathbf{k})\ ;\ \mathbf{k} \in \boldsymbol{Z}^n\}\,, \quad j \in \boldsymbol{Z} \tag{1}$$

of the Hilbert space $L^2(\boldsymbol{R}^n)$ satisfy the conditions (i) and (iii).

A key tool in the study of scaling functions in a frame multiresolution analysis is the Fourier transform. We adopt the convention that the Fourier transform of a function $f \in L^1(\boldsymbol{R}^n) \cap L^2(\boldsymbol{R}^n)$ is defined by

$$\widehat{f}(\mathbf{y}) = \int_{\boldsymbol{R}^n} f(\mathbf{x}) e^{-2\pi i \mathbf{x}\cdot\mathbf{y}} d\mathbf{x}\,.$$

If ϕ is a scaling function of an A-FMRA, observe that $d_A^{-1}\phi(A^{-1}\mathbf{x}) \in V_{-1} \subset V_0$, where $d_A = |\det A|$. By the condition (iv) we express this function in terms of the frame $\{\phi(\mathbf{x}-\mathbf{k})\ ;\ \mathbf{k} \in \boldsymbol{Z}^n\}$ as

$$d_A^{-1}\phi(A^{-1}\mathbf{x}) = \sum_{\mathbf{k}\in \boldsymbol{Z}^n} a_{\mathbf{k}}\phi(\mathbf{x}-\mathbf{k})\,, \quad a_{\mathbf{k}} \in \boldsymbol{C}\,,$$

where the convergence is in $L^2(\boldsymbol{R}^n)$ and $\sum_{\mathbf{k}\in \boldsymbol{Z}^n} |a_{\mathbf{k}}|^2 < \infty$. Taking the Fourier transform, we obtain

$$\widehat{\phi}(A^*\mathbf{t}) = H(\mathbf{t})\widehat{\phi}(\mathbf{t}) \quad \text{a.e. on } \boldsymbol{R}^n \tag{2}$$

where A^* is the adjoint map of A and

$$H(\mathbf{t}) = \sum_{\mathbf{k}\in \boldsymbol{Z}^n} a_{\mathbf{k}} e^{-2\pi i \mathbf{k}\cdot\mathbf{t}}$$

is a function in $L^2(\boldsymbol{T}^n)$ which is called *low pass filter* associated to the scaling function ϕ of an A-FMRA, or shortly *low pass filter* in an A-FMRA.

In Section 2 we give notation and definitions that we use throughout this manuscript. Section 3 reviews a characterization of the scaling functions in an A-FMRA. Section 4 contains some properties of the low pass filters in an A-FMRA. In Section 5, a characterization of the low pass filters associated to a scaling function in an A-PFMRA is given. Finally, Section 6 provides necessary and sufficient conditions on $\boldsymbol{Z}^n$-periodic functions to be a low pass filter in an A-FMRA.

2. Notation and definitions.

Before formulating our results let us introduce some notation and definitions.

$\boldsymbol{T}^n = \boldsymbol{R}^n/\boldsymbol{Z}^n$ and with some abuse of the notation we consider also that $\boldsymbol{T}^n$ is the unit cube $[0, 1)^n$. If we take $f \in L^2(\boldsymbol{T}^n)$ we will understand that f is defined on the whole space $\boldsymbol{R}^n$ as a $\boldsymbol{Z}^n$-periodic function.

We will denote $B_r = \{\mathbf{x} \in \boldsymbol{R}^n \ ; \ |\mathbf{x}| < r\}$. For a set $E \subset \boldsymbol{R}^n$, a point $\mathbf{x} \in \boldsymbol{R}^n$ and a linear map $M : \boldsymbol{R}^n \to \boldsymbol{R}^n$, we will write $\mathbf{x} + E = \{\mathbf{x} + \mathbf{y} \ ; \ \text{for} \ \ \mathbf{y} \in E\}$ and $M(E) = \{M(\mathbf{y}) \ ; \ \text{for } \mathbf{y} \in E\}$. The Lebesgue measure of a measurable set $E \subset \boldsymbol{R}^n$ will be denoted by $|E|_n$ and by χ_E the characteristic function of the set E, i.e., $\chi_E(\mathbf{t})$ takes the value 1 if $\mathbf{t} \in E$ and 0 otherwise.

The following definitions were introduced in [8].

DEFINITION 2.1. We will say that $\mathbf{x} \in \boldsymbol{R}^n$ is a point of A-density for a set $E \subset \boldsymbol{R}^n$, $|E|_n > 0$, if for any $r > 0$

$$\lim_{j\to\infty} \frac{|E \cap (A^{-j}B_r + \mathbf{x})|_n}{|A^{-j}B_r|_n} = 1 \, .$$

DEFINITION 2.2. Let $f : \boldsymbol{R}^n \longrightarrow \boldsymbol{C}$ be a measurable function. We say that $\mathbf{x} \in \boldsymbol{R}^n$ is a point of A-approximate continuity of the function f if there exists $E \subset \boldsymbol{R}^n$, $|E|_n > 0$, such that $\mathbf{x}$ is a point of A-density for the set E and

$$\lim_{\substack{\mathbf{y} \to \mathbf{x} \\ \mathbf{y} \in E}} f(\mathbf{y}) = f(\mathbf{x}) \, .$$

DEFINITION 2.3. A measurable function $f : \boldsymbol{R}^n \to \boldsymbol{C}$ is said to be *A-locally nonzero* at a point $\mathbf{x} \in \boldsymbol{R}^n$ if for any $\varepsilon, r > 0$ there exists $j \in \boldsymbol{N}$ such that

$$|\{\, \mathbf{y} \in A^{-j}B_r + \mathbf{x} \, ; \ f(\mathbf{y}) = 0 \,\}|_n < \varepsilon |A^{-j}B_r|_n \, .$$

Observe that if $A = aI$, where $a > 1$ and I is the identity map on $\boldsymbol{R}^n$, the definition of a point of A-approximate continuity coincides with the well-known definition of *approximate continuity* (cf. [19], [5]).

If $A : \boldsymbol{R}^n \to \boldsymbol{R}^n$ is an expansive linear invertible map such that $A(\boldsymbol{Z}^n) \subset \boldsymbol{Z}^n$, then the quotient group $\boldsymbol{Z}^n/A(\boldsymbol{Z}^n)$ is well defined. We will denote by $\Omega_A \subset \boldsymbol{Z}^n$ a full collection of representatives of the cosets of $\boldsymbol{Z}^n/A(\boldsymbol{Z}^n)$. Recall that there are exactly d_A cosets (see [12] and [22, p. 109]).

