GENERATION OF A STRONGLY CONTINUOUS SEMI-GROUP OPERATORS

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1. Let $\{T(\xi); 0 \leqq \xi < \infty\}$ be a one-parameter semi-group satisfying the following conditions:

(i) $T(\xi)$ is a bounded linear transformation from a complex (B)-space E into itself.

(ii) $T(\xi+\eta) = T(\xi)\cdot T(\eta)$, $T(0) = I$ (the identity transformation).

(iii) $T(\xi)$ tends to I strongly as $\xi \to 0$, but not uniformly.

We define $Ax = \lim_{h\to 0} \frac{1}{h}[T(h) - I]x$ whenever the limit exists and the set of elements x for which Ax exists, will be denoted by $D[A]$.

Let us put

$$R(\lambda; A)x = -\int_0^\infty e^{-\lambda\xi}\, T(\xi)x d\xi, \quad \lambda = \sigma + i\tau,$$

for all $x \in E$, then the representation holds at least for all λ such that $R(\lambda) = \sigma > \log M$, where $M = \sup_{0 \leqq \xi \leqq 1} \| T(\xi) \|$.

We shall consider a problem of E. Hille [1][1)] which is stated as follows:

What properties should an operator A possess in order that it be the infinitesimal generator of a strongly continuous semi-group $\{T(\xi); 0 \leqq \xi < \infty\}$ of bounded linear transformations from a complex (B)-space into itself?

A theorem of E. Hille [1, Theorem 12.2.1] reads as follows.

THEOREM H. *Let A be a closed linear unbounded operator on E into itself whose domain is dense in E. Let the spectrum of A is located in $R(\lambda) = \sigma \leqq 0$ and suppose that*

$$\| R(\sigma + i\tau; A) \| \leqq \frac{1}{\sigma} + \frac{\beta}{\sigma^2}, \quad \sigma > 0, \tag{1}$$

where β is a fixed constant, $\beta \geqq 0$. Then there exists a semi-group $\{T(\xi); 0 \leqq \xi < \infty\}$ such that $T(\xi)$ satisfies the conditions (i)-(iii), $\| T(\xi) \| \leqq e^{\beta\xi}$ *for all $\xi > 0$ and that A is its infinitesimal generator.*

The behavior of the norm, $\| T(\xi) \| \leqq e^{\beta\xi}$, is by no means implied by the conditions (i)-(iii) as may be seen later (Example 1). Therefore this theorem is not the perfect solution of the above problem.

In this paper we shall give a necessary and sufficient condition that the

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1) Numbers in brackets refer to the bibliography at the the end of the paper.

operator A becomes an infinitesimal generator of a semi-group $\{T(\xi)\,;0\leqq \xi<\infty\}$ satisfying the conditions (i)-(iii), in terms of its spectrum and resolvent.

2. We first prove the following theorem.

THEOREM 1. *Let* $\{T(\xi)\,;0\leqq\xi<\infty\}$ *be a one-parameter semi-group satisfying the conditions* (i)-(iii). *Then*

(a) *A is a closed linear operator and $D[A]$ is dense in E;*

(b) *for λ such that $R(\lambda)>\log M$, where $M=\max\,(\sup_{0\leqq\xi\leqq 1}\|T(\xi)\|,\,1)$, we have*

$$(A-\lambda I)R(\lambda\,;A)x=x,\quad x\in E,$$
$$R(\lambda;A)\,(A-\lambda I)x=x,\quad x\in D[A]\,;$$

(c) *for $\sigma>\log M$, we have*

$$\|[R(\sigma\,;A)]^k\|\leqq M[\sigma-\log M]^{-k},\qquad (k=1,2,3,\ldots).\tag{2}$$

