AN EXTENSION OF THE PLANCHEREL FORMULA TO UNIMODULAR GROUPS

HARUO SUNOUCHI

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Several extensions of the Plancherel formula to unimodular locally compact groups have been proposed under some restricted conditions by R. Godement, F. I. Mautner, I. E. Segal and the Russian mathematicians.[1)] These of Mautner [7,8] and Segal [12] were established by the use of the reduction theory of J. von Neumann [10], therefore the separability conditions of groups seem to be essential, and their results are somewhat measure-theoretic. On the other hand, Godement [3] has obtained a Plancherel formula by the method analogous to abelian groups, that is, by constructing the Radon measure on the set of characters. His method seems to be more elegant. But he assumed that $\mathbf{R}^S$ (the definition will be stated below) is of the finite class, and this condition is considerably strong (cf. [3 ; Theorem 6]).

The object of this paper is to give an extension of the Plancherel formula to arbitrary unimodular locally compact groups mostly along the Godement method Our main tool is the $\natural$-operation of arbitrary rings of operators defined in the previous papers [13, 14]. In Godement's paper [3]. the algebra $\mathbf{L}$ of continuous functions with compact supports on the group played an essential role. We replace this algebra $\mathbf{L}$ by the algebra $(\mathbf{R}_0^P)_F$ of the bounded linear operators defined by the bounded elements with some properties. But in the case of the abelian groups, our results do not coincide to the well-known Plancherel formula Therefore, if $\mathbf{R}^S$ is of the finite class, the Godement method is more natural than ours for this purpose. But it is interest that, if we assume $\mathbf{R}^S$ being of the finite class, we obtain the factor decomposition of $\mathbf{R}^S$ as shown in the previous paper [15].

This method of the factor decomposition of rings of operators will give a suggestion for the general one.

As remarked in the previous paper [15], in the double unitary representation of a group by a central Radon measure of the positive type (the definitions will be given below, § 1), our $\mathbf{R}_0^S$ is the maximal Hilbert algebra introduced by H. Nakano[2)]; man can easily see that our treatments are also

1) Numbers in brackets refer to the bibliography at the end of the paper. As for the complete bibliography related to this topic, see Mackey [6]. As the Russian papers are not yet available in this country, we omit them.

2) For the notions of Hilbert algebras, see [15] or H. Nakano, Hilbert algebras, Tôhoku Math. Journ., 2(1950), 4-23. O. Takenouchi, On the maximal Hilbert algebras, ibid 3(1951), 123-131.

available to the arbitrary maximal Hilbert algebras, but we shall not discuss them explicitly.

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1. Double unitary representations.

Let G be a *unimodular* locally compact group, which need not be separable. Following Godement [3],

DEFINITION 1.1. A *double unitary representation* (abr. d. u. r.) of G is a structure $\{\mathfrak{H}, U_s, V_s, S\}$ satisfying the following conditions;

a) $\mathfrak{H}$ is a Hilbert space,

b) $s\to U_s, s\to V_s$ are two continuous unitary representations of G on $\mathfrak{H}$, such that

$$U_sV_t = V_tU_s \qquad \text{for } s, t \in G,$$

c) S is an involution in $\mathfrak{H}$[3] such that

$$V_t = SU_tS^{-1}, \qquad \text{for } t \in G.$$

In the sequel we shall discuss only the following case: let μ be a *central Radon measure of the positive type* on G, that is, μ is a Radon measure satisfying

$$\int f*g(s)d\mu(s) = \int g*f(s)d\mu(s) \qquad \text{for } f, g \in \mathbf{L}, \tag{1.1}$$

$$\int \tilde{f}*f(s)d\mu(s) \geqq 0 \qquad \text{for } f \in \mathbf{L}, \tag{1.2}$$

where $\mathbf{L}$ is an algebra of continuous functions with compact supports on G. $\tilde{f}(s) = f(s^{-1})$; $f*g(s) = \int f(t)g(t^{-1}s)dt$; dt is a Haar measure on G. Then it is known that we can define a d.u.r. of G by such a measure ([3 ;p.16]). For the latter use we shall sketch the construction.