Let us fix $\Omega_{A^*} = \{\mathbf{p}_i\}_{i=0}^{d_A-1}$, where $\mathbf{p}_0 = \mathbf{0}$.

For a given $\phi \in L^2(\boldsymbol{R}^n)$, set

$$\Phi_\phi(\mathbf{t}) = \sum_{\mathbf{k}\in\boldsymbol{Z}^n} |\widehat{\phi}(\mathbf{t}+\mathbf{k})|^2 \tag{3}$$

and denote

$$\mathcal{N}_\phi = \{\mathbf{t} \in \boldsymbol{R}^n \ ; \ \Phi_\phi(\mathbf{t}) = 0\}\,. \tag{4}$$

For a measurable function $f : \boldsymbol{R}^n \to \boldsymbol{C}$ the support of f is defined to be $\mathrm{supp}(f) = \{\mathbf{t} \in \boldsymbol{R}^n ;\ f(\mathbf{t}) \neq 0\}$.

The sets are defined modulo a null measurable set and we will understand some equations as almost everywhere on $\boldsymbol{R}^n$ or $\boldsymbol{T}^n$. Moreover, in order to shorten the notation, we will consider $0/0 = 0$ or $0(1/0) = 0$ in some expressions where such an indeterminacy appears.

For $H \in L^\infty(\boldsymbol{T}^n)$ and for any measurable set $E \subset \boldsymbol{R}^n$ with $E = E + \boldsymbol{Z}^n$, the operator $P_{H,E} : L^1(\boldsymbol{T}^n) \to L^1(\boldsymbol{T}^n)$ defined by

$$P_{H,E}(f)(\mathbf{t}) = \sum_{i=0}^{d_A-1} |H((A^*)^{-1}(\mathbf{t}+\mathbf{p}_i))|^2 f((A^*)^{-1}(\mathbf{t}+\mathbf{p}_i))\chi_{\boldsymbol{R}^n\setminus E}((A^*)^{-1}(\mathbf{t}+\mathbf{p}_i))$$

is linear and continuous.

3. Scaling functions. The definition of low pass filter is closely related with the notion of scaling function, thus for studying the functions $H \in L^2(\boldsymbol{T}^n)$ which are low pass filters we should know if a function $\phi \in L^2(\boldsymbol{R}^n)$ satisfying the refinement equation (2) is a scaling function in an A-FMRA. In this section we review results (Theorem A, Lemma B and Theorem C below) that provide a characterization of the scaling functions in an A-FMRA together.

In Benedetto and Walnut [2], Benedetto and Li [1], Kim and Lim [16] and Casazza, Christensen and Kalton [6] (see also [7]), different versions of a characterization of the functions in $L^2(\boldsymbol{R})$ whose integer translates generate a frame sequence were given. Here we only write a version on $L^2(\boldsymbol{R}^n)$ because the proof is completely similar to the case $L^2(\boldsymbol{R})$.

THEOREM A. *Let $\phi \in L^2(\boldsymbol{R}^n)$. The system $\{\phi(\mathbf{x}-\mathbf{k}) \ ; \ \mathbf{k} \in \boldsymbol{Z}^n\}$ is a frame sequence with frame bounds C and D if and only if*

$$C \le \Phi_\phi(\mathbf{t}) \le D \quad a.e.\ on\ \ \boldsymbol{T}^n \setminus \mathcal{N}_\phi\,. \tag{5}$$

Different versions of the following lemma appeared in various publications (cf. [1], [7], [15]*).*

LEMMA B. *Let $\phi \in L^2(\boldsymbol{R}^n)$ and assume that $\{\phi(\mathbf{x}-\mathbf{k}) \ ; \ \mathbf{k} \in \boldsymbol{Z}^n\}$ is a frame sequence in $L^2(\boldsymbol{R}^n)$. If the subspaces V_j, $j \in \boldsymbol{Z}$, are defined by* (1) *then the following conditions are equivalent*:

a) $V_j \subset V_{j+1}$ *for all* $j \in \boldsymbol{Z}$;

b) $V_0 \subset V_1$;

c) *There exists a function $H \in L^\infty(\boldsymbol{T}^n)$ such that*

$$\widehat{\phi}(A^*\mathbf{t}) = H(\mathbf{t})\widehat{\phi}(\mathbf{t}) \quad a.e.\ on\ \boldsymbol{R}^n\,. \tag{6}$$

In a more general context, the following theorem was proved in [15]. The theorem is formulated here in a modified form.

THEOREM C. *Let V_j, $j \in \boldsymbol{Z}$, be a sequence of closed subspaces in $L^2(\boldsymbol{R}^n)$ satisfying the conditions* (i), (ii) *and* (iv) *with the scaling function ϕ. Then the following conditions are equivalent*:

(A) $\overline{\bigcup_{j\in \boldsymbol{Z}^n} V_j} = L^2(\boldsymbol{R}^n)$;

(B) *The function $\widehat{\phi}$ is A^*-locally nonzero at the origin*;

(C) *The origin is a point of A^*-approximate continuity of the function $|\widehat{\phi}|^2 \cdot (\Phi_\phi)^{-1}$, provided that $|\widehat{\phi}(\mathbf{0})|^2(\Phi_\phi(\mathbf{0}))^{-1} = 1$.*

The following lemma can be found in an implicit form in the paper by de Boor, DeVore and Ron [3], and in particular, it relates frames of translates with Parseval frames of translates.

LEMMA D. *Let $\phi \in L^2(\boldsymbol{R}^n)$ and $V = \overline{\mathrm{span}}\{\phi(\mathbf{x}-\mathbf{k})\ ;\ \mathbf{k} \in \boldsymbol{Z}^n\}$. Then the system $\{\varphi(\mathbf{x}-\mathbf{k})\ ;\ \mathbf{k} \in \boldsymbol{Z}^n\}$ is a Parseval frame for V, where $\varphi \in L^2(\boldsymbol{R}^n)$ is the function defined by $\widehat{\varphi} = \widehat{\phi} \cdot (\Phi_\phi)^{-1/2}$.*

4. Properties of the low pass filters. Let us show some properties of low pass filters associated to a scaling function in an A-FMRA.

