ON THE RELATION BETWEEN HOMOLOGICAL STRUCTURE OF RIEMANNIAN SPACES AND EXACT DIFFERENTIAL FORMS WHICH ARE INVARIANT UNDER HOLONOMY GROUPS. I

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1. Introduction. Let h be a closed subgroup of the proper rotation group $O^+(n)$ in an n dimensional Euclidean space E_n, and y^i be a set of coordinates. Consider an exterior differential form ω of rank p

(1.1) $$\omega = a_{i_1 i_2 \cdots i_p} dy^{i_1} dy^{i_2} \cdots dy^{i_p} \;^{1)}$$

with constant coefficients $a_{i_1 \cdots i_p}$ ($a_{i_1 \cdots i_p}$'s are skew symmetric with respect to their indices). If ω is invariant under every transformation of the group h, we say that ω is invariant under h.

Now, consider an orientable Riemannian manifold M_n of class C''' with positive definite fundamental quadratic form. Let O be an arbitrary point of M_n and h be the homogeneous holonomy group[2] (throughout this paper we shall call it holonomy group for brevity) referred to the natural frame e_i at O with respect to some coordinate neighbourhood of O. Then, we can easily see that h is a closed subgroup of the proper orthogonal group $O^+(n)$. Let P be another point of M_n and C be a curve which combine O with P. Then we can transplant the natural frame at O by Levi-Civita's parallelism in the tangent space at P. We shall denote the frame by $e_i(P, C)$. If we denote the components of the transplanted vector at P of the infinitesimal vector dy^i at O with respect to an arbitrary allowable coordinate neighbourhood x^i at P by dx^i, then we have

(1.2) $$dy^i = b^i_j\, dx^j \qquad |b^i_j| \neq 0.$$

Of course, the constants b^i_j depend on the curve C. Hence if we put the last equation into (1.1), we get an exterior form of the type

(1.3) $$a_{i_1 \cdots i_p}(P, C)\, dx^{i_1} \cdots dx^{i_p}.$$

If we assume that the exterior differential form ω is invariant under the

*) The contents of this paper is essentially the first half of the second posthumous manuscript of late Mr. Iwamoto (Cf. Footnote of the paper 2). Putting the contents of his manuscript in order, adding proofs and translating in English, we publish it here. Of all contents of this paper, the translator takes the responsibility. (Translator : S. Sasaki)

1) Throughout this paper, we assume that indices take the following values :

$$i, j, k = 1, 2, \cdots\cdots, n,$$
$$a, b, c = 1, 2, \cdots\cdots, p \quad (p<n),$$
$$\alpha, \beta, \gamma = p+1, \cdots\cdots, n,$$
$$\lambda, \mu, \nu = 1, 2, \cdots\cdots, m \quad (m<n),$$
$$\xi, \eta, \zeta = m+1, \cdots\cdots, n,$$

2) Cf. H. IWAMOTO, On the structure of Riemannian Spaces whose holonomy groups fix a null system, this Journal (2)1 no. 2 (1950) pp. 109–135.

group h, then we can easily see that the coefficients of (1.3) do not depend on the curve C joining O to P and depend only on P. Hence, if the form (1.1) is invariant under the homogeneous holonomy group h, then there exists an exterior differential form

(1.4) $$\omega = A_{i_1\cdots i_p}(x)\, dx^{i_1}\cdots dx^{i_p}$$

defined over the whole manifold M_n. Hereafter we shall call such differential form as exterior differential form of rank p on M_n invariant under the holonomy group h. It is evident that the coefficients $A_{i_1 i_2\cdots i_p}(x)$ are of class C' and satisfy the remarkable relation

(1.5) $$A_{i_1 i_2\cdots i_p,k} = 0,$$

where comma denotes covariant derivative.

An exterior differential form ω defined over the whole mainfold M_n is called *exact* or *closed* if the exterior derivative ω' vanishes identically and is called *derived* or *a null form* if ω may be regarded as the exterior derivative of another differential form Π which is of rank $p-1$. According to de Rham's theory[3] it is well known that the theory of exact differential forms on any closed orientable manifold M_n is in a close relation with the homological structure of M_n. And according to Hodge's theory[4] of harmonic integrals, when M_n is Riemannian manifold, the same relation holds good even if we restrict the set of exact differential forms only to the set of harmonic differential forms.