Put

$$\mathfrak{u}(\mu) = \left[\ f \in \mathbf{L} \ ; \ \int \tilde{f}*f(s)d\mu(s) = 0 \ \right], \tag{1.3}$$

then, as $\mathfrak{u}(\mu)$ is a two-sided ideal in the algebra $\mathbf{L}$, we obtain a quotient algebra $\mathbf{L}(\mu) = \mathbf{L}/\mathfrak{u}(\mu)$. Denote the canonical mapping of $\mathbf{L}$ on $\mathbf{L}/\mathfrak{u}(\mu)$ by $f\to\mathbf{f}(\mu)$. The expression:

$$< \mathbf{f}(\mu), \mathbf{g}(\mu) > = \int \tilde{g}*f(s)d\mu(s) \tag{1.4}$$

is an inner product on $\mathbf{L}(\mu)$, therefore, by completion with this inner product we obtain a Hilbert space $\mathfrak{H}(\mu)$ in which $\mathbf{L}(\mu)$ is dense. If we define the involution S and unitary operators $U_s(\mu), V_s(\mu)$ by

3) The involution S is such an operator on $\mathfrak{H}$ that $S(\mathbf{x}+\mathbf{y}) = S\mathbf{x}+S\mathbf{y}$, $S(\alpha\mathbf{x}) = \bar{\alpha}S\mathbf{x}$ $SS\mathbf{x} = \mathbf{x}$, $<S\mathbf{x}, S\mathbf{y}> = <\mathbf{y}, \mathbf{x}>$.

$$\mathbf{Sf}(\mu) = \widetilde{\mathbf{f}(\mu)} \qquad \text{for } f \in \mathbf{L}, \tag{1.5}$$

$$U_s(\mu)\mathbf{f}(\mu) = \mathcal{E}_s * \mathbf{f}(\mu), \quad V_s(\mu)\mathbf{f}(\mu) = \mathbf{f} * \mathcal{E}_s{}^{-1}(\mu) \qquad \text{for } f \in \mathbf{L}, \tag{1.6}$$

where $\mathcal{E}_s * f(t) = f(s^{-1}t)$, $f * \mathcal{E}_{s^{-1}}(t) = f(ts)$, then it may be clear that the structure $\{\mathfrak{H}(\mu), U_s(\mu), V_s(\mu), S\}$ is a d.u.r. The above obtained d.u.r. will be called a d.u.r. by μ, and hereafter we omit the notation (μ). If $\mu = \mathcal{E}$ (that is, a measure $+1$ on the unit e of G), we say the above d.u.r. is ***regular***. Mautner and Segal discussed only the regular case.

The d.u.r. by μ is studied by Godement in detail, so refer to [3; Chap I. § 1.]. The principal results[4] [3; Theorem 1] can be stated as follows:

THEOREM 1.1. *In the d. u. r.* $\{\mathfrak{H}, U_s, V_s, S\}$ *by* μ, *let* $\mathbf{R}^s$ *and* $\mathbf{R}^a$ *be the W*-algebras*[5] *generated in* $\mathfrak{H}$ *by* U_s *and* V_s, *respectively*; *then we obtain* $(\mathbf{R}^s)' = \mathbf{R}^a$, $(\mathbf{R}^a)' = \mathbf{R}^s$.

Therefore, denote by $\mathbf{R}^\natural$ the set of all bounded linear operators commute with U_s, V_s, then we obtain $\mathbf{R}^\natural = \mathbf{R}^s \cap \mathbf{R}^a$, that is, $\mathbf{R}^\natural$ is the *center* of $\mathbf{R}^s$ and $\mathbf{R}^a$.

DEFINITION 1.2. If we define the U_f and V_f by

$$U_f = \int U_s f(s) ds, \quad V_f = \int V_s f(s^{-1}) ds, \qquad \text{for } f \in \mathbf{L}, \tag{1.7}$$

then it is well-known that these U_f, V_f are the bounded linear operators on $\mathfrak{H}$ and that they satisfy the relation:

$$U_f \mathbf{g} = V_g \mathbf{f} = f * \mathbf{g} \qquad \text{for } \mathbf{g} (g \in \mathbf{L}). \tag{1.8}$$

Therefore we can define the operator U_x, V_x for any $\mathbf{x} \in \mathfrak{H}$ by

$$U_x \mathbf{f} = V_f \mathbf{x}, \quad V_x \mathbf{f} = U_f \mathbf{x}, \qquad \text{for } \mathbf{f} (f \in \mathbf{L}); \tag{1.9}$$

if these operators U_x and V_x are bounded, then we say that $\mathbf{x}$ is a *bounded element*.

For the bounded elements of $\mathfrak{H}$, the following facts are known (cf. [3; Chap I. § 1]):

LEMMA 1.1. 1°. *If* $\mathbf{x}$ *is bounded, then* $S\mathbf{x}$ *is also bounded and*

$$V_{Sx} = V_x{}^*, \quad U_{Sx} = U_x{}^*. \tag{1.10}$$

2° U_x *and* V_x *are related by*

$$V_{Sx} = SU_x S, \quad U_{Sx} = SV_x S. \tag{1.11}$$

3° $U_x \in \mathbf{R}^s$, $V_x \in \mathbf{R}^a$; *if* $A \in \mathbf{R}^s$ (*or* $\in \mathbf{R}^a$), *then* $A\mathbf{x}$ *is also bounded and*

$$AU_x = U_{Ax}, \quad (AV_x = V_{Ax}) \tag{1.12}$$

$$U_x A = U_{SA^*Sx}, \quad (V_x A = V_{SA^*Sx}). \tag{1.13}$$

4° *Let* $\mathbf{R}_0^s$ $(\mathbf{R}_0^a)$ *be the set of all* U_x (V_x) *for the bounded* $\mathbf{x}$, *then* $\mathbf{R}_0^s$ $(\mathbf{R}_0^a)$

4) As we use only these fundamental properties in this paper, our treatment is also available to the maximal Hilbert algebras; see [15].

