BEURLING'S THEOREM ON EXCEPTIONAL SETS*)

By

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1.

1. Let E be a bounded Borel set of points on z-plane. We distribute a positive mass $d\mu(a)$ of total mass 1 on E and let

$$u(z)=\int_E \log\frac{1}{|z-a|}\,d\mu(a), \qquad (\mu(E)=1),$$

then $u(z)$ is harmonic outside of E. Let V_μ be the upper limit of $u(z)$ for $|z|<\infty$ and $V=\inf\limits_{\mu} V_\mu$, then $C(E)=e^{-V}$ is called the logarithmic capacity of E. Hence if $C(E)>0$, i. e. $V<\infty$, then we can distribute a positive mass $d\mu$ on E, such that $V_\mu<\infty$.

Evans[1] proved the following theorem, which we use in the proof of Theorem 5.

LEMMA 1. (EVANS.) *Let E be a bounded closed set of logarithmic capacity zero on z-plane, then we can distribute a positive mass of total mass 1 on E, such that $u(z)$ tends to $+\infty$, when z tends to any point of E.*

Beurling[2] proved the following important theorems:

THEOREM 1. (BEURLING.) *Let $w=f(z)$ be regular in $|z|<1$ and the area A on w-plane, which is described by $w=f(z)$ ($|z|<1$) be finite, i. e.*

$$A=\iint\limits_{|z|<1}|f'(re^{i\theta})|^2\,r\,dr\,d\theta<\infty,$$

then the set E of points $e^{i\theta}$ on $|z|=1$, such that

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1) G. C. Evans: Potentials and positively infinite singularities of harmonic functions. Monatshefte für Math. u. Phys. **43** (1936). Evans proved for Newtonian potentials and the proof can be easily modified in the case of logarithmic capacity. This is done by K.Noshiro in his paper: Contributions to the theory of the singularities of analytic functions. Jap. Jour. Math. 19 No.4 (1948).

2) Beurling: Ensembles exceptionelles. Acta Math. **72** (1940).

$$\int_0^1 |f'(re^{i\theta})|\, dr = \infty$$

is of logarithmic capacity zero.

Hence the set of points $e^{i\theta}$, such that $\lim\limits_{r\to 1} f(re^{i\theta})$ does not exist or $\lim\limits_{r\to 1} |f(re^{i\theta})| = \infty$ is of logarithmic capacity zero.

THEOREM 2. (BEURLING.) *Let $w = f(z)$ be meromorphic in $|z| < 1$ and the area A on w-sphere, which is described by $w = f(z)$ ($|z| < 1$) be finite, i. e.*

$$A = \iint\limits_{|z|<1} \left(\frac{|f'(re^{i\theta})|}{1 + |f(re^{i\theta})|^2} \right)^2 r\,dr\,d\theta < \infty,$$

then the set E of points $e^{i\theta}$ on $|z| = 1$, such that $\lim\limits_{r\to 1} f(re^{i\theta})$ does not exist is of logarithmic capacity zero.

We will prove the following more general theorems:

THEOREM 3. *Let $w = f(z)$ be regular in $|z| < 1$ and*

$$A = \iint\limits_{|z|<1} |f'(re^{i\theta})|^2\, r\,dr\,d\theta < \infty,$$

then there exists a certain set E on $|z| = 1$, which is of logarithmic capacity zero, such that if $e^{i\theta}$ does not belong to E, then a rectilinear segment, which connects $e^{i\theta}$ to any point z in $|z| < 1$ is mapped on a rectifiable curve on w-plane.

THEOREM 4. *Let $w = f(z)$ be meromorphic in $|z| < 1$ and*

$$A = \iint\limits_{|z|<1} \left(\frac{|f'(re^{i\theta})|}{1 + f(re^{i\theta})^2|^2} \right)^2 r\,dr\,d\theta < \infty,$$

then there exists a certain set E on $|z| = 1$, which is of logarithmic capacity zero, such that if $e^{i\theta}$ does not belong to E, then a rectilinear segment, which connects $e^{i\theta}$ to any point z in $|z| < 1$ is mapped on a rectifiable curve on w-sphere.

Hence the set of points $e^{i\theta}$, such that $\int_0^1 \frac{|f'(re^{i\theta})|}{1 + |f(re^{i\theta})|^2} dr = \infty$ is of logarithmic capacity zero.

