# A DUALITY THEOREM FOR REPRESENTABLE LOCALLY COMPACT GROUPS WITH COMPACT COMMUTATOR SUBGROUP

SHUICHI TAKAHASHI

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The purpose of the present note is to formulate and prove a duality theorem, for representable (or maximally almost periodic) locally compact groups with compact commutator subgroup, which includes Pontrjagin's duality theorem for abelian groups and Tannaka's duality theorem for compact groups.

**1. Definitions of $G^{\Lambda}$ and $G^{\Lambda\Lambda}$.** Let $G$ be a locally compact group. By a representation $D$ we mean a representation of $G$ into the group of unitary matrices $U(r)$ of some degree $r=r(D)$. The totality of representations of $G$ is the dual system $G^{\Lambda}$.

Among the represention of $G$ the linear representations, i. e. with its degree $r(D)=1$, form a group $G^*$ under multiplications. We introduce into $G^*$ a topology defined by the system of neighbourhoods of the identity:

$$W(F, \varepsilon)=\{\chi \mid \chi \in G^*;\ x \in F \text{ implies } |\chi(x)-1|<\varepsilon\}$$

where $F$ is a compact subset of $G$ and $\varepsilon>0$. Furthermore the topology of $G^{\Lambda}$ is, by definition, introduced by the system of neighbourhoods of $D \in G^{\Lambda}$:

$$W(F, \varepsilon) \cdot D.$$

Let $G'$ be the commutator subgroup of $G$, i. e. the closure of the algebraical commutator subgroup; then $G^*$ is the aggregate of characters of abelian group $G/G'$. If $F$ is a compact subset of $G$, then its image by $G \to G/G'$ is compact; and conversely every compact subset of $G/G'$ is obtained by this way (e. g. Weil [5]; § 3). Hence $G^*$ is the dual group of $G/G'$ and so some $W(F, \varepsilon)$ has compact closure. Therefore, we have:

THEOREM I. *The totality of representations of G forms a locally compact space $G^{\Lambda}$ with the above defined topology.*

By a representation of $G^{\Lambda}$ we mean a function on $G^{\Lambda}$ to unitary matrices; $D \to A(D)$ with same degree which satisfies the following four conditions:

| | | |
|---|---|---|
| i) | $A(D_1 \times D_2)=A(D_1) \times A(D_2)$ | kronecker product |
| ii) | $A(D_1+D_2)=A(D_1)+A(D_2)$ | direct sum |
| iii) | $A(P^{-1}DP)=P^{-1}A(D)P$ | transformation by unitarian matrix $P$ |
| iv) | $A(D)$ is continuous on $G^{\Lambda}$. | |

Here unitary matrices are topologized by the norm

$$\|A\|=\sqrt{\sum_{i,j}|a_{ij}|^2} \qquad \text{if} \quad A=(a_{ij})$$

when two matrices $A_1, A_2$ are same degree: and different spaces $U(r), U(s)$ with $r \neq s$ are to be considered as open disjoint subsets. The condition iv) is, by the definition of topology in $G^{\Lambda}$, equivalent to

iv') $\qquad A(D)$ is continuous on $G^*$.

Let $G^{\Lambda\Lambda}$ be the totality of representations of $G^{\Lambda}$; define a product of $A_1, A_2 \in G^{\Lambda\Lambda}$ by

$$A_1A_2(D) = A_1(D)A_2(D) \qquad (D \in G^{\Lambda}).$$

Then $A_1 \cdot A_2$ satisfies i) — iv) hence $A_1 \cdot A_2 \in G^{\Lambda\Lambda}$, therefore, $G^{\Lambda\Lambda}$ becomes a group under this multiplication. We introduce in $G^{\Lambda\Lambda}$ a topology defined by the system of neighbourhoods of the identity:

$$W(F^{\Lambda}, \varepsilon) = \{A \mid A \in G^{\Lambda\Lambda};\ D \in F^{\Lambda} \text{ implies } \| A(D) - I(D) \| < \varepsilon\}$$

where $F^{\Lambda}$ is a compact subset of $G^{\Lambda}$ and $\varepsilon > 0$.