5) By *W*-algebra* we shall mean a weakly closed operator algebra in a Hilbert space, and by *C*-algebra* a uniformly closed one, in the terminology of Segal [11].

is a two-sided ideal in $\mathbf{R}^s$ $(\mathbf{R}^d)$ *and (strongly) dense in* $\mathbf{R}^s$ $(\mathbf{R}^d)$.

5° *For the bounded* $\mathbf{x}, \mathbf{y}$, *the product*:

$$\mathbf{x}*\mathbf{y} = U_x\mathbf{y} = V_y\mathbf{x} \tag{1.14}$$

is well-defined and $\mathbf{x}*\mathbf{y}$ *is also a bounded element in* $\mathfrak{H}$, *satisfying*

$$U_{x*y} = U_xU_y, \quad V_{x\,y} = V_yV_x. \tag{1.15}$$

In this meaning, we shall call this $\mathbf{R}_0^s$, or $\mathfrak{H}_0$, the set of corresponding elements of $\mathfrak{H}$, the ***bounded algebra*** of the **d.** u. r.

DEFINITION 1.3. If $\mathbf{e} \in \mathfrak{H}$ is bounded and U_e is a projection, we call U_e a *bounded projection* in the d. u. r.[6] As be easily seen, U_e is a projection if and only if $S\mathbf{e} = \mathbf{e}$ and $\mathbf{e}*\mathbf{e} = \mathbf{e}$. This suggests the following definitions: an bounded element $\mathbf{x}$ is called *self-adjoint* (abr. s. a.) if $S\mathbf{x} = \mathbf{x}$, and *idempotent* if $\mathbf{x}*\mathbf{x} = \mathbf{x}$.

LEMMA 1.2. *If P is a non-zero projection in* $\mathbf{R}^s$, *then there exists a non-zero s.a. bounded element in the range of P.*

PROOF. As $P \neq 0$ and $\mathbf{L}$ is dense in $\mathfrak{H}$, there exists an element $\mathbf{f} \in \mathbf{L}$ such that $P\mathbf{f} \neq 0$. By 3° and 5° of Lemma 1.1, $P\mathbf{f}$, $SP\mathbf{f}$ and so $P\mathbf{f}*SP\mathbf{f}$ are bounded. $P\mathbf{f}*SP\mathbf{f}$ is the required one. Because, $P\mathbf{f}*SP\mathbf{f} = U_{Pf}SP\mathbf{f} = PU_fSP\mathbf{f}$, implies $P\mathbf{f}*SP\mathbf{f}$ is in the range of P. By the equation $S(P\mathbf{f}*SP\mathbf{f}) = SU_{Pf}(SS)SP\mathbf{f} = V_{Pf}P\mathbf{f} = P\mathbf{f}*SP\mathbf{f}$, it is s. a. Finally we shall show that it is non-zero. If we assume that $P\mathbf{f}*SP\mathbf{f}$ be zero, then $V_{Pf*SPf}\mathbf{g} = 0$ for any $\mathbf{g} \in \mathbf{L}$, therefore we have

$$< V_{Pf*SPf}\mathbf{g}, \mathbf{g} > = < V_{Pf}\mathbf{g}, V_{Pf}\mathbf{g} > = 0 \qquad \text{for } \mathbf{g} \in \mathbf{L};$$

it is easily seen that this implies $P\mathbf{f} = 0$, thus we obtain the contradiction.

Then we have the following theorem by the quite similar manner to [12; Theorem 2]; we omit the proof.

THEOREM 1.2. *Every projection in* $\mathbf{R}^s$ *is the least upper bound of the bounded projections which it bounds.*

LEMMA 1.2. *For any bounded element* $\mathbf{x}$, U_x *is approximated uniformlv by the linear combinations of bounded projections, say* U_{p_n}. *And the same time,* $\mathbf{x}$ *is approximated (strongly) by the linear combinations of the corresponbing s. a. idempotent elements.*

This lemma was given by Segal in the proof of [12; Theorem 4].

2. $\natural$-operations in W^*-algebras.

