NOTE ON DIRICHLET SERIES (III)
ON THE SINGULARITIES OF DIRICHLET SERIES (III)

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9. Theorem V. In this present Note, we shall generalize the Fundamental Theorem 1 established in the previous Note [1][1]. Let us put

$$(1.1)\qquad F(s) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n s) \qquad (s = \sigma + it,\ 0 \leqq \lambda_1 < \lambda_2 < \cdots < \lambda_n \to +\infty).$$

We shall begin with

DEFINITION IV. The subsequence $\{\lambda_{n_k}\}$ $(k = 1, 2, \ldots)$ is called the normal subsequence of density 0, provided that

$$(\mathrm{a})\qquad \lim_{k\to\infty} k/\lambda_{n_k} = 0,$$

$$(\mathrm{b})\qquad \varliminf_{k\to\infty} (\lambda_{n_k} - \lambda_{n_{k-1}}) > 0,\ \varliminf_{\substack{k,n\to\infty \\ n\neq n_k}} |\lambda_{n_k} - \lambda_n| > 0.$$

Then, the Fundamental Theorem 1 is generalized in a following manner.

THEOREM V. *Let* (1.1) *be simply convergent for* $\sigma > 0$. *Then* $s = 0$ *is the singular point for* (1.1), *provided that there exist two sequences* $\{x_k\}$ $(0 < x_k \uparrow \infty)$, $\{\gamma_k\}$ (γ_k : *real*) *such that, for a normal subsequence* $\{\lambda_{n_k}\}$ *of density* 0,

$$(\mathrm{a})\qquad \varlimsup_{k\to\infty} 1/x_k \cdot \log \left| \sum_{\substack{[x_k] \leqq \lambda_n < r_k \\ \lambda_n \overline{\in} \{\lambda_{n_k}\}}} \Re(a_n \exp(-i\gamma_k)) \right| = 0,$$

(b) $\lim_{k\to\infty} \sigma_k/[x_k] = 0$, *where* σ_k : *the number of sign-changes of* $\Re(a_n \exp(-i\gamma_k))$, $\lambda_n \overline{\in} \{\lambda_{n_k}\}$, $\lambda_n \in I_k[[x_k](1-\omega), [x_k](1+\omega)]$ $(0 < \omega < 1)$,
(c) *the sequence* $\Re(a_n \exp(-i\gamma_k))$ $(\lambda_n \overline{\in} \{\lambda_{n_k}\},\ \lambda_n \in I_k)$ *has the normal sign-change in* $\{I_k\}$ $(k = 1, 2, \ldots)$.

By virtue of this theorem, we get

COROLLARY VIII. *If the hypothesis of all theorems established in the previous Note* (I-II) *are satisfied, except for a normal subsequence* $\{\lambda_{n_k}\}$ *of density* 0, *then these theorems are also valid.*

For example, from Corollary 8 we get

COROLLARY IX (S. Izumi, [2]). *Let* (1.1) *be simple convergent for* $\sigma > 0$. *If* $|\arg(a_n)| \leqq \theta < \pi/2$, *except for the normal subsequence* $\{\lambda_{n_k}\}$ *of density* 0, *then* $s = 0$ *is singular for* (1.1).

10. Lemma. For its proof, we need next Lemma.

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1) Vide references placed at the end.

LEMMA. $$0 \leqq \sigma_s - C \leqq \overline{\lim_{n\to\infty}}\, 1/\lambda_n \cdot \log n,$$

where $\sigma_s = \overline{\lim_{x\to\infty}}\, 1/x \cdot \log \left| \sum_{[x]\leqq\lambda_n<x} a_n \right|$, $C = \overline{\lim_{n\to\infty}}\, 1/\lambda_n \cdot \log|a_n|$.