THEOREM II. *$G^{\Lambda\Lambda}$ becomes a representable locally compact group with compact commutator subgroup.*

PROOF. Let $F^{\Lambda}$ be a compact neighbourhood of the identity of $G^*$ and $0 < \varepsilon < 1$, then we can prove that the closure $\overline{W(F^{\Lambda}, \varepsilon)}$ is compact.

For this purpose, corresponds to each $D \in G^{\Lambda}$ the unitary group $U(r(D))$ of degree $r(D)$ and construct the direct product on $G^{\Lambda}$:

$$U_0 = \cdots\cdot \times U(r(D)) \times \cdots\cdot$$

This is a compact group; and the totality of algebraic representations of $G^{\Lambda}$, i.e. those satisfying conditions i) — iii), forms a closed, hence compact, subgroup $U$; and algebraically $G^{\Lambda\Lambda} \subset U$. It is clear that the mapping $G^{\Lambda\Lambda} \rightarrow U$ is continuous; but we shall prove that on $\overline{W(F^{\Lambda}, \varepsilon)}$ the inverse mapping also is continuous and $\overline{W(F^{\Lambda}, \varepsilon)}$ is closed in $U$. These prove compactness of $\overline{W(F^{\Lambda}, \varepsilon)}$.

Let $\varepsilon_1 > 0$ be arbitrarily given; and $n$ an integer $> 0$ such that $|\chi^k - 1| \leq \varepsilon$ $(k = 1, \cdots, n)$ implies $|\chi - 1| < \varepsilon_1$. Take a neighbourhood $V$ of the identity in $G^*$ such that $V^n \subset F^{\Lambda}$; then $\chi \in V$, $A \in \overline{W(F^{\Lambda}, \varepsilon)}$ imply $|A(\chi^k) - 1| = |A(\chi)^k - 1| \leq \varepsilon$ $(k = 1, \cdots, n)$ hence $|A(\chi) - 1| < \varepsilon_1$. Next, let $F_1{}^{\Lambda}$ be an arbitrary compact set in $G^{\Lambda}$; take $D_1, \cdots, D_m \in G^{\Lambda}$ such that

$$F_1{}^{\Lambda} \subset \bigcup_{i=1}^{m} V \cdot D_i.$$

Then $A \in \overline{W(F^{\Lambda}, \varepsilon)}$ and $\| A(D_i) - 1 \| < \varepsilon_1$ $(i = 1, \cdots, m)$ imply

$$\| A(D) - 1 \| \leq \| A(D) - A(D_i) \| + \| A(D_i) - 1 \| < \varepsilon_1\sqrt{r} + \varepsilon_1$$

where $D \in F_1{}^{\Lambda}$ and $D \in V \cdot D_i$ with $r = \max_i r(D_i)$. Hence $A \in W(F_1{}^{\Lambda}, \varepsilon_1\sqrt{r} + \varepsilon_1)$. This means inverse continuity on $\overline{W(F^{\Lambda}, \varepsilon)}$.

Denote by $(\quad)^a$ the closure in $U$; and let $A \in (\overline{W(F^{\Lambda}, \varepsilon)})^a$, then $\chi \in F^{\Lambda}$ implies $|A(\chi) - 1| \leq \varepsilon$, therefore, by the same reasoning as above we conclude that $\chi \in V$ implies $|A(\chi) - 1| < \varepsilon_1$. Hence $A$ is continuous on $G^*$ and $A \in \overline{W(F^{\Lambda}, \varepsilon)}$; i.e. $\overline{W(F^{\Lambda}, \varepsilon)}$ is closed in $U$.

Put $C^{\Lambda\Lambda} = \{A \mid \chi \in G^* \text{ implies } A(\chi) = 1\}$; this is a closed invariant subgroup of $G^{\Lambda\Lambda}$ such that $G^{\Lambda\Lambda}/C^{\Lambda\Lambda}$ is abelian. $C^{\Lambda\Lambda}$ is compact because it is contained in a neighbourhood $W(F^{\Lambda}, \varepsilon)$ with compact closure, therefore, commutator subgroup $(G^{\Lambda\Lambda})' \subset C^{\Lambda\Lambda}$ is compact.