We consider a general W^*-algebra $\mathbf{M}$ on a Hilbert space $\mathfrak{H}$, and denote its *center* by $\mathbf{M}^\natural$. The $\natural$-operation in a W^*-algebra $\mathbf{M}$ has been introduced by Dixmier [1] under the condition that $\mathbf{M}$ is of the finite class, and this notion is extended to an arbitrary W^*-algebra in the previous papers [13, 14].

We say that a projection $P \in \mathbf{M}$ is *finite* if a projection $Q \in \mathbf{M}$, $P \sim Q \leqq P$ implies $Q = P$, and *infinite* in the other case. If the unit element

6) This is no others than the *finite* projection in the termniology of Segal [12].

$I \in \mathbf{M}$ is finite, $\mathbf{M}$ is said to be *of the finite class*, and otherwise *of the infinite class*. As mentioned in the previous paper [13], $\mathbf{M}$ is generally a direct sum of three W^*-algebras, $\mathbf{M}^f$, $\mathbf{M}^i$ and $\mathbf{M}^{pi}$, say; $\mathbf{M}^f$ is of the finite class, $\mathbf{M}^i$ and $\mathbf{M}^{pi}$ are of the infinite class; in $\mathbf{M}^i$, every central projection is infinite but contains a finite projection in it, and $\mathbf{M}^{pi}$ is the other case. Especially $\mathbf{M}^{pi}$ is called *of the purely infinite class*. Hereafter we assume that $\mathbf{M}$ contains no direct summand of the purely infinite class.

By a *central envelope* Z of a projection $P \in \mathbf{M}$, we mean the least central projection containing P. Then there exists a system of finite projections E_α such that corresponding central envelopes Z_α are mutually orthogonal and span the unit I. Denote $E = \Sigma \oplus E_\alpha$, then E is also finite [14; Lemma 1.1]; we shall call this E the *generalised unit* of $\mathbf{M}$. In general, any projection $P \in \mathbf{M}$ is decomposed in a following way with respect to the generalised unit E [14; Lemma 1.2]:

$$P = \sum_{\alpha \in A_1} \oplus E_1^\alpha \oplus F_1 \oplus \sum_{\alpha \in A_2} \oplus E_2^\alpha \oplus F_2 \oplus \cdots \oplus \sum_{\alpha \in A_\mu} \oplus E_\mu^\alpha, \tag{2.1}$$

where $E_\mu^\alpha \sim E_\mu^\beta$, $E_\nu^\alpha \prec E_\mu^\alpha \prec E$ $(\nu > \mu)$, and $F_\mu \prec E_\mu^\alpha$ has no comparable part to the remainders. If the above expression (2.1) ends up with finite terms and every A_i is finite, we say P is *E-finite*; if a operator $A \in \mathbf{M}$ be contained in some E-finite projection P, that is, $AP = PA = A$, then we say A is *E-finite*. With these definitions the results in the previous paper [14; Theorem 2] can be stated as follows:

THEOREM 2.1. *For any E-finite* $A \in \mathbf{M}$, *we can define a mapping* $A \to A^\natural \in \mathbf{M}^\natural$ *satisfying the following properties*:

(i) $A \in \mathbf{M}^\natural$ *implies* $A^\natural = A$,

(ii) $(\alpha A)^\natural = \alpha A^\natural$,

(iii) $(A+B)^\natural = A^\natural + B^\natural$,

(iv) $(AB)^\natural = (BA)^\natural$,

(iv) *For any (not necessarily E-finite)* $C \in \mathbf{M}^\natural$, $(CA)^\natural = CA^\natural$,

(v) *If* A *is s.a. and* $A \geqq 0$, *then* $A^\natural$ *is s.a. and* $A^\natural \geqq 0$,

(v) *If* A *is s.a.*, $A \geqq 0$ *and* $A^\natural = 0$, *then* $A = 0$,

(vi) $(A^*)^\natural = (A^\natural)^*$,

But this $\natural$-operation depends on the choice of the generalised unit of $\mathbf{M}$; if there exists another finite projection $E' \in \mathbf{M}$, of which central envelope spans the unit I, we can define another mapping $A \to A^{\natural\prime}$ for any E'-finite operator A with respect to this E'. *Suppose that E' be E-finite, then any E'-finite operator becomes also E-finite; two operations $A^\natural$ and $A^{\natural\prime}$ are related by*

$$A^{\natural\prime} = (E'^\natural)^{-1} A^\natural. \tag{2.2}$$

We have defined the $\natural$-operation for any finite (not necessarily E-finite) operators in [14], but we shall not use this generalized notion in this paper.