If $\int_0^1 \frac{|f'(re^{i\theta})|}{1 + |f(re^{i\theta})|^2} dr < \infty$ then $\lim\limits_{r\to 1} f(re^{i\theta})$ exists.

2. We use the following lemmas in the proof.

LEMMA 2. *Let $w = f(z)$ be meromorphic in a domain D: $0 < r \leq R$, $0 \leq \theta \leq \theta_0$ ($z = re^{i\theta}$) and take certain three values finite times in D.*

(i) *If* $\lim_{r\to 0} f(r) = \alpha$, $\lim_{r\to 0} f(re^{i\theta_0}) = \beta$ *exist, then* $\alpha = \beta = \omega$ *and* $f(z)$ *tends to* ω *uniformly, when* z *tends to* $z = 0$ *from the inside of* D.

(ii) *If* $\lim_{r\to 0} f(r) = \omega$ *exists, then* $f(z)$ *tends to* ω *uniformly, when* z *tends to* $z = 0$ *from the inside of an angular domain* D_1: $0 < r \leqq R$, $0 \leqq \theta \leqq \theta_0 - \delta$ *for any* $\delta > 0$.

(i) is due to Lindelöf and (ii) is Montel's theorem, when $f(z)$ is bounded in D. The general case can be reduced to this case by means of modular function in the well known way.

REMARK. If $\iint_D \left(\frac{f'(re^{i\theta})}{1+|f(re^{i\theta})|^2}\right)^2 rdrd\theta < \infty$, then $f(z)$ takes almost all values finite times in D, so that satisfies the above condition.

LEMMA 3. (FEJÉR AND F. RIESZ.)[3] *Let* $w = f(z)$ *be regular in* $|z| \leqq 1$, *then*

$$\int_{-1}^{1} |f(z)|\,|dz| \leqq \frac{1}{2}\int_{|z|=1} |f(z)|\,|dz|,$$

where the left hand side is integrated on the diameter $(-1, 1)$ *of* $|z| = 1$.

If we apply the lemma on $f'(z)$, we have

$$\int_{-1}^{1} |f'(z)|\,|dz| \leqq \frac{1}{2}\int_{|z|=1} |f'(z)|\,|dz|,$$

the left hand side is the length of the image of the diameter $(-1, 1)$ and the right hand side is that of $|z| = 1$,

When $f(z)$ is regular in $|z| \leqq 1$, except at $z = 1$ and is continuous in $|z| \leqq 1$, it is easily proved that the same relation holds.

LEMMA 4. *Let* $w = f(z)$ *be regular in a domain* D: $0 < r \leqq R$, $0 \leqq \theta \leqq \theta_0$ $(z = re^{i\theta})$ *and takes certain three values finite times in* D. *If*

$$\int_0^R |f'(r)|\,dr \leqq L < \infty,\quad \int_0^R |f'(re^{i\theta_0})|\,dr \leqq L < \infty,$$

then

$$L(\theta) = \int_0^R |f'(re^{i\theta})|\,dr \leqq L + KR\theta_0 \qquad (0 \leqq \theta \leqq \theta_0),$$

where $K = \max_{0\leqq\theta\leqq\theta_0} |f'(Re^{i\theta})|$.

3) L. Fejér u. F. Riesz: Über einige funktionentheoritische Ungleichungen, Math. Zeits., **11** (1921).

PROOF. Since $\int_0^R |f'(r)|\,dr < \infty$, $\int_0^R |f'(re^{i\theta_0})|\,dr < \infty$, $\lim_{r\to 0} f(r)$, $\lim_{r\to 0} f(re^{i\theta_0})$

exist, so that by lemma 2, $f(z)$ is continuous in the closed domain $\overline{D}$. We map D conformally on a unit circle $|\zeta| < 1$, such that the segment $z = re^{i\theta_0/2}$ $(0 \leqq r < R)$ is mapped on a diameter of $|\zeta| = 1$. Since $|f'(z)|\,|dz|$ is invariant by conformal mapping, we have by lemma 3,

$$L\left(\frac{\theta_0}{2}\right) = \int_0^R |f'(re^{i\theta_0/2})|\,dr \leqq \frac{1}{2}\int_0^R |f(r)|\,dr + \frac{1}{2}\int_0^R |f'(re^{i\theta_0})|\,dr$$

$$+ \frac{1}{2}\int_0^{\theta_0} |f'(Re^{i\theta})|\,R\,d\theta \leqq L + KR\theta_0/2. \tag{1}$$

We divide the interval $(0, \theta_0)$ into 2^n equal parts, then we will prove by induction, that

$$I(\nu\theta_0/2^n) \leqq L + KR\,\theta_0(2^{-1} + \cdots + 2^{-n}), \qquad (\nu = 0, 1, 2, \cdots, 2^n). \tag{2}$$

By (1), (2) holds for $n = 1$.