In the present paper we shall investigate relations between homological structure of any closed orientable Riemannian manifold M_n and exterior differential forms on M_n invariant under the holonomy group h.

2. Fundamental theorems.

THEOREM 1. *Every exterior differential form which is invariant under the holonomy group h is exact.*

PROOF. Let (1.4) be the differential form in consideration. Then, by (1.5), we get

(2.1) $$A_{[i_1 i_2\cdots i_p,k]} = 0,$$

whence we can easily deduce the desired relation $\omega' = 0$.

THEOREM 2. *If a differential form ω which is invariant under the holonomy group h is derived, then ω vanishes identically.*

PROOF. Let ω defined by (1.4) be a derived form of rank p which is invariant under the holonomy group h. We construct from it the dual form ω^*, i. e.

(2.2) $$\omega^* = A_{i_1\cdots i_{n-p}}dx^{i_1}\cdots dx^{i_{n-p}},$$

where we have put

(2.3) $$A_{i_1 i_2\cdots i_{n-p}} = \sqrt{g}\, A^{j_1 j_2\cdots j_p}\varepsilon_{j_1 j_2\cdots j_p i_1 i_2\cdots i_{n-p}}.$$

3) G. DE RHAM, Sur l'analysis situs des variétés à n dimension. J. Math. Pures et Appl. 10 (1931) pp. 115–200.

4) W. V. D. HODGE, The theory and applications of harmonic integrals. (1940).

We can easily see that ω^* is also an invariant differential form of the group h. Hence by Theorem 1, ω^* is also an exact differential form. Consequently, we know that ω is a harmonic differential form in the sense of Hodge.

Now, consider the differential form of rank n

(2.4) $$\Omega = \omega\omega^* = \sqrt{g}\,A_{i_1\dots i_p}\varepsilon_{j_1\cdots j_p k_1\cdots k_{n-p}}A^{j_1\cdots j_p}dx^{i_1}\cdots dx^{i_p}dx^{k_1}\cdots dx^{k_{n-p}}.$$

We can easily verify that Ω is reducible to the following form

(2.5) $$\Omega = A^2\sqrt{g}\,dx^1dx^2\cdots dx^n,$$

where we have put

(2.6) $$A^2 = A_{i_1\dots i_p}A^{i_1\cdots i_p}.$$

As the tensor $A_{i_1\dots i_p}$ saisfies (1.5) by hypothesis, we see that A is constant over the whole manifold M_n. It is not zero unless ω vanishes identically, for A^2 is the square of the tensor $A_{i_1\dots i_p}$. Accordingly, $\int_{M_n}\Omega$ is a harmonic integral and its value does not vanish. Whence, we can deduce that Ω can never be a derived form, for if $\Omega = \Pi'$, we see

$$\int_{M_n}\Omega = \int_{\partial M_n}\Pi = 0,$$

which contradicts to the fact stated above.

Now, by hypothesis, ω is a derived form. Hence, if we put

$$\omega = \omega_1',$$

then we have

$$(\omega_1\omega^*)' = \omega_1'\omega^* = \omega\omega^* = \Omega,$$

which shows that Ω is a derived-form. Hence ω can not be a derived form unless ω vanishes identically. Q. E. D.

Now, consider the totality of exact differential forms of rank p and denote it by Z_p. We can divide elements of Z_p into classes (we shall call them homology classes for brevity) each of which is constituted of forms of rank p which are homologous to each other. Then, according to de Rham's theory, the maximum number of linearly independent homology classes in Z_p is equal to the p-th Betti number of the manifold M_n. On the other hand, a homology class of Z_p may not contain differential forms which are invariant under the holonomy group h. But, if the homology class contains a differential form ω which is invariant under h, then we can easily see, in virtue of Theorem 2, that any other invariant differential forms under h which belong to the homology class in consideration are necessarily constant multiples of ω. Hence we obtain the following theorem:

THEOREM 3. *Let B_p be the p-th Betti number of a closed orientable Riemannian mainfold M_n and B_p' the maximum number of linearly independent (in the sense of algebra) differential forms of rank p which are invariant* *under the holonomy group h, then the following relation holds good*:

(2.7) $$B_p \geqq B'_p.$$ [5)]

It is well known that the direct sum of Z_p's $(p = 0, 1, \cdots, n)$ constitutes a ring R and the direct sum of the set of null forms N_p constitutes a ideal I of R. The residue class ring R/I is a topological invariant of the given manifold M_n. On the other hand we can easily recognize that the direct sum of the set of exact differential forms H_p which are invariant under the holonomy group h constitutes also a subring of R. It will be easily seen that the last subring can be regarded as a subring of R/I. Hence we can restate Theorem 3 in the following way:

THEOREM 4. *In order that we can metrize a given closed orientable manifold of class* C^r $(r \geqq 3)$ *so that its holonomy group is an arbitrary preassigned group* h *(subgroup of the proper orthogonal group* $O^+(n)$*), it is necessary that the residue class ring* R/I *contains a ring isomorphic to the ring* R_0 *which is the set of exterior forms of* E_n *invariant under the given orthogonal group* h *as a subring.*

3. Homological structure of closed orientable Riemannian manifolds whose holonomy groups fix an oriented p dimensional plane.

THEOREM 5. *Assume that the holonomy group* h *of a closed orientable Riemannian manifold* M_n *be reducible (in the field of real numbers) and fixes an oriented* p*-dimensional plane* E_p. *Then both Betti numbers* B_p *and* B_{n-p} *do not vanish.*

PROOF. When the holonomy gronp h fixes an oriented plane E_p, we can choose orthogonal frames at each point of M_n so that the equations which define the Euclidean connexion of the space take the following form :[6)]

(3.1) $$\begin{cases} dP = \omega_a e_a + \omega_\alpha e_\alpha, \\ de_a = \omega_{ab} e_b, \\ de_\alpha = \omega_{\alpha\beta} e_\beta. \end{cases}$$

A part of the equations of structure of the space, i. e. the equations which express the condition that the Euclidean connexion in consideration is torsionless are given by

(3.2) $$\begin{cases} (\omega_a)' = \omega_{ab}\omega_b, \\ (\omega_\alpha)' = \omega_{\alpha\beta}\omega_\beta. \end{cases}$$

Now the p vectors e_a span the invariant plane E_p of the holonomy

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5) Mr. Iwamoto states without any indication of the proof that "If the Riemannian manifold in consideration is symmetric in the sense of Cartan, then B_p is equal to B_p''". I shall prove it in my paper "On a theorem concerning the homological structure and the holonomy groups of closed orientable symmetric spaces", which will be shortly published For the method of proof is quite different from that of this paper. (Translator)

6) M. ABE. Sur la réductibilité du groupe d'holonomie II. Les espaces de Riemann, Proc. Imp. Acad. Tôkyô, 20. pp. 177—182.

group h and $(n-p)$ vectors e_α span the plane E_{n-p} which is completely orthogonal to E_p. From the assumption that the group h fixes the oriented plane E_p we can conclude that the group h also fixes the oriented plane E_{n-p}. Hence the induced group of orthogonal transformations $h(p)$ on E_p and $h(n-p)$ on E_{n-p} by h are subgroups of $O^+(p)$ and $O^+(n-p)$ respectively.

In the next place, consider a set of Pfaffian equations

$$\omega_{p+1} = \cdots\cdots = \omega_n = 0. \tag{3.3}$$

We can easily see that the last equations are completely integrable. Hence there exists a set of variables y^α such that

$$\omega_\alpha = a_{\alpha\beta}(x^i)dy^\beta.$$

It is well known that ∞^{n-p} p-dimensional varieties V_p's defined by $y^\alpha =$ const. are totally geodesic. Through every point of M_n there passes one and only one such variety V_p and p vectors e_a are tangent to it. The same is true for the set of Pfaffian equations $\omega_1 = \cdot\cdot = \omega_p = 0$ and $(n-p)$ vectors e_α. On the other hand, the volume element

$$\Omega = \sqrt{g}\, dx^1 dx^2 \cdots\cdot dx^n \tag{3.4}$$

of the given Riemannian manifold M_n can be written as

$$\Omega = \omega_1\omega_2\cdots\cdot\omega_n. \tag{3.5}$$

It is evident that Ω is invariant under the group of orthogonal transformations $O^+(n)$ and hence under h. We can also see that

$$\omega = \omega_1\omega_2\cdots\cdot\omega_p, \tag{3.6}$$

$$\bar{\omega} = \omega_{p+1}\omega_{p+2}\cdots\cdot\omega_n$$

are volume elements of V_p and V_{n-p} stated above and invariant differential forms of the group $h(p)$ and $h(n-p)$ respectively and hence of h.