Since σ_s is the simple convergence-abscissa of (1.1) by T. Kojima's theorem [3], under the condition $\lim_{n\to\infty} 1/\lambda_n \cdot \log n = 0$, we have

$\sigma_s = C = \overline{\lim_{n\to\infty}}\, 1/\lambda_n \cdot \log|a_n|$. Hence we get

COROLLARY X. (G. Valiron, [4] p. 4). *If* $\lim_{n\to\infty} 1/\lambda_n \cdot \log n = 0$, *the simple convergence-abscissa of* (1.1) *is given by*

$$\overline{\lim_{n\to\infty}}\, 1/\lambda_n \cdot \log|a_n|.$$

PROOF OF LEMMA. Since $\sigma_s = \overline{\lim_{x\to\infty}}\, 1/x \cdot \log \left| \sum_{[x]\leqq\Lambda_n\lambda x} a_n \right| = \overline{\lim_{x\to\infty}}\, 1/[x] \cdot \log \left| \sum_{[x]\leqq\lambda_n<x} a_n \right|$, for any given ε (>0), there exists a constant $X_0(\varepsilon)$ such that

$$\left| \sum_{[x]\leqq\lambda_n<x} a_n \right| < \exp([x](\sigma_s+\varepsilon)) \text{ for } [x] > X_0(\varepsilon). \tag{10.1}$$

If $[x] \leqq \lambda_{n-1} < \lambda_n < x$, then $a_n = \sum_{[x]\leqq\lambda_\nu\leqq\lambda_n} a_\nu - \sum_{[x]\leqq\lambda_\nu\leqq\lambda_{n-1}} a_\nu$.

If $\lambda_{n-1} > [x] \leqq \lambda_n < x$, then $a_n = \sum_{[x]\leqq\lambda_\nu\leqq\lambda_n} a_\nu$. In any case, by (10.1)

$$|a_n| < 2 \exp([x](\sigma_s+\varepsilon)) \qquad \text{for } [x] > X_0(\varepsilon),$$

so that

$$1/\lambda_n \cdot \log|a_n| < \log 2/\lambda_n + [x]/\lambda_n \cdot (\sigma_s+\varepsilon).$$

Since $\lim_{n\to\infty} \lambda_n/[x] = 1$, we get

$$C = \overline{\lim_{n\to\infty}}\, 1/\lambda_n \cdot \log|a_n| \leqq \sigma_s + \varepsilon.$$

Letting $\varepsilon \to 0$, $C \leqq \sigma_s$. Therefore,

$$0 \leqq \sigma_s - C. \tag{10.2}$$

By the definition of C, for any given ε (>0), we can choose $X_1(\varepsilon)$ such that

$$|a_n| < \exp(\lambda_n(C+\varepsilon)) \quad \text{for } \lambda_n > X_1(\varepsilon). \tag{10.3}$$

Putting $N(x) = \sum_{[x]\leqq\lambda_n<x} 1$, by (10.3) we get easily

$$\left| \sum_{[x]\leqq\lambda_n<x} a_n \right| \leqq N(x) \cdot \exp(C+\varepsilon)) \quad \text{for } [x] > X_1(\varepsilon),$$

so that

$$\sigma_s \leqq \overline{\lim_{x\to\infty}} \log^+ N(x)/x + (C+\varepsilon).$$

Letting $\varepsilon \to 0$,

$$\sigma_s - C \leqq \overline{\lim_{x\to\infty}} \log^+ N(x)/x. \tag{10.4}$$

Putting $\theta = \overline{\lim_{n\to\infty}} \log n/\lambda_n$, for any given ε (>0), we have

$$n < \exp(\lambda_n(\theta+\varepsilon)) \quad \text{for } \lambda_n > X_3(\varepsilon). \tag{10.5}$$