Suppose that (2) holds for $n = m$, then

$$L(\nu\theta_0/2^m) \leqq L + KR\,\theta_0(2^{-1} + \cdots + 2^{-m}), \qquad (0 \leqq \nu \leqq 2^m),$$
$$L((\nu + 1)\,\theta_0/2^m) \leqq L + KR\,\theta_0(2^{-1} + \cdots + 2^{-m}), \qquad (0 \leqq \nu \leqq 2^m - 1).$$

Similarly as (1), we have

$$L((2\nu + 1)\,\theta_0/2^{m+1}) \leqq \frac{1}{2} L(\nu\theta_0/2^m) + \frac{1}{2} L((\nu + 1)\,\theta_0/2^m)$$

$$+ \frac{1}{2}\int_{\nu\theta_0/2^m}^{(\nu+1)\theta_0/2^m} |f'(Re^{i\theta})|\,R\,d\theta \leqq L + KR\,\theta_0(2^{-1} + \cdots + 2^{-m})$$

$$+ KR\,\theta_0/2^{m+1} = L + KR\,\theta_0(2^{-1} + \cdots + 2^{-m-1}), \qquad (0 \leqq \nu \leqq 2^m - 1),$$

$$L(2\nu\,\theta_0/2^{m+1}) = L(\nu\theta_0/2^m) \leqq L + KR\,\theta_0(2^{-1} + \cdots + 2^{-m})$$
$$< L + KR\,\theta_0(2^{-1} + \cdots + 2^{-m-1}), \qquad (0 \leqq \nu \leqq 2^m),$$

so that (2) holds for $n = m + 1$, $(0 \leqq \nu \leqq 2^{m+1})$. Hence by induction, (2) holds for any n.

From (2), we have

$$L(\nu\,\theta_0/2^n) \leqq L + KR\,\theta_0. \tag{3}$$

Let θ be any value in $(0, \theta_0)$, then we can find ν_n, such that $\nu_n\,\theta_0/2^n \to \theta$ $(n \to \infty)$, so that by (3),

$$L(\theta) \leqq \varliminf_{n\to\infty} L\left(\frac{\nu_n\theta_0}{2_n}\right) \tag{4}$$

REMARK. Hence

$$L(\theta) \leqq \int_0^R |f'(r)|\, dr + \int_0^R |f'(re^{i\theta_0})|\, dr + KR\,\theta_0. \tag{5}$$

LEMMA 5. *Let* $w=f(z)$ *be meromorphic in a domain* D: $0<r\leqq R$, $0\leqq\theta\leqq\theta_0$ ($z=re^{i\theta}$) *and take certain three values finite times in* D. *If*

$$\int_0^R \frac{|f'(r)|}{1+|f(r)|^2}\, dr < \infty, \quad \int_0^R \frac{|f'(re^{i\theta_0})|}{1+|f(re^{i\theta_0})|}\, dr < \infty,$$

then there exists a constant K, *such that*

$$L(\theta) = \int_0^R \frac{|f(re^{i\theta})|}{1+|f(re^{i\theta})|^2}\, dr \leqq K, \qquad (0\leqq\theta\leqq\theta_0).$$

PROOF. From the hypothesis, $\lim_{r\to 0} f(r)=\alpha$, $\lim_{r\to 0} f(re^{i\theta_0})=\beta$ exist, so that by Lemma 2, $\alpha=\beta=\omega$ and $f(z)$ tends to ω uniformly, when z tends to $z=0$ from the inside of D. By a suitable rotation of w-sphere, we may assume that $\omega=0$ and $f(z)$ is regular and bounded in D, such that $|f(z)|\leqq M$ in D, then by the remark of Lemma 4.

$$\begin{aligned} L(\theta) &= \int_0^R \frac{|f'(re^{i\theta})|}{1+|f(re^{i\theta})|^2}\, dr \leqq \int_0^R |f'(re^{i\theta})|\, dr \leqq \int_0^R |f'(r)|\, dr + \int_0^R |f'(re^{i\theta})|\, dr + KR\,\theta_0 \\ &\leqq (1+M^2)\int_0^R \frac{|f'(r)|}{1+|f(r)|^2}\, dr + (1+M^2)\int_0^R \frac{|f'(re^{i\theta_0})|}{1+|f(re^{i\theta_0})|}\, dr + KR\,\theta_0, \end{aligned}$$

so that $L(\theta)$ is bounded for $0\leqq\theta\leqq\theta_0$.