Finally, $D(A) = A(D)$ $(D \in G^{\Lambda})$ is a representation of $G^{\Lambda\Lambda}$; and $A \neq 1$ implies existence of a $D \in G^{\Lambda}$ such that $A(D) \neq 1$, i.e., $G^{\Lambda\Lambda}$ possess sufficiently many representations. Hence $G^{\Lambda\Lambda}$ is representable. q. e. d.

**2. Canonical representation.**

THEOREM III. *The function $\varphi$ of $G$ into $G^{\Lambda\Lambda}$ defined by*

$$\varphi(a)(D) = D(a) \qquad (D \in G^{\Lambda})$$

*gives a representation of $G$ into $G^{\Lambda\Lambda}$. If $G$ is reprsentable, then $\varphi$ is an one to one correspondence.*

PROOF. That $a \to \varphi(a)$ gives an algebraic representation of $G$ into $G^{\Lambda\Lambda}$ is obvious. We shall prove its continuity.

Let $F^{\Lambda}$ be a compact set in $G^{\Lambda}$ and $\varepsilon > 0$. Consider $W(F, \varepsilon)$, where $F$ is a compact neighbourhood of the identity in $G$, and cover $F^{\Lambda}$ by

$$F^{\Lambda} \subset \bigcup_{i=1}^{m} W(F, \varepsilon) \cdot D_i \qquad (D_i \in G^{\Lambda}).$$

Take a neighbourhood $V \subset F$ such that $x \in V$ implies $\| D_i(x) - 1 \| < \varepsilon$ $(i = 1, \cdots, m)$. Then $a \in V, D \in F^{\Lambda}$ imply

$$\| D(a) - 1 \| \leq \| D(a) - D_i(a) \| + \| D_i(a) - 1 \| < \varepsilon \sqrt{r} + \varepsilon$$

where $D \in W(F, \varepsilon) \cdot D_i$ and $r = \max_i r(D_i)$. Hence $a \in V$ implies $\varphi(a) \in W(F^{\Lambda}, \varepsilon\sqrt{r} + \varepsilon)$. q. e. d.

We say that $\varphi(a)$ is the canonical representation of $G$ into $G^{\Lambda\Lambda}$. The duality theorem, which we shall prove later, asserts that the canonical representation gives, in fact, an isomorphism of two topological groups $G$ and $G^{\Lambda\Lambda}$.

**3. Definition of the algebra $R_G$.** The aggregate of all finite linear combinations with complex coefficients

$$f(x) = \sum \alpha_{ij}^{k} d_{ij}^{k}(x), \qquad D^{k}(x) = (d_{ij}^{k}(x)) \in G^{\Lambda}$$

forms an algebra $R_G$ over the complex number field under pointwise operations.

LEMMA 1. *Let $G$ be a representable locally compact group with a compact subgroup $K$. Put*

$$(R_G, K) = \{f \mid f(K) = 0, \ f \in R_G\}$$

*then we have:*

$$R_K \cong R_G / (R_G, K).$$

PROOF. For any $f \in R_G$ the contraction on $K$: $f \mid K$ is a function in $R_K$. Obviously, $f \to f \mid K$ is a homomorphism of the algebra $R_G$ into $R_K$, whose image in $R_K$ is generated by a system $K_1^{\Lambda}$ of representations of $K$ such that $1 \in K_1^{\Lambda}$; $D_1, D_2 \in K_1^{\Lambda}$ imply $\bar{D}_i$, the conjugate representations, and irreducible constituents of $D_1 \times D_2$ are in $K_1^{\Lambda}$. Moreover, by the assumption of representability of $G$, there exists for any $x \in K$ different from the identity 1 a $D$

$\in K_1^\wedge$ such that $D(x) \neq D(1)$. Hence by Kampen's theorem $K_1^\wedge = K^\wedge$ and the above homomorphism is onto; and its kernel is precisely $(R_G, K)$. q. e. d.