Now return to the case of the d. u. r. of G by μ. When $\mathbf{R}^s$ is of the finite class, there exist the interesting relations between the $\natural$-operation of $\mathbf{R}^s$ and the elements of Hilbert space or the structure of G; these facts are discussed in detail by Godement [3; Chap. I]. Finally we shall note an important

LEMMA 2.1. *Any bounded projection in* $\mathbf{R}^s$ *is finite in the sense of the* W^**-algebra.*

PROOF. Let U_e be a bounded projection in $\mathbf{R}^s$, and let P be a projection in $\mathbf{R}^s$ such that $P \leqq U_e$, $P \sim U_e$. Then there exists a partially isometric operator $W \in \mathbf{R}^s$ such that $WW^* = P$, and $W^*W = U_e$. $P = WW^*U_e = U_{WW^*e}$; evidently this implies $WW^*e \neq 0$. As W and W^* are partially isometric, $\| e \| = \| WW^*e \|$; WW^*e is an image of e by the projection $WW^* = P$, so that $e = WW^*e$ or we have $P = U_e$. This completes the proof.

Thus we can see that $\mathbf{R}^s$ has no direct summand of the purely infinite class, by Theorem 1.2, and that the above theorem is available for $\mathbf{R}^s$.

3. **Traces on $*$-algebras.**

By *$*$-algebra* we shall mean, as usual, an algebra which has an operation A^* satisfying 1° $(\alpha A + \beta B)^* = \bar{\alpha}A^* + \bar{\beta}B^*$, 2° $(AB)^* = B^*A^*$, 3° $A^{**} = A$.

DEFINITION 3.1. A linear functional σ on a $*$-algebra $\mathbf{A}$ is called a *state* if $\sigma(A^*A) \geqq 0$ for $A \in \mathbf{A}$, and a *trace* if it is a state and satisfies $\sigma(AB) = \sigma(BA)$ for $A, B \in \mathbf{A}$. A trace (or state) is said to be *bounded* if there exists a constant M such that

(3.1) $$|\sigma(A)| \leqq M\sigma(A^*A), \qquad \text{for } A \in \mathbf{A}.$$

The *double unitary representations of a $*$-algebra* by the trace (for the definition of the d. u. r.[7] of a $*$-algebra, see [15; Definition 3.1]) have been already discussed by Nakamura [9]. He treated only a C^*-algebra with the unit element, but most part of his results holds true in the case of a $*$-algebra with a trace. (cf. also [2; Chap. II]).

Next we shall prove the following generalization of Lemma 15 of [3], which is also interest in the theory of the W^*-algebras.

THEOREM 3.1. *Let* $\mathbf{M}$ *be a* W^**-algebra without a part of the purely infinite class, and let* $\mathbf{M}_F$ *be a* $*$*-algebra generated by the E-finite operators in* $\mathbf{M}$. *Then there exists a maximal two-sided ideal in* $\mathbf{M}_F$; *moreover, there exists a one-to-one correspondence between the maximal two-sided ideals and the maximal ideals in* $\mathbf{M}^\natural$.

First, we shall prove some lemmas.

LEMMA 3.1. *Let* $\mathfrak{m}$ *be a two-sided ideal in* $\mathbf{M}_F$, *and let* $\mathfrak{m}^\natural$ *be the image of* $\mathfrak{m}$ *by the* $\natural$*-operation, then* $\mathfrak{m}^\natural$ *is an ideal in* $\mathbf{M}^\natural$.

7) It must be noted here that in this case, U_A is not necessarily a unitary operator, but it satisfies only the relation $U_{A^*} = U_A{}^*$.

PROOF. If $A,\ B \in \mathbf{m}^\natural$, then $A+B \in \mathbf{m}^\natural$, $\alpha A \in \mathbf{m}^\natural$ are evident. Let $T \in \mathbf{M}^\natural$, $A \in \mathbf{m}$. then $TA^\natural = (TA)^\natural$ by Theorem 2.1. (ivβ), and $TA \in \mathbf{M}_F$, so $TA^\natural \in \mathbf{m}^\natural$. Thus it is sufficient to prove that $\mathbf{m}^\natural \neq \mathbf{M}^\natural$. Suppose $\mathbf{m}^\natural = \mathbf{M}^\natural$, then it implies that the unit $I \in \mathbf{m}^\natural$, or it implies that there exists a projection $E' \in \mathbf{m}$ such that $E' \sim E$, by the definition of the ♮-operation. But, generally, if a projection P is contained in a two-sided ideal $\mathbf{m}$, and if a projection $Q \sim P$, then $Q \in \mathbf{m}$. Indeed, denote by W the partially isometric operator which gives the equivalence $Q \sim P$, then W and its adjoint W^* are contained in $\mathbf{M}_F$ because W and W^* are contained in an E-finitep rojection $P \cup Q$. Thus we obtain $Q = QQ = WW^*WW^* = WPW^* \in \mathbf{m}$. By this fact, any E-finite projection is contained in $\mathbf{m}$ and we obtain $\mathbf{m} = \mathbf{M}_F$; this is a contradiction.