PROPOSITION 4.1. *Let $\phi \in L^2(\boldsymbol{R}^n)$ be a scaling function of an A-FMRA where the system $\{\phi(\mathbf{x}-\mathbf{k})\ ;\ \mathbf{k} \in \boldsymbol{Z}^n\}$ is a frame sequence with frame bounds C and D. Then there exists H, a low pass filter associated to ϕ, such that*

$$C/D \le \sum_{i=0}^{d_A-1} |H((A^*)^{-1}(\mathbf{t}) + (A^*)^{-1}\mathbf{p}_i)|^2 \le D/C \quad \text{a.e. } \mathbf{t} \in \boldsymbol{R}^n \setminus \mathcal{N}_\phi\,, \tag{7}$$

and

$$\sum_{i=0}^{d_A-1} |H((A^*)^{-1}(\mathbf{t}) + (A^*)^{-1}\mathbf{p}_i)|^2 = 0 \quad \text{a.e. } \mathbf{t} \in \mathcal{N}_\phi\,. \tag{8}$$

PROOF. Let $\widetilde{H}$ be a low pass filter associated to the scaling function ϕ. If we denote by H the $\boldsymbol{Z}^n$-periodic measurable function defined by $H(\mathbf{t}) = \widetilde{H}(\mathbf{t})$ on $\boldsymbol{R}^n \setminus \mathcal{N}_\phi$ and 0 on $\mathcal{N}_\phi$, we have that $H \in L^2(\boldsymbol{T}^n)$ and $\widehat{\phi}(A^*\mathbf{t}) = H(\mathbf{t})\widehat{\phi}(\mathbf{t})$ a.e. on $\boldsymbol{R}^n$, thus H is a low pass filter associated to ϕ. To check that H satisfies (7) and (8), we do the computations

$$\begin{aligned}
\Phi_\phi(\mathbf{t}) &= \sum_{\mathbf{k}\in\boldsymbol{Z}^n} |H((A^*)^{-1}\mathbf{t} + (A^*)^{-1}\mathbf{k})\widehat{\phi}((A^*)^{-1}\mathbf{t} + (A^*)^{-1}\mathbf{k})|^2 \\
&= \sum_{i=0}^{d_A-1}\sum_{\mathbf{q}\in\boldsymbol{Z}^n} |H((A^*)^{-1}\mathbf{t} + (A^*)^{-1}\mathbf{p}_i + \mathbf{q})|^2 |\widehat{\phi}((A^*)^{-1}\mathbf{t} + (A^*)^{-1}\mathbf{p}_i + \mathbf{q})|^2 \qquad (9)\\
&= \sum_{i=0}^{d_A-1} |H((A^*)^{-1}\mathbf{t} + (A^*)^{-1}\mathbf{p}_i)|^2\, \Phi_\phi((A^*)^{-1}\mathbf{t} + (A^*)^{-1}\mathbf{p}_i)
\end{aligned}$$

a.e. on $\boldsymbol{R}^n$, where the third equality is true because H is $\boldsymbol{Z}^n$-periodic. Thus, by Theorem A and taking into account that $H(\mathbf{t}) = 0$ a.e. on $\mathcal{N}_\phi$, we obtain (7) and (8). □

PROPOSITION 4.2. *If a function $H \in L^2(\boldsymbol{T}^n)$ is a low pass filter associated to a scaling function ϕ in an A-FMRA where the system $\{\phi(\mathbf{x}-\mathbf{k})\ ;\ \mathbf{k} \in \mathbf{Z}^n\}$ is a frame sequence with frame bounds C and D, then any function $G \in L^2(\boldsymbol{T}^n)$ such that $|H(\mathbf{t})| = |G(\mathbf{t})|$ a.e. on $\boldsymbol{R}^n \setminus \mathcal{N}_\phi$ is a low pass filter associated to some scaling function φ in an A-FMRA, where the system $\{\varphi(\mathbf{x}-\mathbf{k})\ ;\ \mathbf{k} \in \mathbf{Z}^n\}$ is a frame sequence with the same frame bounds C and D.*

PROOF. In the proof of [21, Theorem 2], it is shown that for the following $\boldsymbol{Z}^n$-periodic function

$$m_G(\mathbf{t}) = \begin{cases} G(\mathbf{t})/H(\mathbf{t}) & \text{if } \mathbf{t} \in \{\mathbf{v} \in \boldsymbol{R}^{\mathbf{n}} \setminus \mathcal{N}_\phi\ ;\ \mathbf{H}(\mathbf{v}) \neq 0\} \\ 1 & \text{otherwise}, \end{cases}$$

there exists a measurable function μ on $\boldsymbol{R}^n$ such that

$$|\mu(\mathbf{t})| = 1 \quad \text{and} \quad m_G(\mathbf{t}) = \mu(A^*\mathbf{t})\overline{\mu(\mathbf{t})}, \tag{10}$$

where the two above equalities should be understood almost everywhere on $\boldsymbol{R}^n$.

We are going to check that G is a low pass filter associated to a scaling function φ of some A-FMRA defined by $\widehat{\varphi} = \mu\widehat{\phi}$. For this purpose, we show that the conditions in Theorem A, Lemma B and Theorem C are satisfied by φ. It is clear that if one replaces ϕ by φ, then the condition (5) in Theorem A holds with the same frame bounds C and D, and also the condition (B) in Theorem C is satisfied. Let us prove that the condition c) in Lemma B holds. We have

$$\begin{aligned} \widehat{\varphi}(A^*\mathbf{t}) &= \mu(A^*\mathbf{t})\widehat{\phi}(A^*\mathbf{t}) = \mu(A^*\mathbf{t})H(\mathbf{t})\widehat{\phi}(\mathbf{t}) \\ &= \mu(A^*\mathbf{t})\overline{\mu(\mathbf{t})}H(\mathbf{t})\mu(\mathbf{t})\widehat{\phi}(\mathbf{t}) \\ &= m_G(\mathbf{t})H(\mathbf{t})\widehat{\varphi}(\mathbf{t}) = G(\mathbf{t})\widehat{\varphi}(\mathbf{t}) \quad \text{a.e. on } \boldsymbol{R}^n. \end{aligned} \tag{11}$$

Let $\widetilde{G}(\mathbf{t}) = 0$ on $\mathcal{N}_\varphi$ and $\widetilde{G}(\mathbf{t}) = G(\mathbf{t})$ on $\boldsymbol{R}^n \setminus \mathcal{N}_\varphi$. Then we obtain

$$\widehat{\varphi}(A^*\mathbf{t}) = \widetilde{G}(\mathbf{t})\widehat{\varphi}(\mathbf{t}) \quad \text{a.e. on } \boldsymbol{R}^n.$$