Now, as Ω is the volume element of M_n, $\Omega \neq 0$, hence $\omega \neq 0$ and $\bar{\omega} \neq 0$. Accordingly, we see, in virtue of Theorem 2, that both ω and $\bar{\omega}$ are not derived forms. Consequently, we can conclude that $B_p > 0$ and $B_{n-p} > 0$.

THEOREM 6. *When there exist a closed orientable manifold S in the set of totally geodesic varieties V_p defined in the proof of Theorem 5, then the manifold S can not be homologous to zero.*

PROOF. Assume that S be the boundary of a $(p+1)$ dimensional region D, then we can orient D and S so that $\partial D = S$. By Stokes' theorem, we obtain

$$\int_D \omega' = \int_S \omega.$$

On the other hand, as ω is a differential form invariant under the holonomy group h, ω is exact. Hence we get

$$\int_S \omega = 0.$$

However, as ω is volume element of the closed orientable manifold S,

$\int \omega$ can not vanish. Hence we meet a contradiction. Consequently, there does not exist any region D such that $\partial D = S$. Q. E. D.

4. Homological structure of closed orientable Riemannian manifolds whose holonomy groups fix a null system which is commutative with the fundamental polarity. At an arbitrary point P_0 of an orientable Riemannian mainfold M_n, vectors in the tangent Euclidean space $E_n(P_0)$ constitute an n-dimensional vector space $R_n(P_0)$. If we understand contravariant vectors X^i and covariant vectors U_i in $R_n(P_0)$ as if they were homogeneous point - and hyperplanecoordinates, the $R_n(P_0)$ may be regarded as an $(n-1)$-dimensional projective space P_{n-1}.

Let us assume that the dimension of the manifold in consideration is even, and put it $2n$ for convenience sake. Let S_{AB} be a skew symmetric covariant tensor at P_0 with non vanishing determinant. Then the correspondence

$$X^A \rightarrow S_{AB}X^B$$

may be regarded as a null-correlation in P_{2n-1}. On account of this fact we shall hereafter call such $S = (S_{AB})$ as a null system at P_0 in M_{2n}.

Now, if the holonomy group h of M_{2n} constructed at P_0 fixes the nullsystem S, then we can, as was done in § 1, transplant S to each point of M_{2n} by the Euclidean connexion of the Riemannian manifold M_{2n}. Hence, we get a field of skew symmetric tensor $S_{AB}(x)$ with non vanishing determinant over the whole manifold M_{2n}. We shall call such tensor field as a null-system which is invariant under the holonomy group h.

On the other hand, the fundamental tensor $G = (g_{AB})$ of M_{2n} at P_0 determines a polarity in P_{2n-1}. We shall call it the fundamental polarity and denote it by G. It is evident that the polarity G is invariant under the holonomy group h. When the components of S and G satisfy the relation

$$G^{-1}S = -S^{-1}G\,, \tag{4.1}$$

then we shall say that the fundamental polarity and null-correlation are commutative. When the holonomy group h of the Riemannian manifold fixes a null-system S, two fields of tensors G and S defined over the whole manifold M_{2n} satisfy also the last relation. In my previous paper[7] I investigated the local structure of Riemannian spaces whose holonomy groups fix a null system which is commutative with the fundamental polarity. If the holonomy group h is irreducible in the field of real numbers and reducible in the field of complex numbers, there exists a null system S having the above stated property. Of course, the components of the skew symmetric tensor satisfy the following equation

$$S_{AB,C} = 0. \tag{4.2}$$

THEOREM 7. *If the holonomy group h of a closed orientable Riemannian*

7) Cf. loc. cit. 2).

manifold M_{2n} fixes a null system which is commutative with the fundamental polarity of M_{2n}, then even dimensional Betti numbers of M_{2n} are not equal to zero.