LEMMA 3.2. *Let* $\mathbf{n}$ *be an ideal in* $\mathbf{M}^\natural$, *then*

$$\mathbf{m} = \{A \in \mathbf{M}_F\ ;\ (AT)^\natural \in \mathbf{n} \quad \textit{for all } T \in \mathbf{M}_F\} \tag{3.2}$$

is a two-sided ideal in $\mathbf{M}_F$.

PROOF. If $A, B \in \mathbf{m}$, then $A+B$, $\alpha A \in \mathbf{m}$ are evident. Let $A \in \mathbf{m}$, $S \in \mathbf{M}_F$, then we obtain AS, $SA \in \mathbf{m}$; because, for any $T \in \mathbf{M}_F$, $(AST)^\natural \in \mathbf{n}$ and $(SAT)^\natural = (ATS)^\natural \in \mathbf{n}$. $\mathbf{m} \neq \mathbf{M}_F$ is as follows: the unit $I \notin \mathbf{n}$, so that for the generalized unit E, $I = E^\natural = (EE)^\natural \notin \mathbf{n}$, therefore we obtain $E \notin \mathbf{m}$.

PROOF OF THE THEOREM. Let $\mathbf{n}$ be a maximal ideal in $\mathbf{M}^\natural$, then $\mathbf{m}$, given by (3.2) is a two-sided ideal in $\mathbf{M}_F$. Let there exists a two-sided ideal $\mathbf{m}'$ containing $\mathbf{m}$, then $(\mathbf{M}')^\natural$ is an ideal in $\mathbf{M}^\natural$ by Lemma 3.1. Suppose a maximal ideal $\mathbf{n}_1$ in $\mathbf{M}^\natural$ containing $(\mathbf{m}')^\natural$, and denote by $\mathbf{m}_1$ the two-sided ideal in $\mathbf{M}_F$ given by (3.2) for $\mathbf{n}_1$. Then $\mathbf{m} \subseteqq \mathbf{m}' \subseteqq \mathbf{m}_1$ is clear, so that $\mathbf{n} \subseteqq \mathbf{n}_1$ and we have $\mathbf{n} = \mathbf{n}_1$ as $\mathbf{n}$ is maximal. Therefore we obtain $\mathbf{m} = \mathbf{m}'$, that is, $\mathbf{m}$ is maximal. Thus we see that there is a maximal two-sided ideal in $\mathbf{M}_F$, and it corresponds to the maximal ideal in $\mathbf{M}^\natural$.

Conversely, let $\mathbf{m}$ be a maximal ideal in $\mathbf{M}_F$, and suppose that the corresponding $\mathbf{m}^\natural$ is not maximal in $\mathbf{M}^\natural$. Then there is a maximal ideal $\mathbf{n}$ containing $\mathbf{m}^\natural$ properly. Consider the two-sided ideals $\mathbf{m}'$ and $\mathbf{m}_1$ in $\mathbf{M}_F$, given by (3.2) for $\mathbf{m}^\natural$ and $\mathbf{n}$, respectively, then clearly $\mathbf{m}_1 \supseteqq \mathbf{m}' \supseteqq \mathbf{m}$. But there exists a $A \in \mathbf{M}_F$ such that $A^\natural \in \mathbf{n} - \mathbf{m}^\natural$, therefore there exists a $B \in \mathbf{M}^\natural$ such that $BA^\natural \notin \mathbf{m}^\natural$, this implies $A \in \mathbf{m}_1 - \mathbf{m}'$; this fact contradicts the maximality of $\mathbf{m}$. Thus we obtained the proof.

By the above argument, we see that A is contained in the maximal $\mathbf{m}$ if and only if $(AT)^\natural \in \mathbf{m}^\natural$ for all $T \in \mathbf{M}_F$.

As we have defined the ♮-operation for the $*$-algebra $\mathbf{M}_F$ in § 2, put

$$f(A) = f'(A^\natural) \qquad \text{for } A \in \mathbf{M}_F, \tag{3.3}$$

where f' is a trace on $\mathbf{M}^\natural$, then we obtain a trace on $\mathbf{M}_F$.