In a similar way as in the proof of Proposition 4.1 we obtain that $\widetilde{G} \in L^\infty(\boldsymbol{T}^n)$. Hence, the condition c) of Lemma B is satisfied. □

PROPOSITION 4.3. *Let $\theta \in L^2(\boldsymbol{R}^n)$ and $H \in L^\infty(\boldsymbol{T}^n)$ satisfy $\widehat{\theta}(A^*\mathbf{t}) = H(\mathbf{t})\widehat{\theta}(\mathbf{t})$ a.e. Then Φ_θ is a fixed point for the operator $P_{H,\mathcal{N}_\theta}$.*

PROOF. By the Monotone Convergence Theorem, one observes that the $\boldsymbol{Z}^n$-periodic function Φ_θ belongs to $L^1(\boldsymbol{T}^n)$. In addition, that Φ_θ is a fixed point for the operator $P_{H,\mathcal{N}_\theta}$ follows if one replaces ϕ by θ in (9). □

An interesting case for us is the study of low pass filters associated to a scaling functions in an A-PFMRA. Thus, in the following we write some properties of those low pass filters.

An immediate corollary of Proposition 4.1 is that if $\phi \in L^2(\boldsymbol{R}^n)$ a scaling function of an A-PFMRA, then there exists H, a low pass filter associated to ϕ, such that

$$\sum_{i=0}^{d_A-1} |H((A^*)^{-1}(\mathbf{t}) + (A^*)^{-1}\mathbf{p}_i)|^2 = 1 \text{ or } 0 \quad \text{a.e. on } \boldsymbol{R}^n . \tag{12}$$

PROPOSITION 4.4. *Let H be a low pass filter associated to a scaling function ϕ of an A-PFMRA such that* (12) *holds. Then we have the following.*

i) *Setting $|H(\mathbf{0})| = 1$, the origin is a point of A^*-approximate continuity of $|H|$. Moreover, any point $(A^*)^{-1}\mathbf{p}_i$, $i = 1, \dots, d_A - 1$, is a point of A^*-approximate continuity of $|H|$ if we set $|H((A^*)^{-1}\mathbf{p}_i)| = 0$.*

ii)

$$|\widehat{\phi}(\mathbf{t})| = \prod_{j=1}^{\infty} |H((A^*)^{-j}\mathbf{t})| , \quad a.e. \text{ on } \boldsymbol{R}^n .$$

For the proof of i), one only needs to use a refinement equation $\widehat{\phi}(A^*\mathbf{t}) = H(\mathbf{t})\widehat{\phi}(\mathbf{t})$ a.e. and the A^*-approximate continuity of $|\widehat{\phi}|$ at the origin if we set $|\widehat{\phi}(\mathbf{0})| = 1$ together with (12).

We skip the proof of ii) because, having in mind that $\Phi_\phi = \chi_{\boldsymbol{R}^n \setminus \mathcal{N}_\phi}$, it is completely similar to the case of low pass filters of an A-MRA in [21].

PROPOSITION 4.5. *Let $H \in L^\infty(\boldsymbol{T}^n)$ such that* (12) *holds. If the infinite product $\prod_{j=1}^{\infty} |H((A^*)^{-j}\mathbf{t})|$ converges almost everywhere on $\boldsymbol{R}^n$, then*

a) *the function $\widehat{\theta}(\mathbf{t})$ belongs to $L^2(\boldsymbol{R}^n)$ and $\| \widehat{\theta} \|_{L^2(\boldsymbol{R}^n)} \le 1$;*

b) *$\Phi_\theta(\mathbf{t}) \le 1$ a.e. on $\boldsymbol{R}^n$,*

where the function θ is defined by $\widehat{\theta}(\mathbf{t}) := \prod_{j=1}^{\infty} |H((A^*)^{-j}\mathbf{t})|$.

PROOF. Letting a $\boldsymbol{Z}^n$-periodic measurable function G such that $|G(\mathbf{t})| \ge |H(\mathbf{t})|$ a.e. on $\boldsymbol{R}^n$ and $\sum_{i=0}^{d_A-1} |G(\mathbf{t} + (A^*)^{-1}\mathbf{p}_i)|^2 = 1$ a.e. on $\boldsymbol{R}^n$, for instance

$$G(\mathbf{t}) = \begin{cases} H(\mathbf{t}) & \text{if } \sum_{i=0}^{d_A-1} |H(\mathbf{t} + (A^*)^{-1}\mathbf{p}_i)|^2 = 1 \\ 1/d_A & \text{otherwise}, \end{cases}$$

we have that $\prod_{j=1}^{\infty} |G((A^*)^{-j}\mathbf{t})|$ converges a.e. on $\boldsymbol{R}^n$. Thus, we know that the function $\widehat{g}(\mathbf{t}) = \prod_{j=1}^{\infty} |G((A^*)^{-j}\mathbf{t})|$ belongs to $L^2(\boldsymbol{R}^n)$ and $\| \widehat{g} \|_{L^2(\boldsymbol{R}^n)} \le 1$ by Bownik [4] (cf. [9], [14]), thus the condition a) follows from $\widehat{g}(\mathbf{t}) \ge \widehat{\theta}(\mathbf{t})$ a.e. on $\boldsymbol{R}^n$. Furthermore, the condition b) holds because $\Phi_\theta(\mathbf{t}) \le \Phi_g(\mathbf{t}) \le 1$ a.e. on $\boldsymbol{R}^n$, where the second inequality was proved in the proof of main result in [21]. □

5. On low pass filters in an A-PFMRA. We present necessary an sufficient conditions on the functions $H \in L^\infty(\boldsymbol{T}^n)$ to be low pass filters associated to a scaling functions in an A-PFMRA. For this purpose, we suppose that the infinite product

$$\prod_{j=1}^{\infty} |H((A^*)^{-j}\mathbf{t})| \tag{13}$$

converges almost everywhere on $\boldsymbol{R}^n$ and we are going to look for a scaling function ϕ of an A-PFMRA which satisfies the condition

$$|\widehat{\phi}(\mathbf{t})| = \prod_{j=1}^{\infty} |H((A^*)^{-j}\mathbf{t})| .$$

Hence, according to Theorem C, we should also suppose that $|\widehat{\phi}|$ is A^*-locally nonzero at the origin. In order not to repeat those conditions, let $\mathbf{H}_A$ be the class of all functions $H \in L^\infty(\boldsymbol{T}^n)$ such that the infinite product (13) converges almost everywhere on $\boldsymbol{R}^n$ and is A^*-locally nonzero at the origin.