PROOF. Let us put

$$\omega = S_{AB}dx^A dx^B \tag{4.3}$$

and consider n differential forms

$$\omega_{2p} = \underbrace{\omega\omega\cdots\cdot\omega}_{p} \quad (1 \leqq p \leqq n). \tag{4.4}$$

Then. ω_{2n} can be written also as

$$\omega_{2n} = \Psi dx^1 dx^2 \cdots\cdot dx^{2n}, \tag{4.5}$$

where we have put

$$\Psi = \sum \text{sign} \begin{pmatrix} A_1 A_2 \cdot\cdot A_{2n} \\ 1 \;\; 2 \;\cdot\cdot\; 2n \end{pmatrix} S_{A_1A_2}S_{A_3A_4}\cdots\cdot S_{A_{2n-1}A_{2n}}.$$

On the other hand, it is well known that

$$|S_{AB}| = \Psi^2.$$

However, as (4.1) shows us, we have

$$|g_{AB}|^2 = |S_{AB}|^2.$$

Hence, we get

$$\Psi^2 = |g_{AB}|.$$

Accordingly, we see that ω_{2n} is the volume element of the Riemannian manifold in consideration:

$$\omega_{2n} = \sqrt{g}\;\; dx^1 dx^2 \cdots\cdot dx^{2n}. \tag{4.6}$$

Consequently ω_{2n} does not vanish.

Now, let us fix the index p and suppose that ω_{2p} is a derived form. Then as ω_{2p} is also an invariant differential form of the holonomy group h, we see, by virtue of Theorem 2, that ω_{2p} is identically zero. Hence, ω_{2n} is identically zero too, which contradicts that ω_{2n} is the volume element of the Riemannian manifold in consideration. Accordingly, ω_{2p} is not a derived form. Consequently, we can conclude, by virtue of Theorem 3, that $B_{2p} > 0$. Q. E. D.

In the next place we shall consider an even dimensional submanifold

$$M_{2m}: \qquad x^A = x^A(u^1, u^2, \cdots, u^{2m}), \qquad (m < n),$$

and put

$$\begin{cases} g_{\lambda\mu} = g_{AB}B^A_\lambda B^B_\mu \\ S_{\lambda\mu} = S_{AB}B^A_\lambda B^B_\mu \end{cases} \tag{4.7}$$

where B^A_λ means

$$B^A_\lambda = \frac{\partial x^A}{\partial u^\lambda}.$$

When the relation

$$g^{AB}S_{BC}B^C_\nu = g^{\lambda\mu}S_{\mu\nu}B^A_\lambda \tag{4.8}$$

is satisfied at every point of M_{2m}, we shall say that the M_{2m} in consideration is a ***proper submanifold*** of M_{2n}.

Let us first consider the geometrical meaning of the "proper" submanifold. The equation (4.8) can be written also as

$$S_{BC}B^C_\nu = g_{AB}g^{\lambda\mu}S_{\mu\nu}B^A_\lambda.$$

If we contract an arbitrary vector X^B with both sides of the last equation we get

$$S_{BC}X^B B^C_\nu = g^{\lambda\mu}S_{\mu\nu}(g_{AB}B^A_\lambda X^B).$$

On the other hand, we see that

$$|g^{\lambda\mu}S_{\mu\nu}| = |g^{\lambda\mu}|\,|S_{\mu\nu}| \neq 0.$$

Hence, we can conclude that two equations

$$S_{BC}X^B B^C_\nu = 0 \tag{4.9}$$

and

$$g_{BC}X^B B^C_\nu = 0 \tag{4.9$'$}$$

are equivalent to each other.

If we write equation (4.1) in detail we get

$$g^{AB}S_{BC}g^{CD}S_{DE} = -\delta^A_E. \tag{4.10}$$

Now, putting

$$B^\lambda_A = g_{AF}g^{\lambda\nu}B^F_\nu \tag{4.11}$$

and contracting $B^E_\mu B^\lambda_A$ with both sides of (4.10), we get

$$g^{\lambda\nu}B^B_\nu B^E_\mu S_{BC}g^{CD}S_{DE} = -(g_{AF}g^{\lambda\nu}B^F_\nu)\,B^A_\mu. \tag{4.12}$$