3. PROOF OF THEOREM 4.

We will prove Theorem 4, since Theorem 3 can be proved similarly. Let $e^{i\theta}$ be a point on $|z|=1$ and K be a circle $\left|z-\dfrac{3e^{i\theta}}{4}\right|=\dfrac{1}{4}$, which has a radius $\dfrac{1}{4}$ and touches $|z|=1$ internally at $e^{i\theta}$.

Let l be a segnent through $e^{i\theta}$, which makes an angle $\psi\left(-\dfrac{\pi}{2}<\psi<\dfrac{\pi}{2}\right)$ with the radius of $|z|=1$ through $e^{i\theta}$ and l_ψ be the part of l, which is contained in K.

We put

$$L(\psi) = \int_{l_\psi} \frac{|f'(z)|}{1+|f(z)|^2} |dz|, \tag{1}$$

$$\chi(\theta) = \int_{-\pi/2}^{\pi/2} L(\psi) \cos\psi \, d\psi. \tag{2}$$

$L(\psi)$ is the length of the image of l_ψ on w-sphere by $w = f(z)$.

First we will prove that the set E of points $e^{i\theta}$ on $|z| = 1$, such that $\chi(\theta) = \infty$ is of logarithmic capacity zero.

Suppose that $C(E) > 0$, then E contains a closed sub-set of positive capacity, so that we may suppose that E is closed. Since $C(E) > 0$, we can distribute a positive mass of total mass 1 on E, such that for $|z| < \infty$,

$$u(z) = u(re^{i\theta}) = \int_E \log \frac{1}{|re^{i\theta} - e^{i\varphi}|} d\mu(\varphi) \leqq V_\mu < \infty. \tag{3}$$

$u(z)$ is harmonic in $|z| < 1$ and its Fourier expansion is

$$u(z) = \sum_{n=1}^{\infty} \frac{r^n}{n} (h_n \cos n\theta + k_n \sin n\theta), \tag{4}$$

where

$$h_n = \int_E \cos n\theta d\mu(\theta), \quad k_n \quad \int_E \sin n\theta \, d\mu(\theta), \tag{5}$$

so that

$$\int_E u(re^{i\theta}) \, d\mu(\theta) = \sum_{n=1}^{\infty} \frac{r^n}{n} (h_n^2 + k_n^2) \leqq V_\mu,$$

whence

$$\sum_{n=1}^{\infty} \frac{1}{n} (h_n^2 + k_n^2) \leqq V_\mu. \tag{6}$$

Hence

$$\iint_{|z|<1} \left(\frac{\partial u}{\partial r}\right)^2 r dr d\theta = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n} (h_n^2 + k_n^2) \leqq \frac{\pi}{2} V_\mu. \tag{7}$$

If we put

$$I = \iint_{|z|<1} \frac{|f'(re^{i\theta})|}{1+|f(re^{i\theta})|^2} \frac{\partial u}{\partial r} r dr d\theta, \tag{8}$$

then

$$|I|^2 \leqq \iint_{|z|<1} \left(\frac{|f'(re^{i\theta})|}{1+|f(re^{i\theta})|^2} \right)^2 r dr d\theta \iint_{|z|<1} \left(\frac{\partial u}{\partial r} \right)^2 r dr d\theta \leqq \frac{\pi}{2} AV_\mu < \infty. \quad (9)$$

Now on $|z| = r$, we have

$$\frac{\partial u}{\partial r} r d\theta = - \int_E d \arg (re^{i\theta} - e^{i\varphi}) d\mu(\varphi),$$

so that

$$I = \int_E d\mu(\varphi) \int_0^1 dr \int_0^{2\pi} \frac{|f'(re^{i\theta})|}{1+|f(re^{i\theta})|^2} (- d \arg (re^{i\theta} - e^{i\varphi})).$$

Hence if we put

$$I(\varphi) = \int_0^1 dr \int_0^{2\pi} \frac{|f'(re^{i\theta})|}{1+|f(re^{i\theta})|^2} (- d \arg (re^{i\theta} - e^{i\varphi})), \quad (10)$$

we have

$$I = \int_E I(\varphi) d\mu(\varphi). \quad (11)$$

We will prove that $I(\varphi) = \infty$ for any point $e^{i\varphi} \in E$.