Moreover, let Π_A be the class of all real-valued bounded measurable functions f on $\boldsymbol{R}^n$ such that $f(\mathbf{0}) = 1$ and the origin is a point of A^*-approximate continuity of f.

We prove the following.

THEOREM 5.1. *Let $H \in \mathbf{H}_A$ satisfy* (12) *and let $\theta \in L^2(\boldsymbol{R}^n)$ be defined by $\widehat{\theta}(\mathbf{t}) := \prod_{j=1}^{\infty} |H((A^*)^{-j}\mathbf{t})|$. Then the following conditions are equivalent*:

A) *The function $|H|$ is a low pass filter associated to a scaling function θ of an A-PFMRA.*

B) $\|\theta\|^2_{L^2(\boldsymbol{R}^n)} = |\boldsymbol{T}^n \setminus \mathcal{N}_\theta|_n$.

C) *The only function $f \in L^1(\boldsymbol{T}^n) \bigcap \Pi_A$ which is a fixed point of the operator $P_{H,\mathcal{N}_\theta}$ is $f \equiv \chi_{\boldsymbol{R}^n \setminus \mathcal{N}_\theta}$.*

Observe that a characterization of all low pass filters in an A-PFMRA follows from Theorem 5.1 and Proposition 4.2.

The rest of the section is devoted to prove Theorem 5.1. We need the following auxiliary results.

The following proposition is proved in [21].

PROPOSITION E. *Let $f : \boldsymbol{R}^n \to \boldsymbol{C}$ be a measurable function and $\mathbf{y} \in \boldsymbol{R}^n$ a point of A-approximate continuity of f. Then, there exists an increasing sequence of natural numbers $\{j_k\}_{k=1}^{\infty} \subset \boldsymbol{N}$, $j_{k+1} > j_k$, such that*

$$\lim_{k \longrightarrow \infty} f(A^{-j_k}\mathbf{x} + \mathbf{y}) = f(\mathbf{y}) \quad a.e. \ on \ \boldsymbol{R}^n .$$

The following lemma is a slight modification of [21, Corollary 1].

LEMMA F. *Let $H \in L^\infty(\boldsymbol{T}^n)$ such that* (12) *holds and let $\widehat{\theta}(\mathbf{t}) = \prod_{j=1}^{\infty} |H((A^*)^{-j}\mathbf{t})|$ a.e. on $\boldsymbol{R}^n$. Then either $\widehat{\theta}$ is not A^*-locally nonzero at the origin or the origin is a point of A^*-approximate continuity of $\widehat{\theta}$ if we set $\widehat{\theta}(\mathbf{0}) = 1$.*

The following result is proved in [8]. Note that the equality (ii) in the following lemma does not appear in the original result but it is an immediate consequence of the proof of (i).

LEMMA G. *Let $g \in L^2(\boldsymbol{T}^n)$, let $A : \boldsymbol{R}^n \to \boldsymbol{R}^n$ be a fixed linear invertible map such that $A(\boldsymbol{Z}^n) \subset \boldsymbol{Z}^n$ and let $\hat{A} : \boldsymbol{T}^n \to \boldsymbol{T}^n$ be the induced endomorphism. Then*

(i) $\int_{\boldsymbol{T}^n} g(\hat{A}\mathbf{t})d\mathbf{t} = \int_{\boldsymbol{T}^n} g(\mathbf{t})d\mathbf{t}$,

(ii) $\int_{[0,1]^n} g(\mathbf{t})d\mathbf{t} = d_A^{-1} \int_{[0,1]^n} \sum_{i=0}^{d_A-1} g(A^{-1}\mathbf{t} + A^{-1}\mathbf{p}_i)d\mathbf{t}$.

PROOF OF THEOREM 5.1. Let us begin with the proof of A) to B). Since $\{\theta(\mathbf{x} - \mathbf{k})\ ;\ \mathbf{k} \in \boldsymbol{Z}^n\}$ is a Parseval frame sequence, $\Phi_\theta(\mathbf{t}) = \chi_{\boldsymbol{R}^n \setminus \mathcal{N}_\theta}(\mathbf{t})$ a.e. on $\boldsymbol{R}^n$, then

$$\|\widehat{\theta}\|^2_{L^2(\boldsymbol{R}^n)} = \sum_{\mathbf{k}\in \boldsymbol{Z}^n} \int_{[0,1]^n+\mathbf{k}} |\widehat{\theta}(\mathbf{t})|^2 d\mathbf{t} = \int_{[0,1]^n} \Phi_\theta(\mathbf{t})d\mathbf{t} = |\boldsymbol{T}^n \setminus \mathcal{N}_\theta|_n\,. \tag{14}$$

The implication B) to A) is proved as follow. According to the condition b) in Proposition 4.5 we have that $\Phi_\theta(\mathbf{t}) \le 1$ a.e. on $\boldsymbol{R}^n$. Further, by (14) we have that

$$|\boldsymbol{T}^n \setminus \mathcal{N}_\theta|_n = \|\theta\|^2_{L^2(\boldsymbol{R}^n)} = \|\widehat{\theta}\|^2_{L^2(\boldsymbol{R}^n)} = \int_{[0,1]^n} \Phi_\theta(\mathbf{t})d\mathbf{t}\,.$$

Hence $\Phi_\theta(\mathbf{t}) = \chi_{\boldsymbol{T}^n\setminus\mathcal{N}_\theta}(\mathbf{t})$ a.e. on $\boldsymbol{R}^n$ and therefore A) holds.

We prove A) to C). Let us check that $\chi_{\boldsymbol{R}^n\setminus\mathcal{N}_\theta}$ is a fixed point of the operator $P_{H,\mathcal{N}_\theta}$ and also it belongs to $L^1(\boldsymbol{T}^n) \bigcap \varPi_A$.