LEMMA 2. *Let $G$ be a representable locally compact group with compact invariant subgroup $K$. Consider $R_G/K$ as a subalgebra of $R_G$ and the least ideal containing $(R_{G/K}, K)$, in the notation of Lemma* 1, *be* id. $(R_{G/K}, K)$. *Then* id. $(R_{G/K}, K)(\subset (R_G, K))$ *is dense in* $(R_G, K)$ *by the uniform norm*:

$$\| f \| = \sup_{x \in G} | f(x) |$$

*on* $R_G$.

PROOF. Let $\widetilde{G}$ be the associated compact group to $G$ (in the sense of Weil [5]; § 31) and the image of $K$ by the representation $G \to \widetilde{G}$ be $\widetilde{K}$. Then $K \cong \widetilde{K}$ and $\widetilde{K}$ is an invariant subgroup of $\widetilde{G}$ such that $\widetilde{G}/\widetilde{K}$ is the associated compact group to $G/K$. Hence

$$R_G \cong R_{\widetilde{G}}, \ \ (R_G, K) \cong (R_{\widetilde{G}}, \widetilde{K}), \ \ (R_{G/K}, K) \cong (R_{\widetilde{G}/\widetilde{K}}, \widetilde{K}).$$

Therefore, it is sufficient to prove the lemma for compact group $G$ with closed invariant subgroup $K$.

Let $\varepsilon > 0$ and $f \in (R_G, K)$ be given, we shall prove that $f$ is $\varepsilon$-uniformly approximated by a function in id. $(R_{G/K}, K)$. For any $x_1 \in G$ there must exists an $f_1 \in (R_{G/K}, K)$ such that $f(x_1) = f_1(x)$. For, if $x_1 \in K$ then $f(x_1) = 0$ and the assertion is trivial; if $x_1 \bar{\in} K$ the assertion follows from representability of $G/K$. By uniform continuity of $f$ and $f_1$, there exists a neighbourhood $V_1$ of $x_1$ such that

$$x \in V_1 \text{ implies } | f(x) - f_1(x) | < \frac{\varepsilon}{2}.$$

To each point $x_1 \in G$ we associate a neighbourhood $V_1$ and a function $f_1 \in (R_{G/K}, K)$. Then the compact group $G$ is covered by a finite sum

$$G \subset \bigcup_{i=1}^{n} V_i$$

with associated points $x_1, \cdots, x_n$ and functions $f_1, \cdots, f_n \in (R_{G/K}, K)$.

Take a Dieudonné partition $e_1, \cdots, e_n$ to the covering $V_1, \cdots, V_n$, i.e., functions on $G$ such that

$$1 = e_1(x) + \cdots + e_n(x) \qquad (x \in G)$$

$0 \leq e_i(x) \leq 1$ and $x \bar{\in} V_i$ implies $e_i(x) = 0$.

$$\left| f(x) - \sum_{i=1}^{n} e_i(x) f_i(x) \right| < \frac{\varepsilon}{2}.$$

Put $M = \max(\varepsilon, \sup \| f - f_i \|, \| f \|)$ then, by the approximation theorem, we can find $g_i(x) \in R_G$ such that

$$| e_i(x) - g_i(x) | < \frac{\varepsilon}{6nM} \qquad (x \in G, \ i = 1, \cdots, n).$$

Then

$$|f(x) - \Sigma g_i(x) f_i(x)| \leq |f(x) - \Sigma g_i(x) f(x)| + |\Sigma g_i(x) f(x) - \Sigma g_i(x) f_i(x)|$$

$$\leq \frac{\varepsilon}{6M}\|f\| + \sum_{x\in V_i} |g_i(x)|\ |f(x)-f_i(x)| + \sum_{x\bar{\in} V_i} |g_i(x)|\ |f(x)-f_i(x)|.$$