Suppose that $z = 1$ $(\varphi = 0)$ belongs to E, then

$$\chi(0) = \infty. \quad (12)$$

If we put

$$-\psi = \arg (re^{i\theta} - 1), \quad \left(-\frac{\pi}{2} < \psi < \frac{\pi}{2} \right), \quad (13)$$

then

$$I(0) = \int_0^1 dr \int_{\theta=0}^{\theta=2\pi} \frac{|f'(re^{i\theta})|}{1+|f(re^{i\theta})|^2} d\psi. \quad (14)$$

Since $\sin\psi = \dfrac{r \sin\theta}{\sqrt{1 + r^2 - 2r\cos\theta}}$, we have on $|z| = r$,

$$d\psi = r \frac{\cos\theta - r}{1 + r^2 - 2r\cos\theta} d\theta. \quad (15)$$

Hence if we put $\cos^{-1} r = \theta_0$, then $d\psi \geqq 0$ for $|\theta| \leqq \theta_0$ and $d\psi \leqq 0$ for $\theta_0 \leqq |\theta| \leqq \pi$, so that

$$I(0)=\int_0^1 dr \int_{|\theta|\leqq\theta_0} \frac{|f'(re^{i\theta})|}{1+|f(re^{i\theta})|^2}d\psi - \int_0^1 dr \int_{\theta_0\leqq|\theta|\leqq\pi} \frac{|f'(re^{i\theta})|}{1+|f(re^{i\theta})|^2}|d\psi|. \tag{16}$$

We remark that the point $|z|=re^{i\theta}\,(\theta_0\leqq|\theta|\leqq\pi)$ lies outside a circle $\left|z-\frac{1}{2}\right|=\frac{1}{2}$.

Since for $\theta_0\leqq|\theta|\leqq\pi$, $0\leqq\frac{r-\cos\theta}{1+r^2-2r\cos\theta}\leqq\frac{1}{1+r}\leqq1$, we have $|d\psi|\leqq r\,d\theta$, so that

$$\int_0^1 dr \int_{\theta_0\leqq|\theta|\leqq\pi} \frac{|f'(re^{i\theta})|}{1+|f(re^{i\theta})|^2}|d\psi| \leqq \iint_{|z|\leqq1} \frac{|f'(re^{i\theta})|}{1+|f(re^{i\theta})|^2}r\,dr\,d\theta \leqq \sqrt{\pi A}.$$

Hence

$$I(0)\geqq\int_0^1 dr \int_{|\theta|\leqq\theta_0} \frac{|f'(re^{i\theta})|}{1+|f(re^{i\theta})|^2}d\psi-\sqrt{\pi A}$$
$$=\iint_{|z-\frac{1}{2}|\leqq\frac{1}{2}} \frac{|f'(z)|}{1+|f(z)|^2}dr\,d\psi-\sqrt{\pi A}. \tag{17}$$

Now we change variables in the double integral (17) from (r,ψ) to (t,ψ) by $z=re^{i\theta}=1-te^{-i\psi}$, then

$$dr\,d\psi=\frac{\cos\psi-t}{\sqrt{1+t^2-2t\cos\psi}}dt\,d\psi.$$

Since $1+t^2-2t\cos\psi\leqq1$ in $\left|z-\frac{1}{2}\right|\leqq\frac{1}{2}$, we have

$$\iint_{|z-\frac{1}{2}|\leqq\frac{1}{2}} \frac{|f'(z)|}{1+|f(z)|^2}dr\,d\psi=\int_{-\pi/2}^{\pi/2}d\psi\int_0^{\cos\psi}\frac{|f'(z)|}{1+|f(z)|^2}\frac{\cos\psi-t}{\sqrt{1+t^2-2t\cos\psi}}dt$$
$$\geqq\int_{-\pi/2}^{\pi/2}\frac{\cos\psi}{2}d\psi\int_0^{(\cos\psi)/2}\frac{|f'(z)|}{1+|f(z)|^2}dt=\frac{1}{2}\chi(0)=\infty,$$

so that from (17), $I(0)=\infty$. Similarly we have $I(\varphi)=\infty$ for any $e^{i\varphi}\in E$. Hence $I=\infty$, which contradicts (9), so that $C(E)=0$.