PROOF. For the proof of (a) and (b), see Hille's book [1]. We shall prove (c). Since $\sup_{0\leqq\xi\leqq 1}\|T(\xi)\|\leqq M$, we have $\|T(\xi)\|\leqq M^{1+\xi}$. Now

$$[R(\lambda\,;A)]^k x=-\int_0^\infty e^{-\xi_1\lambda}T(\xi_1)\{[R(\lambda\,;A)]^{k-1}\}x d\xi_1$$

$$=(-1)^2\int_0^\infty e^{-\xi_1\lambda}d\xi_1\int_0^\infty e^{-\xi_2\lambda}\,T(\xi_1+\xi_2)\{[R(\lambda\,;A)]^{k-2}\}x d\xi_2$$

$$=(-1)^k\int_0^\infty e^{-\xi_1\lambda}d\xi_1\int_0^\infty e^{-\xi_2\lambda}\,d\xi_2\cdots\int_0^\infty e^{-\xi_k\lambda}\,T(\xi_1+\xi_2+\cdots+\xi_k)x\,d\xi_k.$$

Hence

$$\|[R(\sigma\,;A)]^k\|\leqq\int_0^\infty e^{-\xi_1\sigma}\,d\xi_1\cdot\int_0^\infty e^{-\xi_2\sigma}d\xi_2\cdots\int_0^\infty e^{-\xi_k\sigma}\|T(\xi_1+\xi_2+\cdots+\xi_k)\|\,d\xi_k$$

$$\leqq M\prod_{i=1}^k\int_0^\infty e^{-\xi_i\sigma}M^{\xi_i}\,d\xi_i=M[\sigma-\log M]^{-k}.\qquad\text{Q.E.D.}$$

3. We prove now the main theorem.

THEOREM 2. (i′) *Let A be a closed linear unbounded operator on E into itself whose domain is dense in E.* (ii′) *There is a constant $M\geqq 1$ such that the spectrum of A is located in $R(\lambda)=\sigma\leqq\log M$, and such that for $\sigma>\log M$,*

$$\|[R(\sigma;A)]^k\|\leqq M[\sigma-\log M]^{-k}\qquad (k=1,2,\ldots.)$$

where $R(\sigma\,;A)$ is the resolvent of A. Then there exists a semi-group $\{T(\xi)\,;0\leqq\xi<\infty\}$ such that $T(\xi)$ satisfies the conditions (i)-(iii) *and that A is its infinitesimal generator.*

We have, from the assumption (ii′) of the above theorem,

$$(3)\qquad \begin{cases}(A-\lambda I)R(\lambda;A)x=x, & x\in E,\\ R(\lambda;A)(A-\lambda I)x=x, & x\in D[A],\end{cases}$$

for all $R(\lambda)=\sigma>\log M$.

For the proof of this theorem[2] we need the following lemmas.

LEMMA 1. *Under the assumption of Theorem* 2, $D[A^2]$ *is dense in* E.

This result is due to E. Hille [1, Lemma 12.2.1].

LEMMA 2. *Under the assumption of Theorem* 2, *we have*

$$\lim_{n\to\infty} A(-nR(n;A)y)=Ay, \quad y\in D[A^2].$$

PROOF. From the result (3) we have $-nR(n;A)y=y-R(n;A)Ay$ for all $z\in D[A^2]$. Hence

$$A(-nR(n;A)y)=Ay-AR(n;A)Ay.$$

Since $y\in D[A^2]$, $Ay\in D[A]$. Therefore we have $AR(n;A)Ay=R(n;A)A^2y$. Thus it follows that

$$\| A(-nR(n;A)y-Ay\| \leqq M(n-\log M)^{-1}\|A^2y\|. \quad \text{Q.E.D.}$$

PROOF OF THEOREM 2. Let us put

$$(4)\qquad I_n=-nR(n;A), \quad n>\log M.$$

Then I_n is a bounded linear transformation from E into itself and

$$(5)\qquad \|I_n^k\|=\|n^k[R(n;A)]^k\|\leqq M[n(n-\log M)^{-1}]^k \qquad (k=1,2,\dots),$$

$$(6)\qquad AI_nx=n(I_n-I)x, \quad x\in E.$$

Since

$$\exp(\xi AI_n)x=\exp\xi nI_n\cdot\exp(-n\xi)x=\sum_{k=0}^{\infty}\frac{[\xi nI_n]^k}{k!}\exp(-n\xi)x$$

we have

$$(7)\qquad \|\exp(\xi AI_n)\|\leqq\sum_{k=0}^{\infty}n^k\xi^k\frac{\|I_n^k\|}{k!}\exp(-n\xi)\leqq M\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\xi n^2}{n-\log M}\right)^k\exp(-n\xi)$$
$$=M\exp\left(\frac{\xi\log M}{1-\frac{1}{n}\log M}\right).$$