The submanifold M_{2m} has $2(n-m)$ mutually orthogonal normal vectors at each point of it. If we denote these vectors by $\underset{(\xi)}{N^A}$ ($\xi = 1, 2, \cdots, 2(n-m)$), we can write g^{CD} in the following form:

$$g^{CD} = g^{\rho\sigma}B^C_\rho B^D_\sigma + \sum_\xi \underset{(\xi)}{N^C}\,\underset{(\xi)}{N^D}. \tag{4.13}$$

If we put the last relation in (4.12) and notice that (4.9) and (4.9)′ are equivalent to each other, we get the relation

$$g^{\lambda\nu}S_{\nu\rho}g^{\rho\sigma}S_{\sigma\mu} = -\delta^\lambda_\mu \tag{4.14}$$

which is analogous to (4.10). Accordingly, the null-system $S_{\lambda\mu}$ defined over the whole manifold M_{2m} is commutative with the polarity defined by the fundamental tensor $g_{\lambda\mu}$.

The Euler-Schouten's tensor $B^A_{\lambda,\mu}$ is perpendicular to B^A_μ. Hence, by virtue of the fact that (4.9) and (4.9)′ are equivalent, we get

$$S_{AB}B^A_{\lambda,\nu}B^B_\mu = 0. \tag{4.15}$$

Accordingly, we see that the following relation holds good:

$$S_{\lambda\mu,\nu} = (S_{AB}B^A_\lambda B^B_\mu)_{,\nu} = 0. \tag{4.16}$$

Consequently, the holonomy group of the proper submanifold M_{2m} also fixes a null-system $S_{\lambda\mu}$ which is commutative with the fundamental polarity $g_{\lambda\mu}$.

We derived the relation (4.14) from (4.8). But, conversely, we can derive the relation (4.8) from (4.14). To show this, we put first (4.13) into (4.12) and making use of (4.14), we get

$$\sum_\xi g^{\lambda\nu}B^B_\nu B^E_\mu S_{BC}\underset{(\xi)}{N^C}\,\underset{(\xi)}{N^D}S_{DE} = 0.$$

The last equation can be written also

$$\sum_{\xi} g^{\lambda\nu}(S_{BC}B^B_\nu \underset{(\xi)}{N^C})(S_{DE}B^E_\mu \underset{(\xi)}{N^D}) = 0,$$

hence we get

$$(S_{BC}B^B_\nu \underset{(\xi)}{N^C})(S_{DE}B^E_\mu \underset{(\xi)}{N^D}) = 0.$$

If we consider the special case such that $\nu = \mu$, the last equation becomes

$$\sum_{\xi}(S_{BC}B^B_\mu \underset{(\xi)}{N^C})^2 = 0$$

whence we get

(4.17) $$S_{BC}B^B_\mu \underset{(\xi)}{N^C} = 0.$$

If we notice the last relation, we can easily see that the equation

$$g^{AB}S_{BC}B^C_\nu \doteq (g^{\lambda\mu}B^A_\lambda B^B_\mu + \underset{(\xi)}{N^A}\underset{(\xi)}{N^B})S_{BC}B^C_\nu$$

reduces to (4.8). Q. E. D.

In my previous paper, I proved the following facts: If the holonomy group h of a Riemannian manifold M_{2n} fixes a null-system S which is commutative with the fundamental polarity G, then the differential equations

(4.18) $$\left(S_{BC}g^{CD}\frac{\partial\phi}{\partial x^D}dx^B\right)' = 0$$

admits n functionally independent solutions ϕ^i. From the last equation we see that $S_{BC}g^{CD}\frac{\partial\phi^i}{\partial x^D}$'s are gradient vectors of n functions. We denote these functions by ψ^i and call them the conjugate functions of ϕ^i. On account of (4.1) we can easily see that ϕ^i are conjugate functions of ψ^i. In my previous paper mentioned above, we also proved that, if we put

(4.19) $$Z^i = \phi^i + \sqrt{-1}\,\psi^i,$$

the fundamental quadradic form of the Riemannian manifold in consideration can be written in Kähler's form

(4.20) $$ds^2 = \frac{\partial^2 U}{\partial z^i \partial \bar{z}^j}dz^i d\bar{z}^j,$$

where $\bar{z}^j$ is conjugate complex of z^j.