Hence there exists a certain set E on $|z|=1$, which is of logarithmic capacity zero, such that if $e^{i\theta}$ does not belong to E, then $\chi(\theta)<\infty$. If $\chi(\theta)<\infty$, we have from (2), $L(\psi)<\infty$ for almost all ψ, so thar by Lemma 5, $L(\psi)<\infty$ for all ψ, which proves the Theorem.

Remark. If in the proof, we replace $|f'(z)|/(1+|f(z)|^2)$ by $|f'(z)|$ and use Lemma 4 instead of Lemma 5, we have Theorem 3.

2.

Let $w=f(z)$ be meromorphic in $|z|<1$ and F be its Riemann surface spread over w-spere K. Let a be a point on K and K_ρ be a spherical disc of radius ρ with a as its center. Let $s(\rho)$ be the total area of the part of F, which lies above K_ρ.

If

$$\bar{n}(a)=\overline{\lim_{\rho\to 0}}\, s(\rho)/(\pi\rho^2)<\infty, \tag{1}$$

then a is called *an ordinary value of* $f(z)$.

Evidently $n(a)\leqq \bar{n}(a)$, where $n(a)$ is the number of zero points of $f(z)-a$ in $|z|<1$.

If

$$A=\iint_{|z|<1}\left(\frac{|f'(re^{i\theta})|}{1+|f(re^{i\theta})|^2}\right)^2 r\,dr\,d\theta<\infty, \tag{2}$$

then by Lebesgue's theorem, $n(a)=\bar{n}(a)$ for almost all a on K.

Beurling proved: Let $w=f(z)$ be meromorphic in $|z|<1$ and $A<\infty$ and a be an ordinary value of $f(z)$, then the set of E of points $e^{i\theta}$ on $|z|=1$, such that $\lim_{r\to 1} f(re^{i\theta})=a$ is of logarithmic capacity zero. We will prove the following more general theorem:

THEOREM 5. *Let $w=f(z)$ be meromorphic in $|z|<1$ and take certain three values finite times in $|z|<1$ and a be an ordinary value of $f(z)$. Then the set E of points $e^{i\theta}$ on $|z|=1$, such that $\lim_{r\to 1} f(re^{i\theta})=a$ is of logarithmic capacity zero.*

PROOF. Without loss of generality, we may suppose that $a=0$. Since $n(0)\leqq\bar{n}(0)$, $f(z)$ has only a finite number of zero points $z_1,\cdots,z_n$ in $|z|<1$. If ρ is small, then the part of the Riemann surface F of $w=f(z)$ above a disc $|w|(1+|w|^2)^{-1/2}\leqq\rho$ is mapped on domains $D_\rho^{(1)},\cdots,D_\rho^{(n)}$, Δ_ρ, where $D_\rho^{(i)}$ contains z_i and is bounded by a Jordan curve lying in $|z|<1$ and Δ_ρ consists of connected domains $\{\Delta_\rho^{(\nu)}\}$, which have boundary points on $|z|=1$ and at every boundary point in $|z|<1$, $|f(z)|\,(1+|f(z)|^2)^{-1/2}=\rho$.

Then by definition, for a suitable constant K,

$$K\rho^2\geqq s(\rho)\geqq\iint_{\Delta_\rho}\left(\frac{|f'(re^{i\theta})|}{1+|f(re^{i\theta})|^2}\right)^2 r\,dr\,d\theta. \tag{3}$$

Suppose that $C(E)>0$, then as in the proof of Theorem 4, if we put

$$I=\iint_{\Delta\rho}\frac{|f'(re^{i\theta})|}{1+|f(re^{i\theta})|^2}\frac{\partial u}{\partial r}\,rd\,rd\,\theta, \tag{4}$$

then

$$|I|^2\leqq\iint_{\Delta\rho}\left(\frac{|f'(re^{i\theta})|}{1+|f(re^{i\theta})|^2}\right)^2 rd\,rd\,\theta\iint_{\Delta\rho}\left(\frac{\partial u}{\partial r}\right)^2 rd\,rd\,\theta\leqq K\rho^2\iint_{\Delta\rho}\left(\frac{\partial u}{\partial r}\right)^2 rd\,rd\,\theta.$$