Now let $\mathbf{m}$ be a maximal two-sided ideal in $\mathbf{M}_F$, then $\mathbf{m}^\natural$ is a maximal ideal in $\mathbf{M}^\natural$ $\cdot$ it is well-known that in $\mathbf{M}^\natural$ there exists the one-to-one correspondence between the maximal ideals and the characters, and they are related by the condition: $A \in \mathfrak{m}^\natural$ if and only if $\mathcal{X}'(A) = 0$, for a character $\mathcal{X}'$. Therefore, by the above remark, we obtain that $A \in \mathfrak{m}$ if and only if $\mathcal{X}'((AT)^\natural) = 0$ for all $T \in \mathbf{M}_F$; by (3.3), $A \in \mathfrak{m}$ is equivalent to $\mathcal{X}(AT) = 0$ for all $T \in \mathbf{M}_F$. That is, $A \in \mathfrak{m}$ is equivalent to $\mathcal{X}(A^*A) = 0$, by the Schwarz inequality. Thus, the maximal two-sided ideal in $\mathbf{M}_F$ is characterised by the trace introduced by the character of $\mathbf{M}^\natural$. In this sense, we shall call such a trace a *character* of $\mathbf{M}_F$. Then the following theorem is obtained:

THEOREM 3.2. *There exists a one-to-one correspondence between the characters $\mathcal{X}$ of $\mathbf{M}_F$ and the characters $\mathcal{X}'$ of $\mathbf{M}^\natural$, and they are related by*

$$\mathcal{X}(A) = \mathcal{X}'(A^\natural). \tag{3.4}$$

4. An extension of the Plancherel formula.

Consider again the d.u.r. of G by the central Radon measure of the positive type μ. Because $\mathbf{R}^s$ has no part of the purely infinite class, as remarked in §2, $\mathbf{R}^s$ is a direct sum of two W^*-algebras: of the finite class $(\mathbf{R}^s)^f$ and of the infinite class $(\mathbf{R}^s)^i$. As the generalized unit E of $\mathbf{R}^s$, we take $E = I^f \oplus \Sigma \oplus U_{e_\alpha}$, where $I^f = Z_0$ is the unit element of $(\mathbf{R}^s)^f$, and U_e^α are the bounded projection defined by e_α; the possibility of this choice is due to Theorem 1.2. Let the corresponding central envelopes be Z_α, then the unit element I^i of $(\mathbf{R}^s)^i$ is spanned by Z_α. This system $\{Z_0; U_{e_\alpha}, Z_\alpha\}$ will be called the *defining system of the $\natural$ operation.* In $(\mathbf{R}^s)^f$, we can also take some system of bounded projections U_{e_β} such that $\Sigma \oplus U_{e_\beta} \oplus \Sigma \oplus U_{e_\alpha} = E$, and define a $\natural$-operation with respect to this system, then we can discuss the both parts by the unified method. But such $\natural$-operation has some pathological properties as shown in [14; §3], so it is preferable to treat as heer.

As be well-known, the set of all characters of $\mathbf{R}^\natural$ is a compact (totally-disconnected) space $\overline{\Omega}$ in the weak topology; by the above partition I^f and I^i of the unit I the space $\overline{\Omega}$ is decomposed into the direct sum of two compact spaces $\overline{\Omega}^f$ and $\overline{\Omega}^i$, say. By Theorem 3.2, there is the one-to-one correspondences between the characters of $\mathbf{R}^s_F$, generated by the E-finite operators in $\mathbf{R}^s$, and the characters of $\mathbf{R}^\natural$. If we introduce the weak topology in the set of the characters of $\mathbf{R}^s_F$, then by (3.3), we obtain a homeomorphism between them, because $\overline{\Omega}$ is compact; denote by $\overline{\mathbf{X}}$ the set of characters of $\mathbf{R}^s_F$, then $\overline{\mathbf{X}}$ becomes compact and $\overline{\mathbf{X}} = \overline{\mathbf{X}}^f \oplus \overline{\mathbf{X}}^i$, each of which corresponds to $\overline{\Omega}^f$ and $\overline{\Omega}^i$, respectively. In each $\mathbf{R}^s_\alpha = Z_\alpha \mathbf{R}^s$ $(\alpha \neq 0)$, any E-finite operator is defined by a bounded element in $\mathfrak{H}$, but this is not the case in $(\mathbf{R}^s)^f$. Therefore it is convenient to introduce the $*$-algebra $(\mathbf{R}^s_0)_F$, generated by the E-finite operators in $\mathbf{R}^s_0$, (see, §1); we shall denote by $(\mathfrak{H}_0)_F$ the set of the corresponding element of $\mathfrak{H}$ to the operators in $(\mathbf{R}^s_0)_F$. It is clear that $(\mathbf{R}^s_0)^f = (\mathbf{R}^s_0)^f_F$ and $(\mathbf{R}^s)^i = (\mathbf{R}^s_0)^i_F$. Now let us contract the character $\mathcal{X}$ in $\overline{\mathbf{X}}$ to $(\mathbf{R}^s_0)_F$

and consider this as a trace σ_χ on $(\mathbf{R}_0^s)_F$; if we introduce also the weak topology in the set of the traces on $(\mathbf{R}_0^s)_F$, then the mapping $\chi \rightarrow \sigma_\chi$ is continuous, and $\overline{\mathbf{X}}$ is compact, so that the image $\widetilde{\mathbf{X}}$ of $\overline{\mathbf{X}}$ is also compact; omit the trace $\sigma \equiv 0$ on $(\mathbf{R}_0^s)_F$, then we obtain a locally compact space $\mathbf{X}$. Clearly, $\mathbf{X} = \mathbf{X}^{f} \oplus \mathbf{X}^{i}$, and $\mathbf{X}^{i} = \overline{\mathbf{X}^{i}}$; thus we obtain a locally compact space, which plays a role of the dual object of the Plancherel formula.