Now, let us consider the generalized Cauchy-Riemann's equation

(4.21) $$\frac{\partial\Psi}{\partial x^A} = S_{AB}g^{BC}\frac{\partial\Phi}{\partial x^C}.$$

If we contract B^A_λ with both sides of the last equation we get

$$\frac{\partial\Psi}{\partial x^A}B^A_\lambda = S_{AB}B^A_\lambda\left(g^{\mu\nu}B^B_\mu B^C_\nu + \sum_{\xi}\underset{(\xi)}{N^B}\underset{(\xi)}{N^C}\right)\frac{\partial\Phi}{\partial x^C},$$

which reduces, by virtue of (4.17), to the following equation:

(4.22) $$\frac{\partial\Psi}{\partial u^\lambda} = S_{\lambda\mu}g^{\mu\nu}\frac{\partial\Phi}{\partial u^\nu}.$$

Accordingly, if we take m sets of independent solutions $\Phi^\lambda + \sqrt{-1}\,\Psi^\lambda$ of the generalized Cauchy-Riemann's equations (4.21) of M_{2n}, then they are also those of the generalized Cauchy-Riemann's equations (4.22) of M_{2m}.

Hence, if we put $z^\lambda = \Phi^\cdot + \sqrt{-1}\,\Psi^\lambda$, then z^λ's are complex coordinates of M_{2m} and the induced metric on M_{2m} from M_{2n} can be written as

$$d\sigma^2 = \frac{\partial^2 V}{\partial z^\lambda \partial \bar{z}^\mu} dz^\lambda d\bar{z}^\mu. \tag{4.23}$$

If we denote the $(n-m)$ sets of remaining solutions of (4.19) which are independent to each other and to $\Phi^\lambda + \sqrt{-1}\,\Psi^\lambda$ by $\Phi^\xi + \sqrt{-1}\,\Psi^\xi$, the manifold M_{2m} can be represented by

$$Z^\xi \equiv \Phi^\xi + \sqrt{-1}\Psi^\xi = \text{const.},$$

consequently, we can conclude the following theorem:

THEOREM 8. *If a submanifold M_{2m} in an orientable Riemannian manifold M_{2n} whose homogeneous holonomy group fixes a null system S which is commutative with the fundamental polarity is proper, then M_{2m} is an analytic subvariety in M_{2n} with respect to complex coordinate systems of M_{2n}. The converse is also true.*

We shall now prove the following theorem:

THEOREM 9. *Suppose that the holonomy group h of a closed orientable Riemannian manifold M_{2n} fixes a null-system S which is commutative with the fundamental polarity of M_{2n}. If M_{2n} admits a closed orientable proper submanifold M_{2m} such that $g_{\lambda\mu}$ and $S_{\lambda\mu}$ are regular over the whole manifold M_{2m}, then the manifold M_{2m} can not be homologous to zero.*

PROOF. Take the differential form ω_{2m} defined by (4.4) on the submanifold M_{2m}. Then, by virtue of the relation

$$dx^A = B^A_\lambda du^\lambda,$$

ω_{2m} can be written as

$$\omega_{2m} = \underbrace{\tilde{\omega}\tilde{\omega}\cdots\tilde{\omega}}_{m} \tag{4.24}$$

where we have put

$$\tilde{\omega} = S_{\lambda\mu} du^\lambda du^\mu. \tag{4.25}$$

Hence, as in the proof of Theorem 7, we can see that ω_{2m} is the volume element of the submanifold M_{2m}. Accordingly, by virtue of the fact that M_{2m} is closed orientable, it is evident that

$$\int_{M_{2m}} \omega_{2m} \neq 0. \tag{4.26}$$

The last inequality shows that M_{2m} in consideration can not be the boundary of a $(2m+1)$ dimensional region in M_{2n}. For, if we assume that M_{2m} is the boundary of a $(2m+1)$ dimensional region D, then we get

$$\int_{M_{2m}} \omega_{2m} = \int_{\partial D} \omega_{2m} = \int_{D} \omega'_{2m}.$$

However, as ω_{2m} is invariant under the holonomy group h, ω_{2m} is a closed form. Hence we get

$$\int_{M_{2m}} \omega_{2m} = 0,$$

which contradicts to (4.26). Consequently, the submanifold M_{2m} in consideration can not be homologous to zero. Q. E. D.