Since

$$\iint_{|z|<1}\left(\frac{\partial u}{\partial r}\right)^2 rd\,rd\,\theta<\infty$$

we have

$$\lim_{\rho\to 0}\iint_{\Delta\rho}\left(\frac{\partial u}{\partial r}\right)^2 rd\,rd\,\theta=0,$$

so that

$$|I|\leqq\varepsilon\,\rho,\qquad\text{where } \varepsilon\to 0 \text{ with } \rho\to 0. \tag{5}$$

As in the proof of Theorem 4, we have

$$I=\int_E I(\varphi)\,d\mu(\varphi), \tag{6}$$

where

$$I(\varphi)=\int_0^1 dr\int_{re^{i\theta}\varepsilon\Delta\rho}\frac{|f'(re^{i\theta})|}{1+|f(re^{i\theta})|^2}(-d\arg(re^{i\theta}-e^{i\varphi})). \tag{7}$$

Suppose that $z=1$ ($\varphi=0$) belongs to E, then $\lim_{r\to 1}f(r)=0$, hence by Lemma 2, $\lim f(r)=0$ uniformly, when z tends to $z=1$ in an angular domain ω, which has its vertex at $z=1$ and symmetrical to the radius of $|z|=1$ through $z=1$ and is of aparture $\pi/2$, so that the part of ω in the vicinity of $z=1$ belongs to Δ_ρ.

Let $\Delta_\rho(1)$ be the common part of ω and Δ_ρ, then if ρ is small, $\Delta_\rho(1)$ lies in a circle $\left|z-\frac{1}{2}\right|=\frac{1}{2}$, so that as the proof of Theorem 4, we have

$$I(0)\geqq\iint_{\Delta\rho(1)}\frac{|f(z)|}{1+|f(z)|^2}dr\,d\psi-\left[\iint_{\Delta\rho}\left(\frac{|f'(re^{i\theta})|}{1+|f(re^{i\theta})|^2}\right)^2 rd\,rd\,\theta\iint_{\Delta\rho}\left(\frac{\partial u}{\partial r}\right)^2 rd\,rd\,\theta\right]^{1/2}$$

$$\geqq \iint_{\Delta_\rho(1)} \frac{|f'(z)|}{1+|f(z)|^2} dr\, d\psi - \varepsilon_1 \rho \geqq \int_{-\pi/4}^{\pi/4} \frac{\cos\psi}{2} d\psi \int_{l_\psi} \frac{|f'(z)|}{1+|f(z)|^2} dt - \varepsilon_1\rho, \qquad (8)$$

where $\varepsilon_1 \to 0$ with $\rho \to 0$ and l_ψ is the part of the line $\psi = \text{const.}$, which is contained in $\Delta_\rho(1)$. As remarked before, l_ψ contains a segment, which connects $z = 1$ to a boundary point of Δ_ρ, so that the im-ge of l_ψ on w-sphere contains an arc, which connects $w = 0$ with a point on a circle $|w|/(1 + |w|^2)^{1/2} = \rho$, so that

$$\int_{l_\psi} \frac{|f'(z)|}{1+|f(z)|^2} dt \geqq \rho,$$

hence $I(0) \geqq \rho/\sqrt{2} - \varepsilon_1\rho$. Similarly we have $I(\varphi) \geqq \rho\sqrt{2} - \varepsilon_1\rho$ for any $e^{i\theta} \in E$, so that from (6), $I \geqq \rho/\sqrt{2} - \varepsilon_1\rho$.

From (5), we have

$$\rho/\sqrt{2} - \varepsilon_1\rho \leqq I \leqq \varepsilon\rho \quad \text{or} \quad 1/\sqrt{2} - \varepsilon_1 \leqq \varepsilon,$$

which is absurd, since $\varepsilon \to 0$, $\varepsilon_1 \to 0$ with $\rho \to 0$. Hence $C(E) = 0$.

3.

Let D be a simply connected domain on w-plane, which does not contain $w = \infty$ as an inner point and Γ be its boundary.