Since θ is a scaling function in an A-PFMRA we have $\Phi_\theta(\mathbf{t}) = \chi_{\boldsymbol{R}^n\setminus\mathcal{N}_\theta}(\mathbf{t})$ a.e., thus according to Proposition 4.3 we have that

$$P_{H,\mathcal{N}_\theta}(\chi_{\boldsymbol{R}^n\setminus\mathcal{N}_\theta}) = \chi_{\boldsymbol{R}^n\setminus\mathcal{N}_\theta}\,. \tag{15}$$

Moreover, setting $\chi_{\boldsymbol{R}^n\setminus\mathcal{N}_\theta}(\mathbf{0}) = 1$, that the origin is a point of A^*-approximate continuity of $\chi_{\boldsymbol{R}^n\setminus\mathcal{N}_\theta}$ is true because $\widehat{\theta}$ is A^*-locally nonzero at the origin. Thus according to Lemma F, the origin is a point of A^*-approximate continuity of $\widehat{\theta}$ if we set $\widehat{\theta}(\mathbf{0}) = 1$, and finally, using $\widehat{\theta}(\mathbf{t}) \le \Phi_\theta(\mathbf{t}) = \chi_{\boldsymbol{R}^n\setminus\mathcal{N}_\theta}(\mathbf{t})$ a.e. on $\boldsymbol{R}^n$ the assertion follows.

We now show that if $f \in L^1(\boldsymbol{T}^n) \bigcap \varPi_A$ is a fixed point of the operator $P_{H,\mathcal{N}_\theta}$, then $f \equiv \chi_{\boldsymbol{R}^n\setminus\mathcal{N}_\theta}$.

Let us first prove that $f(\mathbf{t}) \ge \chi_{\boldsymbol{R}^n\setminus\mathcal{N}_\theta}(\mathbf{t})$ a.e. on $\boldsymbol{R}^n$. For this purpose, it suffices to prove that given any measurable set $J \subset [0,1]^n$ we have $\int_J f(\mathbf{t})d\mathbf{t} \ge \int_J \chi_{\boldsymbol{R}^n\setminus\mathcal{N}_\theta}(\mathbf{t})d\mathbf{t}$.

In order to shorten the notation in the following computation, we mention that since f is a fixed point of the operator $P_{H,\mathcal{N}_\theta}$, the equality (15) says

$$P_{H,\mathcal{N}_\theta}(f)(\mathbf{t}) = f(\mathbf{t}) = 0 \quad \text{a.e. on } \mathcal{N}_\theta\,. \tag{16}$$

Let $J \subset [0,1]^n$ be a measurable set and let $S = J + \boldsymbol{Z}^n$. Using the equalities $P_{H,\mathcal{N}_\theta}(f) = f$ and (16), we obtain

$$\begin{aligned}\int_J f(\mathbf{t})d\mathbf{t} &= \int_{[0,1]^n} \chi_S(\mathbf{t}) P_{H,\mathcal{N}_\theta}(f)(\mathbf{t})d\mathbf{t}\\ &= \int_{[0,1]^n} \chi_S(\mathbf{t}) \sum_{i=0}^{d_A-1} |H((A^*)^{-1}(\mathbf{t}+\mathbf{p}_i))|^2 f((A^*)^{-1}(\mathbf{t}+\mathbf{p}_i))d\mathbf{t}\,.\end{aligned}$$

Since $\chi_S(A^*\mathbf{t})\,|H(\mathbf{t})|^2\, f(\mathbf{t})$ is a $\boldsymbol{Z}^n$-periodic bounded function, according to the condition (ii) in Lemma G we have

$$\int_J f(\mathbf{t})d\mathbf{t} = d_A \int_{[0,1]^n} \chi_S(A^*\mathbf{t})\,|H(\mathbf{t})|^2\, f(\mathbf{t})d\mathbf{t}$$

$$= d_A \int_{[-1/2,1/2]^n} \chi_S(A^*\mathbf{t})\, |H(\mathbf{t})|^2\, f(\mathbf{t}) d\mathbf{t}\,.$$

Putting $A^*\mathbf{t} = \mathbf{v}$ and taking into account $P_{H,\mathcal{N}_\theta}(f) = f$, we have

$$\int_J f(\mathbf{t})d\mathbf{t} = \int_{\boldsymbol{R}^n} \chi_S(\mathbf{v}) \left| H((A^*)^{-1}\mathbf{v}) \right|^2 P_{H,\mathcal{N}_\theta}(f)((A^*)^{-1}\mathbf{v}) \chi_{[-1/2,1/2]^n}((A^*)^{-1}\mathbf{v}) d\mathbf{v}\,.$$

Given $N \in \boldsymbol{N}$, repeating the above calculations and using the condition $A^*(\boldsymbol{Z}^n) \subset \boldsymbol{Z}^n$, we obtain

$$\int_J f(\mathbf{t})d\mathbf{t} = \int_{\boldsymbol{R}^n} \Gamma_N f(\mathbf{t}) d\mathbf{t}\,,$$

where

$$\Gamma_N f(\mathbf{t}) = \chi_S(\mathbf{t}) \prod_{j=1}^{N} \left| H((A^*)^{-j}\mathbf{t}) \right|^2 f((A^*)^{-N}\mathbf{t}) \chi_{[-1/2,1/2]^n}((A^*)^{-N}\mathbf{t})\,.$$

Since the origin is a point of A^*-approximate continuity of $\chi_{[-1/2,1/2]^n} f$, by Proposition E, there exists an increasing sequence $\{l_N\}_{N=1}^{\infty} \subset \boldsymbol{N}$ such that

$$\lim_{N\longrightarrow\infty} \Gamma_{l_N} f(\mathbf{t}) = \chi_S(\mathbf{t}) \prod_{j=1}^{\infty} \left| H((A^*)^{-j}\mathbf{t}) \right|^2 \quad \text{a.e. on } \boldsymbol{R}^n\,. \tag{17}$$

By Fatou's lemma and (17),

$$\begin{aligned}\int_J f(\mathbf{t})d\mathbf{t} &= \lim_{N\to\infty} \int_{\boldsymbol{R}^n} \Gamma_{l_N} f(\mathbf{t})d\mathbf{t} \geq \int_{\boldsymbol{R}^n} \lim_{N\to\infty} \Gamma_{l_N} f(\mathbf{t})d\mathbf{t} \\ &= \int_{\boldsymbol{R}^n} \chi_S(\mathbf{t}) \prod_{j=1}^{\infty} \left| H((A^*)^{-j}\mathbf{t}) \right|^2 d\mathbf{t} = \int_{\boldsymbol{R}^n} \chi_S(\mathbf{t}) |\widehat{\theta}(\mathbf{t})|^2 d\mathbf{t} \\ &= \sum_{\mathbf{k}\in\boldsymbol{Z}^n} \int_{J+\mathbf{k}} |\widehat{\theta}(\mathbf{t})|^2 d\mathbf{t} = \int_J \Phi_\theta(\mathbf{t})d\mathbf{t} = \int_J \chi_{\boldsymbol{R}^n\setminus\mathcal{N}_\theta}(\mathbf{t})d\mathbf{t}\,.\end{aligned}$$