By the definition of $T_n(\xi)$, we have

$$(8)\qquad T_n(\xi)x-x=\int_0^{\xi}T_n(\zeta)AI_nxd\zeta.$$

Since AI_n commute with $T_m(\xi)$ for all integers $m,n>\log M$, it follows that

2) We wish to express our cordial thanks to Prof. K. Yoshida, who has kindly pointed us, that the proof of our Theorem 2 is similar to his investigation in K. YOSHIDA, On the differentiability and the representation of one-parameter semigroup of linear operators, Journ. Math. Soc. Japan, 1 (1948).

$$\|[T_m(\xi) - T_n(\xi)]\, x\| = \left\| \int_0^\xi \frac{d}{d\zeta}[\exp(\xi-\zeta)AI_n T_m(\zeta)x]\, d\zeta \right\|$$
$$= \left\| \int_0^\xi [\exp(\xi-\zeta)AI_n]\, T_m(\zeta)(AI_m - AI_n)x\, d\zeta \right\|$$
$$\leqq M^2 \exp\left(\frac{\xi \log M}{1 - \frac{1}{n}\log M} + \frac{\xi \log M}{1 - \frac{1}{m}\log M} \right) \xi \, \|(AI_m - AI_n)x\|.$$

By Lemma 2, we have

$$\lim_{m,n\to\infty} \| T_m(\xi)x - T_n(\xi)x \| = 0, \quad x \in D[A^2].$$

Thus, by (7) and Lemma 1, for all $\xi > 0$ and all $x \in E$

(9) $$\lim_{n\to\infty} T_n(\xi)x$$

exists.

We shall now define $T(\xi)$ as the limit (9). Then we have, by (7),

(10) $$\| T(\xi) \| \leqq M^{1+\xi}, \quad \xi > 0.$$

It is obvious that $\{T(\xi); 0 \leqq \xi < \infty\}$ satisfies the conditions (i) and (ii). By (8) and Lemma 2, we have

(11) $$T(\xi)x - x = \int_0^\xi T(\zeta)Ax\, d\zeta, \quad x \in D[A^2].$$

Thus is follows that $\lim_{\xi\to 0} \| T(\xi)x - x \| = 0$ for $x \in D[A^2]$. By the Banach-Steinhaus theorem, we have the condition (iii). It follows from (11) that A is the infinitesimal generator of $\{T(\xi); 0 \leqq \xi < \infty\}$. This completes the proof of Theorem 2.

Let us consider the assumption (1) of Theorem H. If $\| R(\sigma; A) \| \leqq 1/\sigma + \beta/\sigma^2$, then $\| R(\sigma; A) \| \leqq (\sigma - \beta)^{-1}$ for $\sigma > \beta$. Hence $\| [R(\sigma; A)]^k \| \leqq (\sigma - \beta)^{-k} \leqq e^\beta (\sigma - \beta)^{-k}$ for $\sigma > \beta$. Hence (1) implies (1′). Hence Theorem 2 implies Theorem H.

As a particular case of Theorem 2, we get

THEOREM 3. *Let $\{T(\xi); 0 \leqq \xi < \infty\}$ be a semi-group satisfying the assumptions* (i)-(iii) *and let $\| T(\xi) \| \leqq M$ for all $\geqq 0$. Then*

(a′) *A is a closed linear unbounded operator on E into itself whose domain is dense in E;*

(b′) *there is a constant $M \geqq 1$ such that the spectrum of A is located in $\sigma = R(\lambda) \leqq 0$ and such that for $\sigma > 0$*

$$\| [R(\sigma; A)]^k \| \leqq M \sigma^{-k} \qquad (k = 1, 2, \cdots),$$

where $R(\sigma; A)$ is the resolvent of A.