Denoting by λ_ν the maximum exponent in the interval $[x] \leqq \lambda_n < x$, by (10.5)

$$N(x) < \sum_{\lambda_n < x} 1 = \nu < \exp(x(\theta+\varepsilon)).$$

Therefore,

$$\overline{\lim_{x\to\infty}} \log^+ N(x)/x \leqq \theta + \varepsilon.$$

Letting $\varepsilon \to 0$,

$$\overline{\lim_{x\to\infty}} \log^+ N(x)/x \leqq \theta. \tag{10.6}$$

Thus, by (10.2), (10.4) and (10.6), we obtain

$$0 \leqq \sigma_s - C \leqq \overline{\lim_{n\to\infty}} \log n/\lambda_n.$$

11. Proof of Theorem V. Let us put

$$F_1(s) = \sum_{k=1}^{\infty} a_{n_k} \exp(-\lambda_{n_k} s), \quad F_2(s) = \sum_{k=1}^{\infty} \Re(a_{n_k} \exp(-i\gamma_k)) \exp(-\lambda_{n_k} s),$$

$$F_3(s) = \sum_{n=1,\ n \notin \{n_k\}}^{\infty} a_n \exp(-\lambda_n s).$$

Denote by $\sigma_s(F_1)$, $\sigma_s(F_2)$ $\sigma_s(F_3)$ and $\sigma_s(F)$ the simple convergence-abscisses of F_1, F_2, F_3 and F respectively. By (a) and Lemma, we have

$$\sigma_s(F_2) = \overline{\lim_{k\to\infty}} 1/\lambda_{n_k} \cdot \log|\Re(a_{n_k} \exp(-i\gamma_k))| \leqq \overline{\lim_{k\to\infty}} 1/\lambda_{n_k} \cdot \log|a_{n_k}|$$

$$= \sigma_s(F_1) \leqq \overline{\lim_{n\to\infty}} 1/\lambda_n \cdot \log|a_n| \leqq \sigma_s(F) = 0,$$

so that

$$\sigma_s(F_2) \leqq \sigma_s(F_1) \leqq 0. \tag{11.1}$$

By (11.1), we have evidently $\sigma_s(F_3) \leqq 0$. Hence, the following four cases are possible:

(A) $\sigma_s(F_1) < 0$, $\sigma_s(F_3) = 0$, (B) $\sigma_s(F_1) = 0$, $\sigma_s(F_3) < 0$,
(C) $\sigma_s(F_2) = \sigma_s(F_1) = \sigma_s(F_3) = 0$, (D) $\sigma_s(F_2) < \sigma_s(F_1) = \sigma_s(F_3) = 0$.

CASE (A): By the Fundamental Theorem 1, $s=0$ is singular for $F_3(s)$. Since $s=0$ is regular for $F_1(s)$, $s=0$ is also singular for $F(s)$.

CASE (B): By Carlson-Landau-Száz's theorem ([3] pp. 140–141), $\sigma = 0$ is the natural boundary for $F_1(s)$, a fortiori $s = 0$ is singular for $F_1(s)$. Since $s = 0$ is regular for $F_3(s)$, $s = 0$ is also singular for $F(s)$.

CASE (C): In this case, we have

$$\varlimsup_{k\to\infty} 1/\lambda_{n_k} \cdot \log |\Re(a_{n_k} \exp(-i\gamma_k))| = 0. \tag{10.2}$$

Putting $N(t) = \sum_{\lambda_{n_k} < t} 1$, by $k = o(\lambda_{n_k})$ we get evidently

$$N(t) = o(\lambda_{n_{N(t)}}) = o(t). \tag{10.3}$$

Case (C) is further classified into cases:

CASE (C_1): There exists a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ such that

(i) no λ_{n_k} belongs to $[x_{k_i}] \leqq \lambda_n < x_{k_i}$ ($i = 1, 2, \ldots$),

(ii) $\varlimsup_{i\to\infty} 1/x_{k_i} \cdot \log \left| \sum_{[x_{k_i}] \leqq \lambda_n < x_{k_i}} \Re(a_n \exp(-i\gamma_{k_i})) \right| = 0.$

CASE (C_2): $[x_k] \leqq \lambda_n < x_k$ ($k = 1, 2, \ldots$) contains at least one of $\{\lambda_{n_k}\}$.