We now see that $f(\mathbf{t}) \leq \chi_{\boldsymbol{R}^n\setminus\mathcal{N}_\theta}(\mathbf{t})$ a.e. on $\boldsymbol{R}^n$. We consider the auxiliary function $g(\mathbf{t}) = (3/2)\chi_{\boldsymbol{R}^n\setminus\mathcal{N}_\theta}(\mathbf{t}) - (1/2)f(\mathbf{t})$ and observe that $g \in \Pi_A \cap L^1(\boldsymbol{T}^n)$ if we set $g(\mathbf{0}) = 1$.

Since $P_{H,\mathcal{N}_\theta}(g) = g$, $(3/2)\chi_{\boldsymbol{R}^n\setminus\mathcal{N}_\theta}(\mathbf{t}) - (1/2)f(\mathbf{t}) \geq \chi_{\boldsymbol{R}^n\setminus\mathcal{N}_\theta}(\mathbf{t})$ a.e. It follows that $f(\mathbf{t}) \leq \chi_{\boldsymbol{R}^n\setminus\mathcal{N}_\theta}(\mathbf{t})$ a.e. on $\boldsymbol{R}^n$ and this finishes the proof.

To prove C) to A), we should see that conditions (5) in Theorem A, the condition c) in Lemma B and the condition (B) in Theorem C are satisfied by the function θ. Observe that according to the hypotheses, the condition c) in Lemma B and the condition (B) in Theorem C hold. Now we show that the function Φ_θ defined by (3) belongs to $L^1(\boldsymbol{T}^n) \bigcap \Pi_A$ and is a fixed point for the operator $P_{H,\mathcal{N}_\theta}$. Thus by the condition C) we will have that $\Phi_\theta(\mathbf{t}) = \chi_{\boldsymbol{R}^n\setminus\mathcal{N}_\theta}$ a.e. on $\boldsymbol{T}^n$, and the proof of Theorem 5.1 will be finished.

First of all, according to Proposition 4.3 we know that Φ_θ is a fixed point for the operator $P_{H,\mathcal{N}_\theta}$.

Furthermore, obviously $0 \leq \Phi_\theta(\mathbf{t})$ a.e. on $\boldsymbol{R}^n$ and Φ_θ is $\boldsymbol{Z}^n$-periodic. Moreover, by b) in Proposition 4.5, $\Phi_\theta(\mathbf{t}) \leq 1$ a.e. on $\boldsymbol{R}^n$.

It remains to prove that the origin is a point of A^*-approximate continuity of Φ_θ if we set $\Phi_\theta(\mathbf{0}) = 1$. By hypothesis, $\widehat{\theta}$ is A^*-locally nonzero at the origin, thus according to Lemma F, the origin is a point of A^*-approximate continuity of $\widehat{\theta}$ if we set $\widehat{\theta}(\mathbf{0}) = 1$. Hence, the inequalities $\widehat{\theta}(\mathbf{t}) \le \Phi_\theta(\mathbf{t}) \le 1$ yield the required assertion. □

6. On low pass filters in an A-FMRA. We are ready to show a criterion on the functions $H \in L^\infty(\boldsymbol{T}^n)$ which are low pass filters associated to a scaling function ϕ in an A-FMRA. Our conditions involve an appropriate subset of $L^1(\boldsymbol{T}^n)$ where we look for fixed points of the operator $P_{H,\mathcal{N}_\phi}$.

We denote by

$$\mathbf{L}_A = \{f \in L^2(\boldsymbol{R}^n)\ ;\ \widehat{f} \text{ is } A^*\text{-locally nonzero at the origin}\}\,.$$

Moreover, given a real-valued function g in $L^1(\boldsymbol{T}^n)$, let $\Delta_{A,g}$ be the class of all non negative measurable functions, h, such that

(a) both h and $1/h$ are essentially bounded on the support of g;

(b) $h(\mathbf{t}) = g(\mathbf{t})f(\mathbf{t})$ where $f \in \Pi_A$.

We prove the following.

THEOREM 6.1. *Let $H \in L^\infty(\boldsymbol{T}^n)$. The two following conditions are equivalent*:

(I) *The function H is a low pass filter in an A-FMRA.*

(II) (α) *There exists $\phi \in \mathbf{L}_A$ such that $|\widehat{\phi}(\mathbf{t})| = |H((A^*)^{-1}\mathbf{t})||\widehat{\phi}((A^*)^{-1}\mathbf{t})|$ a.e.*;
(β) *The only fixed point of the operator $P_{H,\mathcal{N}_\phi}$ in the set $L^1(\boldsymbol{T}^n) \cap \Delta_{A,\Phi_\phi}$ is Φ_ϕ.*

PROOF. We prove (I) to (II). Since H is a low pass filter associated to a scaling function ϕ in an A-FMRA, by (2) and (B) in Theorem C, the condition (α) holds. It remains to check that (β) is satisfied. According to Theorem A, the function Φ_ϕ belongs to $L^1(\boldsymbol{T}^n) \cap \Delta_{A,\Phi_\phi}$, and further, by Proposition 4.3, we have $P_{H,\mathcal{N}_\phi}(\Phi_\phi) = \Phi_\phi$.