Conversely, let A be an operator satisfying (a′) *and* (b′). *Then there exists a semi-group $\{T(\xi); 0 \leqq \xi < \infty\}$ such that $T(\xi)$ satisfies the condition* (i)-(iii),

$\|T(\xi)\| \leqq M$ for all $\xi \geqq 0$ and that A is its infinitesimal generator.

By Theorem 1 and Theorem 2, we get a perfect solution of E. Hille's problem.

4. We shall extend above theorem to the n-parameter semi-group.

Let E_n be a reai Euclidean space of n-dimension with usual metric and $e_1, e_2, \cdots, e_n$ be its independent unit vectors.

Let A_i be the infinitesimal generator of $T(\xi e_i)$. We have then the following theorem.

THEOREM 4. *Let $\{T(\xi_1, \xi_2, \cdots, \xi_n); 0 \leqq \xi_1 < \infty,\ 0 \leqq \xi_2 < \infty, \cdots, 0 \leqq \xi_n < \infty\}$ be an n-parameter semi-group satisfying the following conditions:*

(ia) *$T(\xi_1, \xi_2, \cdots, \xi_n)$ is a bounded linear transformation from E into itself;*

(iia) *$T(\xi_1, \xi_2, \cdots, \xi_n) \cdot T(\eta_1, \eta_2, \cdots, \eta_n) = T(\xi_1 + \eta_1, \xi_2 + \eta_2, \cdots, \xi_n + \eta_n)$; $T(0, 0, \cdots, 0) = I$ ($=$ the identity transformation),*

(iiia) *$T(\xi_1, \xi_2, \cdots, \xi_n)$ tends to I strongly as $(\xi_1, \xi_2, \cdots, \xi_n) \to (0, 0, \cdots, 0)$, but not uniformly. Then we have*

(ib) *$A_i\ (i = 1, 2, \cdots n)$ is a closed linear unbounded operator on E into itself whose domain is dense in E;*

(iib) *there is a constant $M \geqq I$ such that the spectrum of A_i is located in $R(\lambda) = \sigma \leqq \log M$ and that, for all $1 \leqq i \leqq n$,*

$$\|[R(\sigma; A_i)]^k\| \leqq M[\sigma - \log M]^{-k}, \quad \sigma > \log M \qquad (k = 1, 2, \cdots),$$

where $R(\sigma; A_i)$ is the resolvent of A_i;

(iiib) *$R(\sigma; A_i)R(\mu; A_j) = R(\mu; A_j)R(\sigma; A_i)$ for $i, j = 1, 2, \cdots$.*

Conversely, let A_i $(i = 1, 2, \cdots, n)$ be operators satisfying the conditions (ib)-(iiib). *Then there exists a semi-group $\{T(\xi_1, \xi_2, \cdots, \xi_n); 0 \leqq \xi_i < \infty,\ i = 1, 2, \cdots, n\}$ such that $T(\xi_1, \xi_2, \cdots, \xi_n)$ satisfies the conditions* (ia)-(iiia) *and all the generators of $\{T(\xi_1, \xi_2, \cdots, \xi_n); 0 \leqq \xi_i < \infty,\ i = 1, 2, \cdots, n\}$ are of the form $A = \sum_{k=1}^{n} \xi_k A_k$.*

For the proof, it is sufficient to note that $\bigcap_{i=1}^{n} D[A_i^2]$ is dense in E, where $D[A_i^2]$ denotes the domain of A_i^2. Then Theorem 4 is immediately obtained from Theorems 1 and 2.

5. We shall show, by examples, that the conditions (i)-(iii) do not imply $\|T(\xi)\| \leqq e^{3\xi}$.

EXAMPLE 1. Let $E = L[(0, \infty); e^{-\sqrt{t}}]$ be the class of all measurable functions on $(0, \infty)$ such that

$$\|x\| = \int_0^\infty |x(t)|\ e^{-\sqrt{t}}\ dt < \infty.$$

Then E is a (B)-space. Let $\{T(\xi); 0 \leqq \xi < \infty\}$ be the semi-group of right translations on E. Then $\{T(\xi); 0 \leqq \xi < \infty\}$ satisfies the conditions (i)-(iii) and $\|T(\xi)\| = e^{\sqrt{\xi}}$. Hence $T(\xi)$ does not satisfy the Hille's condition $\|T(\xi)\| \leqq e^{\beta\xi}$ for any $\beta \geqq 0$.