We map D conformally on $|z| < 1$ by $w = f(z)$. Let a be an access ble boundary point of D and e be the set of points $e^{i\theta}$ on $|z| = 1$, such that $\lim_{r \to 1} f(re^{i\theta}) = a$. Since a is an ordinary value of $f(z)$, we have by Theorem 5, as Beurling remarked, e is of logarithmic capacity zero. We will prove:

THEOREM 6. *Let E be a closed set of accessible boundary points on Γ, which is of logarithmic capacity zero and E correspond to a set e on $|z| = 1$, then e is of logarithmic capacity zero.*

PROOF. Since any simply connected domain can be mapped on a bounded domain, we may assume that D is bounded.

Since E is closed, by Lemma 1, we can distribute a positive mass $d\mu(a)$ of total mass 1 on E, such that

$$u(w) = \int_E \log \frac{1}{|w - a|} d\mu(a) \qquad (\mu(E) = 1) \qquad (1)$$

tends to ∞, when w tends to any point of E. Hence the niveau curve C_r: $u(w) = \text{const.} = r$ consists of a finite number of Jordan curves. which cluster to E as $r \to \infty$.

If we put

$$\int_E \log \frac{1}{|w-a|} d\mu(a) = u(w) + iv(w), \tag{2}$$

then

$$\int_{C_r} dv = \int_{C_r} \frac{\partial v}{\partial s} ds = \int_{C_r} \frac{\partial u}{\partial \nu} ds = 2\pi, \tag{3}$$

where ds is the arc element and ν is the inner normal of C_r.

Let

$$t = t(z) = \int_E \log \frac{1}{f(z)-a} d\mu(a) = \int_E \log \frac{1}{w-a} d\mu(a) = u + iv,$$

then $t(z)$ is regular in $|z| < 1$. Since $u \to \infty$, as w tends to E and D is bounded, we can find a positive constant $c > 0$, such that $u(w) + c \geqq 1$ for any point w of D, hence if we put

$$\zeta = \zeta(z) = (t(z) + c)^{1/3}, \tag{5}$$

then $\zeta(z)$ is regular in $|z| < 1$. Let A be the area on ζ-plane, which is described by $\zeta = \zeta(z)$ ($|z| < 1$), then since $d\zeta = \frac{1}{3} \frac{dt}{(t+c)^{2/3}}$, we have

$$A = \frac{1}{9} \iint_\Delta \frac{du\, dv}{((u+c)^2 + v^2)^{2/3}}, \tag{6}$$

where Δ is the Riemann surface on $t = (u+iv)$-plane, which is described by $t = u(w) + iv(w)$, when w varies in D.

Let $C_r(D)$ be the part of C_r contained in D, then by (3),

$$\int_{C_r(D)} dv \leqq 2\pi,$$

so that

$$A \leqq \frac{1}{9} \int_{1-c}^{\infty} du \int_{C_r(D)} \frac{dv}{((u+c)^2 + v^2)^{2/3}} \leqq \frac{1}{9} \int_{1-c}^{\infty} \frac{du}{(u+c)^{4/3}} \int_{C_r(D)} dv$$
$$\leqq \frac{2\pi}{9} \int_1^{\infty} \frac{d\tau}{\tau^{4/3}} = \frac{2\pi}{3} < \infty.$$

Hence by Theorem 1, the set e' of point $e^{i\theta}$ on $|z| = 1$, such that $\lim_{r\to 1} |\zeta(re^{i\theta})| = \infty$ is of logarithmic capacity zero. From

$$\int_E \log \frac{1}{w-a}\, d\mu(a) + c = \int_E \log \frac{1}{f(z)-a}\, d\mu(a) + c = \zeta^s(z),$$

we see that e' coincide with e. Hence $C(e) = 0$. q. e. d.

In the general case, where E is not closed, if the boundary of D is a Jordan curve Γ, then we can prove that $C(e) = 0$ as follows.

Suppose that $C(e) > 0$, then e contains a closed subset e', such that $C(e') > 0$. Let e' correspond to E' on Γ, then E' is closed and $C(E') = 0$. Hence by Theorem 6, $C(e') = 0$, which contradicts $C(e') > 0$, so that $C(e) = 0$. Hence we have:

THEOREM 7. *Let Γ be a Jordan curve on w-plane and E be a set of logarithmic capacity zero on Γ. If we map the inside of Γ on $|z| < 1$ conformally, then E corresponds to a set of logarithmic capacity zero on $|z| = 1$.*

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