We see now the uniqueness. According to Lemma D, the function $\varphi \in L^2(\boldsymbol{R}^n)$ defined by $\widehat{\varphi} = \widehat{\phi}\cdot(\Phi_\phi)^{-1/2}$ is a scaling function in an A-PFMRA with, G, a bounded low pass filter associated defined by

$$G(\mathbf{t}) = H(\mathbf{t})(\Phi_\phi(\mathbf{t}))^{1/2}(\Phi_\phi(A^*\mathbf{t}))^{-1/2}\,.$$

On the other hand, let $h \in L^1(\boldsymbol{T}^n) \cap \Delta_{A,\Phi_\phi}$ be a fixed point of the operator $P_{H,\mathcal{N}_\phi}$, then

$$\begin{aligned}\Phi_\phi(\mathbf{t})f(\mathbf{t}) = h(\mathbf{t}) &= P_{H,\mathcal{N}_\phi}(h)(\mathbf{t})\\ &= \sum_{i=0}^{d_A-1} |H((A^*)^{-1}(\mathbf{t}+\mathbf{p}_i)|^2 h((A^*)^{-1}(\mathbf{t}+\mathbf{p}_i))\chi_{\boldsymbol{R}^n\setminus\mathcal{N}_\phi}((A^*)^{-1}(\mathbf{t}+\mathbf{p}_i))\\ &= \sum_{i=0}^{d_A-1} |H((A^*)^{-1}(\mathbf{t}+\mathbf{p}_i)|^2 \Phi_\phi((A^*)^{-1}(\mathbf{t}+\mathbf{p}_i))f((A^*)^{-1}(\mathbf{t}+\mathbf{p}_i))\,,\end{aligned}$$

where $f \in L^1(\boldsymbol{R}^n) \cap \Pi_A$. Thus, having in mind that $\mathcal{N}_\varphi = \mathcal{N}_\phi$ (except a null measurable set), we have $f = P_{G,\mathcal{N}_\varphi}(f)$, so according to the condition (ii) in Proposition 4.4

and by Theorem 5.1, we really know that f must be $\chi_{\boldsymbol{R}^n \setminus \mathcal{N}_\phi}$. We have just proved that if $h \in L^1(\boldsymbol{T}^n) \cap \Delta_{A,\Phi_\phi}$ is a fixed point of the operator $P_{H,\mathcal{N}_\phi}$, then $h \equiv \Phi_\phi$.

To prove (II) to (I), we should see that the function θ defined by $\widehat{\theta}(\mathbf{t}) = |\widehat{\phi}(\mathbf{t})|$, where ϕ is given in (α), is a scaling function in an A-FMRA. Hence $|H|$ is a low pass filter associated to θ, and finally we conclude that H is a low pass filter in an A-FMRA according to Proposition 4.2. By (α) the function θ satisfies the conditions c) in Lemma B and (B) in Theorem C where one replaces ϕ by θ. Furthermore, since $\Phi_\phi = \Phi_\theta$ then (β) tells us that (5) in Theorem A holds. We have really proved that θ is a scaling function in an A-FMRA. □

REFERENCES

[1] J. J. BENEDETTO AND S. LI, The theory of multiresolution analysis frames and applications to filter banks, Appl. Comput. Harmon. Anal. 5 (1998), 389–427.

[2] J. J. BENEDETTO AND D. F. WALNUT, Gabor frames for L^2 and related spaces, Wavelets: mathematics and applications, 97–162, Stud. Adv. Math., CRC, BocaRaton, FL, 1994.

[3] C. DE BOOR, R. A. DEVORE AND A. RON, The structure of finitely generated shift invariant spaces in $L^2(\boldsymbol{R}^d)$, J. Funct. Anal. 119 (1994), 37–78.

[4] M. BOWNIK, Tight frames of multidimensional wavelets, Dedicated to the memory of Richard J. Duffin. J. Fourier Anal. Appl. 3 (1997), 525–542.

[5] A. BRUCKNER, Differentiation of real functions, Lecture Notes in Mathematics 659, Springer, Berlin, 1978.

[6] P. G. CASAZZA, O. CHRISTENSEN AND N. J. KALTON, Frames of translates, Collect. Math. 52 (2001), 35–54.

[7] O. CHRISTENSEN, An introduction to frames and Riesz bases, Birkhäser, Boston, 2003.

[8] P. CIFUENTES, K. S. KAZARIAN AND A. SAN ANTOLÍN, Characterization of scaling functions in a multiresolution analysis, Proc. Amer. Math. Soc. 133 (2005), 1013–1023.

[9] I. DAUBECHIES, Ten lectures on wavelets, CBMS-NSF Regional Conf. Ser, in Appl. Math. 61, SIAM, Philadelphia, 1992.

[10] V. DOBRIĆ, R. F. GUNDY AND P. HITCZENKO, Characterizations of orthonormal scale functions: a probabilistic approach, J. Geom. Anal. 10 (2000), 417–434.

[11] R. DUFFIN AND A. SCHAEFFER, A class of nonharmonic Fourier series, Trans. Amer. Math. Soc. 72 (1952), 341–366.

[12] K. GRÖCHENING AND W. R. MADYCH, Multiresolution analysis, Haar bases and self-similar tillings of R^n, IEEE Trans. Inform. Theory 38 (1992), 556–568.

[13] R. F. GUNDY, Low-pass filters, martingales, and multiresolution analyses, Appl. Comput. Harmon. Anal. 9 (2000), 204–219.

[14] E. HERNÁNDEZ AND G. WEISS, A first course on Wavelets, Stud. Adv. Math., CRC Press, Boca Raton, FL, 1996.

[15] K. S. KAZARIAN AND A. SAN ANTOLIN, Characterization of scaling functions in a frame multiresolution analysis in H^2_G, Topics in classical analysis and applications in honor of Daniel Waterman, 118–140, World Sci. Publ., Hackensack, NJ, 2008.

[16] H. O. KIM AND J. K. LIM, Frame multiresolution analysis, Commun. Korean Math. Soc. 15 (2000), 285–308.

[17] S. MALLAT, Multiresolution approximations and wavelet orthonormal bases for $L^2(R)$, Trans. Amer. Math. Soc. 315 (1989), 69–87.

[18] Y. MEYER, Ondelettes et opérateurs. I, Hermann, Paris, 1990 (English Translation: Wavelets and operators, Cambridge University Press, 1992).

[19] I. P. NATANSON, Theory of functions of a real variable, London, vol. II, Frederick Ungar Publishing Co., New

York, 1961.

[20] M. PAPADAKIS, H. SIKIĆ AND G. WEISS, The characterization of low pass filters and some basic properties of wavelets, scaling functions and related concepts, J. Fourier Anal. Appl. 5 (1999), 495–521.

[21] A. SAN ANTOLÍN, Characterization of low pass filters in a multiresolution analysis, Studia Math. 190 (2009), 99–116.

[22] P. WOJTASZCZYK, A mathematical introduction to wavelets, London Math. Soc. Stud. Texts 37, Cambridge University Press, Cambridge, 1997.

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO
UNIVERSIDAD DE ALICANTE
03080 ALICANTE
SPAIN

E-mail address: angel.sanantolin@ua.es