In Case (C_1), denoting by σ'_{k_i} the number of $\Re(a_n \exp(-i\gamma_{k_i}))$, $\lambda_n \in I_{k_i}[[x_{k_i}](1-\omega), [x_{k_i}](1+\omega)]$, by (10.3) we get

$$\sigma'_{k_i} \leqq \sigma_{k_i} + N([x_{k_i}])(1+\omega) + O(1) = o([x_{k_i}]).$$

By (b) of the definition, the sequence $\Re(a_n \exp(-i\gamma_k))$ ($\lambda_n \in I_{k_i}$) has the normal sign-change. Hence, by the Fundamental Theorem 1, $s = 0$ is singular for $F(s)$.

In Case (C_2), without any loss of generality we can assume

$$[x_k] = [\lambda_{n_k}]. \tag{10.4}$$

Therefore, by (10.2), (10.4) and the similar arguments as above, the Fundamental Theorem 2 ascertains the existence of singularity of $F(s)$ at $s = 0$.

CASE (D): Since $\sigma_s(F_2) < 0$, for suitable ε (>0), we can put

$$|\Re(a_{n_k} \exp(-i\gamma_k))| < \exp(-\varepsilon\lambda_{n_k}). \tag{10.5}$$

By the assumption (a), there exists a subsequence $\{x_{k_i}\}$ such that

$$\lim_{i\to\infty} 1/x_{k_i} \cdot \log \left| \sum_{\substack{[x_{k_i}] \leqq \lambda_n < x_{k_i} \\ \lambda_n \in \{\lambda_{n_k}\}}} \Re(a_n \cdot \exp(-i\gamma_{k_i})) \right| = 0.$$

Hence, we can put

$$\left| \sum_{\substack{[x_{k_i}] \leqq \lambda_n < x_{k_i} \\ \lambda_n \in \{\lambda_{n_k}\}}} \Re(a_n \exp(-i\gamma_k)) \right| = \exp(\alpha(i) x_{k_i}), \ \lim_{i\to\infty} \alpha(i) = 0. \tag{10.6}$$

Putting $\varliminf_{k\to\infty} (\lambda_{n_k} - \lambda_{n_{k-1}}) > h > 0$, by (10.5) and (10.6)

$$\left| \sum_{[x_{k_i}] \leqq \lambda_n < x_{k_i}} \Re(a_n \exp(i\gamma_{k_i})) \right| = \left| \sum_{[x_{k_i}] \leqq \lambda_n < x_{k_i}, \lambda_n \bar{\in} \{\lambda_{n_k}\}} - \sum_{[x_{k_i}] \leqq \lambda_n < x_{k_i}, \lambda_n \in \{\lambda_{n_k}\}} \right|$$

$$> \exp(\alpha(i) x_{k_i} - 1/h \cdot \exp(-\varepsilon(x_{k_i} - 1))$$

$$= \exp(\alpha(i)x_{k_i})\{1 - \exp(\varepsilon - (\varepsilon + \alpha(i))x_{k_i}) \cdot 1/h\}$$
$$> 1/2 \cdot \exp(\alpha(i)x_{k_i}),$$

so that

$$\overline{\lim_{i\to\infty}}\, 1/x_{k_i} \cdot \log \left| \sum_{[x_{k_i}] \leqq \lambda_n < x_{k_i}} \Re(a_n \exp(-i\gamma_{k_i})) \right| \geqq \lim_{i\to\infty} \alpha(i) = 0. \tag{10.7}$$

On the other hand, we have evidently

$$\overline{\lim_{i\to\infty}}\, 1/x_{k_i} \cdot \log \left| \sum_{[x_{k_i}] \leqq \lambda_n < x_{k_i}} \Re(a_n \exp(-i\gamma_{k_i})) \right| \leqq \lim 1/x \cdot \log \left| \sum_{[x] \leqq \lambda_n < n} a_n \right| = 0$$

(by T. Kojima's theorem). Thus, by (10.7),

$$\overline{\lim_{i\to\infty}}\, 1/x_{k_i} \cdot \log \left| \sum_{[x_{k_i}] \leqq \lambda_n < x_{k_i}} \Re(a_n \exp(-i\gamma_k)) \right| = 0.$$

Hence, by the similar arguments as in Case (C_1), the Fundamental Theorem 1 ascertains the singularity of $F(s)$ at $s = 0$. q.e.d.

References

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