Since

$$\left|1-\sum_{x\in V_i} g_i(x)\right| \leq \left|\sum e_i(x) - \sum g_i(x)\right| + \left|\sum_{x\bar{\in} V_i} g_i(x)\right| < \frac{\varepsilon}{6M} + \frac{\varepsilon}{6M} = \frac{\varepsilon}{3M}$$

we have

$$\left|f(x) - \sum g_i(x) f_i(x)\right| \leq \frac{\varepsilon}{6} + \left(1+\frac{\varepsilon}{3M}\right)\frac{\varepsilon}{2} + \frac{\varepsilon}{6} < \varepsilon$$

for all $x\in G$ and $\Sigma\, g_i(x)f_i(x) \in \mathrm{id.}(R_{G/K}, K)$. q.e.d.

4. **Duality theorem.** For the proof of duality theorem we use the algebra $R_G$ and its representations. By a representation of $R_G$ we mean a homomorphism $A$ of the algebra $R_G$ into the field of complex numbers such that

$$A(\bar{f}) = \overline{A(f)} \qquad (f\in R_G)$$

where bar indicates conjugate complex.

From this we see at once that algebraical representations of $G^\Lambda$, i.e., satisfying conditions i) – iii) of §1 and representations of the algebra $R_G$ are one to one by the correspondence:

$$D(x) = (d_{ij}(x)) \rightarrow A(D) = (A(d_{ij})).$$

LEMMA 3. *Any representation of $R_G$ is continuons with respect to the uniform norm*

$$\| f \| = \sup_{x\in G} |f(x)|.$$

PROOF. Let $\widetilde{G}$ be the associated compact group to $G$ then $R_G \cong R_{\widetilde{G}}$ which preserves uniform norm. Therefore, by Tannaka's duality theorem, we can find $\widetilde{a} \in \widetilde{G}$ such that

$$A(f) = f(\widetilde{a}) \qquad (f\in R_{\widetilde{G}}).$$

Hence $A$ is continuous on $R_{\widetilde{G}}$, therefore, on $R_G$. q.e.d.

THEOREM IV (Duality theorem). *Let $G$ be a representable locally compact group with compact commutator subgroup. Then the canonical representation $\varphi(a)$ defined by*

$$\varphi(a)(D) = D(a) \qquad (D\in G^\Lambda)$$

*gives an isomorphism of $G$ and $G^{\Lambda\Lambda}$ as topological groups*:

$$G \cong G^{\Lambda\Lambda}.$$

PROOF. Let $K$ be the compact commutator subgroup of $G$, and put

$$K_1 = \{A \mid A(\chi) = 1, \chi\in G^*\} \subset G.$$

If $b\in K$ then $\varphi(b)(\chi) = \chi(b) = 1$, hence, $\varphi(b)\in K_1$. Conversely, any $A\in K_1$, considered as a representation of the algebra $R_G$, annihilates the subalgebra $(R_{G/K}, K)$ because this is of the form

$$\Sigma\, \alpha_i(\chi_i(x) - 1) \qquad (\chi_i \in G^*).$$

Hence it annihilates the smallest ideal id. $(R_{G/K}, K)$ containing $(R_{G/K}, K)$. By Lemma 2, id. $(R_{K/G}, K)$ is dense in $(R_G, K)$ with respect to the uniform norm; and, by Lemma 3, $A$ is continuous with respect to this norm, therefore, it annihilates the subalgebra $(R_G, K)$. Hence, $A \in K_1$ can be considered as a representations of

$$R_K \cong R_G/(R_G, K).$$

From Tannaka's duality theorem, there exists a $b \in K$ such that

$$A(f) = f(b) \qquad f \in R_G/(R_G, K).$$

Since $A(g) = g(b) = 0$ for any $g \in (R_G, K)$ we have

$$A(f) = f(b) \qquad f \in R_G.$$

Hence, by definition, $A = \varphi(b)$. Thus we have proved $\varphi(K) = K_1$.