We shall now show that the condition (i′) of Theorem 2 can not be replaced by $\|R(\sigma; A)\| \leqq M/\sigma$.

EXAMPLE 2. Let E_2^n $(n = 1, 2, \cdots)$ be a sequence of two dimensional complex Euclidean spaces, where the norm of $x_n = (y, z)$ in E_2^n is defined by $\|x_n\| = (|y|^2 + n|z|^2)^{1/2}$. By E, we denote the set of all sequences $\{x_n \in E_2^n\}$ such that $\Sigma \|x_n\| < \infty$, with the norm $\|\{x_n\}\| = \Sigma \|x_n\|$. We now define a semi-group $T_n(\xi)x_n = x_n' = (y', z')$ from E_2^n into E_2^n such that

$$y' = \exp[-(n + in^3)\xi](y \cos n\xi - z \sin n\xi),$$
$$z' = \exp[-(n + in^3)\xi](y \sin n\xi + z \cos n\xi).$$

It is obvious that $\|T_n(\xi)\| \leqq n^{1/2} \exp(-n\xi)$. If we define a semi-group $\{T(\xi); 0 < \xi < \infty\}$ from E into itself by $T(\xi)\{x_n\} = \{T_n(\xi)x_n\}$, then we have $\|T(\xi)\| \leqq (2e\xi)^{-1/2}$ and $\{T(\xi) \;;\; 0 < \xi < \infty\}$ is a semi-group satisfying the following conditions [2]:

(ic) $T(\xi)$ is strongly measurable for $\xi > 0$.

(iic) $\displaystyle\int_0^\infty \exp(-\lambda\xi) \|T(\xi)\| \, d\xi$ exists for $\lambda > 0$,

(iiic) $\displaystyle\lim_{\eta \to 0} \frac{1}{\eta} \int_0^\eta T(\xi)x \, d\xi = x$ for all $x \in E$,

(ivc) $\displaystyle\limsup_{\xi \to 0} \|T(\xi)\| = \infty$.

From the conditions (ic)-(ivc), we can see that the infinitesimal generator A is a closed linear operator, $D[A]$ is dense in E and that the spectrum of A is located in $R(\lambda) = \sigma \leqq 0$ [1, § 11.8]. Conditions (iiic) means that $T(\xi)$ is strongly $(c, 1)$-ergodic at $\xi = 0$. Thus $T(\xi)$ is strongly Abel-ergodic at $\xi = 0$. Hence there exists a constant $M_0 > 0$ such that $\sigma\|R(\sigma : A)\| \leqq M_0$ for $1 \leqq \sigma < \infty$ [1, Theorem 14.7.3]. Since $\|T(\xi)\| \leqq (2e\xi)^{-1/2}$, we have

$$\|\sigma R(\sigma; A)\| \leqq \sigma \int_0^\infty e^{-\sigma\xi} (2e\xi)^{-\frac{1}{2}} \, d\xi = \sqrt{\frac{\pi\sigma}{2e}} \leqq \sqrt{\frac{\pi}{2e}} \qquad \text{for } 0 < \sigma \leqq 1.$$

Hence there exists a constant $M > 0$ such that $\sigma\|R(\sigma; A)\| \leqq M$ for $0 < \sigma < \infty$. On the other hand, the condition (ivc) shows that $T(\xi)$ does not imply the condition (iii).

BIBLIOGRAPHY

[1] E. HILLE, Functional analysis and semi-groups, Amer. Math. Soc. Coll. Publ., New York, 1948.

[2] R. S. PHILLIPS, A note on ergodic theory, Proc. Amer. Math. Soc., 2(1951).

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