5. An application of Theorem 7.

THEOREM 10. *Any complex m dimensional algebraic variety in an n dimensional complex projective space can not be homologous to zero.*

PROOF. Let P_n be a complex projective space of n dimensions and $(z_0, z_1, \cdots, z_n)$ be a set of homogeneous coordinates of it. We shall call the group of all projective transformations of P_n which fixes the Hermitian form $z_0\bar{z}_0 + z_1\bar{z}_1 + \cdots + z_n\bar{z}_n$ the unitary group $U(n+1)$. Let us denote the point $(1, 0, \cdots, 0)$ by O. Then the subgroup u of the unitary group $U(n+1)$ which fixes the point O is constituted from transformations of the type

$$z_0' = a_{00}z_0, \quad z_i' = \sum a_{ij}z_j$$

provided that the coefficients satisfy the following conditions:

$$a_{00}\bar{a}_{00} = 1, \quad \sum a_{ij}\bar{a}_{ik} = \delta_{jk}.$$

The group u contains transformations which does not operate P_n effectively. They are transformations of the type

$$z_0' = e^{i\theta}z_0, \qquad z_k' = e^{i\theta}z_k,$$

and constitute an invariant subgroup N of $U(n+1)$. The residue class group $G = U(n+1)/N$ operate effectively on P_n and the transformations of $g \equiv u/N$ are given by the following equation:

$$z_0' = z_0, \quad \sum z_i' = a_{ij}z_j, \tag{5.1}$$

provided that the coefficients satisfy the following conditions:

$$\sum a_{ij}\bar{a}_{ik} = \delta_{jk}. \tag{5.2}$$

As G is a transitive group which operates on P_n and g is a closed subgroup of G, the set of groups G and g determines a homogeneous space on the manifold P_n. G and g are called the group of structure and the group of isotropy of the homogeneous space.

Now, we can easily verify that the transformation σ_0 in P_n defined by

$$\sigma_0: \qquad z_0' = z_0, \qquad z_i' = -z_i$$

is an involutive automorphism of G and O is an isolated invariant point of σ_0, and g is the set of all transformations of G which are invariant under σ_0. Hence the homogeneous space in consideration is a symmetric space. Moreover, as g is a linear group which is defined by algebraic relations between their coefficients, it is evident that g is compact. Consequently, the symmetric space in consideration is a symmetric Riemannian space.

We can easily calculate the linear group of isotropy g^* of the symmetric Riemannian space. It is the unitary group $U(n)$. The metric of the symmetric Riemannian space is easily seen to be

$$ds^2 = \sum \omega_{0i}\bar{\omega}_{0i}, \tag{5.3}$$

where ω_{0i} is the components of the infinitesimal transformation $\delta_{\rho\sigma}+\omega_{\sigma\rho}$ of G. It is clear that the line element of the Riemannian space in consideration is invariant under the linear group of isotropy g^*.

If we consider the real representation of the linear group of isotropy g^*, all the transformations of the group g^* fixes a positive definite quadratic from G which is the real representation of the Hermitian form. Moreover, as g^* is $U(n)$, the real representation of g^* fixes the collineation

$$I=\begin{pmatrix}0 & -E\\ E & 0\end{pmatrix} \tag{5.4}$$

and hence the null system

$$S=GI=IG. \tag{5.5}$$

On the other hand, in symmetric Riemannian space, the linear group of isotropy g^* coincides with the holonomy group h of the space in consideration. Hence, we can conclude that the holonomy group h of the symmetric Riemannian space in consideration fixes a null system S which is commutative with the fundamental polarity G.

We must also prove the orientability of the symmetric Riemannian space in consideration i. e. the orientability of the complex projective space P_n. If we call points in P_n such that $z_n=0$ as points at infinity, then the set of all points at infinity is homeomorphic to P_{n-1} and the set of all finite points is homeomorphic to real $2n$ dimensional Euclidean space E_{2n}. As the dimension of the former is $2n$-2 it does not effect on the orientability of P_n. Hence P_n is orientable as well as E_{2n}.

The above results admit us to apply Theorem 9, for the real representation of algebraic varieties are proper submanifolds of the symmetric Riemannian space in consideration. Consequently the proof is completed.

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