Let $A \in G^{\wedge\wedge}$ be given arbitrarily, by Pontrjagin's duality theorem, we can find an $a \in G$ such that

$$A(\chi) = \chi(a) \qquad (\chi \in G^*).$$

Then $A \cdot \varphi(a)^{-1}(\chi) = 1$ $(\chi \in G^*)$, i.e., $A \cdot \varphi(a)^{-1} \in K_1$. From what we have proved, there exists a $b \in K$ such that

$$A \cdot \varphi(a)^{-1} = \varphi(b).$$

Hence

$$A = \varphi(b)\varphi(a) = \varphi(ba) \qquad ba \in G.$$

This proves that the canonical representation $\varphi(a)$ maps $G$ onto $G^{\wedge\wedge}$.

Finally, we shall prove that $\varphi^{-1}(A)$ is also continuous. For this purpose, take a compact neighbourhood $V$ of the idenity in $G$, and consider the group $H$ generated by $V \cdot K$:

$$H = (V \cdot K)^{\infty}.$$

This is an open and closed subgroup of $G$ such that $G/H$ is discrete abelian. Put

$$H_1 = \{A \mid A(\chi) = 1,\ \chi \in (G/H)^*\}.$$

It is obvious that $\varphi(H) \subset H_1$. For any $A \in H_1$ corresponds an $a \in G$ such that

$$A = \varphi(a).$$

If $a \overline{\in} H$, there would exist $\chi \in (G/H)^*$ such that $\chi(a) \neq 1$, hence

$$A(\chi) = \chi(a) \neq 1.$$

Therefore, $a \in H$ and $\varphi(H) = H_1$ is proved.

Since $G/H$ is discrete, its dual group $(G/H)^*$ is compact, therefore, for any $\varepsilon > 0$

$$W((G/H)^*, \varepsilon) \subset H_1.$$

This means that $H_1$ is an open and closed subgroup in $G^{\wedge\wedge}$. It is clear that $H$ ie covered by a countable compact set, therefore, by a general theorem (e.g. Bourbaki [1]; § 5 exercice 18) or Tannaka [3]; § 44) $\varphi(a)$ is an open mapping on $H$. This, with the fact that $H_1$ is open, completes the proof that $\varphi(a)$ is open throughout $G$. Therefore, $\varphi^{-1}(A)$ is continuous from $G^{\wedge\wedge}$ to $G$. q. e. d.

5. **Some examples.** Consider a locally compact connected group $G$. The fact, that if $G$ is representable then its commutator subgroup must be compact, is a consequence of a theorem of Freudenthal (e. g. Weil [5]: § 32) which asserts that such a group is a direct product

$$G = R^p \times K$$

of a vector group $R^p$ with a compact group $K$.

But there existsa locally compact connected group with compact commutator subgroup which are not representable in a compact group. For example (Communicated by Mr. M. Gotô)

$$G = \left\{ \begin{pmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{pmatrix} \;\middle|\; a, b \text{ reals}\,;\ c \text{ real mod } 1 \right\}.$$

Next, consider non-connected groups; then two assumptions: representability and to have compact commutator subgroup are independent. For example (Kuranishi [2])

$$G = \left\{ \begin{pmatrix} \varepsilon & \alpha \\ & 1 \end{pmatrix} \;\middle|\; \varepsilon = \pm 1, \quad \alpha \text{ real} \right\}$$

is representable but its commutator subgroup is not compact.

Conversely, the group

$$G = \left\{ T^2 = \{(s, t) |\ s, t \text{ reals mod } 1\}, S \right\}$$

with the relation

$$S(s, t)S^{-1} = (s + t, t)$$

has compact commutator group, which is also open, but not representable.

Finally, we shall give an example of representable locally compact groups with compact commutator subgroup which is not a direct product of abelian and compact groups. This is

$$G = \{U(r),\ S\}$$

where $U(r)$ is the group of unitary matrices of degree $r$, with the relation

$$SUS^{-1} = \bar{U} \qquad (U \in U(r)).$$

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MATHEMATICAL INSTITUTE, TÔHOKU UNIVERSITY, SENDAI.