As $\sigma \in \mathbf{X}$ is a trace on the $*$-algebra $(\mathbf{R}_0^s)_F$, we obtain a d. u. r. as stated in §3. That is, $\mathfrak{u}(\sigma) = \{U_x;\ \sigma(U_x{}^*U_x) = 0\}$ is a two-sided ideal in $(\mathbf{R}_0^s)_F$, therefore we obtain a canonical mapping $U_x \rightarrow \mathbf{x}(\sigma)$ to the quotient algebra. Put

$$(4.1) \qquad < \mathbf{x}(\sigma),\ \mathbf{y}(\sigma) > = \sigma(U_y{}^* U_x),$$

this is an inner product on the quotient algebra; by completion with this inner product we obtain a Hilbert space $\mathfrak{H}(\sigma)$. To construct the d. u. r., it is sufficient to put

$$(4.2) \qquad U_x(\sigma)\mathbf{y}(\sigma) = \mathbf{x}*\mathbf{y}(\sigma), \quad V_x(\sigma)\mathbf{y}(\sigma) = \mathbf{y}*\mathbf{x}(\sigma),$$

$$(4.3) \qquad S(\mathbf{x}(\sigma)) = (S\mathbf{x})(\sigma).$$

These reasons are quite analogous to the one sketched in §1 for the d. u. r. of the group; for detail, see Nakamura [9]. Thus we can correspond to each $\sigma \in \mathbf{X}$ a Hilbert space $\mathfrak{H}(\sigma)$. Since we have introduced in $\mathbf{X}$ the weak topology, for each $U_x \in (\mathbf{R}_0^s)_F$, $\sigma(U_x)$ is a continuous function with respect to σ: the vector-function $\mathbf{x}(\sigma)$, defined on $\mathbf{X}$ to $\mathfrak{H}(\sigma)$, is continuous; by the construction mentioned above, the set of $\mathbf{x}(\sigma)$ is dense in each $\mathfrak{H}(\sigma)$, so that the vector-functions $\mathbf{x}(\sigma)$ form a fundamental family of the continuous vector-functions Λ in the sense of Godement [2; Chap. III].

If $\sigma \in \mathbf{X}$ approches to the infinity, then evidently $\mathbf{x}(\sigma) \rightarrow 0$; thus the vector-function $\mathbf{x}(\sigma)$ has an analogous property with the ordinary Fourier transform. Now our object is to generalize the Plancherel theorem in the following form:

THEOREM 4.1. *Let G be a unimodular locally compact group, and let a d. u. r.* $\{\mathfrak{H}, U_s, V_s, S\}$ *be constructed by a central Radon measure of the positive type* μ *on G. Then there exist a locally compact space* $\mathbf{X}$ *and a measure* $\hat{\mu}$ *on* X, *possessing the following properties*:

a) *for any* $\mathbf{x}, \mathbf{y} \in (\mathfrak{H}_0)_F$,

$$(4.4) \qquad < \mathbf{x}, \mathbf{y} > = \int_{\mathbf{X}} < \mathbf{x}(\sigma), \mathbf{y}(\sigma) > d\hat{\mu}(\sigma),$$

b) $\mathfrak{H}$ *is isomorphic to* L^2_Λ.

Here we shall freely use the notion of the continuous sums of the Hilbert spaces proposed by Godement [2][8], but some generalizations are necessary.

8) The notion of the continuous sum of Banach spaces was already discussed by Kondo [4. 5] and his results are stronger than Godement's in some points, but the latters seems to be more suited to our purposes.

DEFINITION 4.1. We call the *fundamental family of the continuous vector-functions* (la famille fondamentale de champs de vecteurs continus) the set Λ of the vector-functions on $\mathbf{X}$, satisfying the following axioms: (Λ_1): Λ is a linear subspace of the space of all the vector-functions defined on X; (Λ_2): for any $\mathbf{X} \in \Lambda$, the scalar function $\|\mathbf{x}(\sigma)\|$ is continuous on $\mathbf{X}$; (Λ_3); for any $\sigma \in \mathbf{X}$, the $\mathbf{x}(\sigma)\,(\mathbf{x} \in \Lambda)$ are dense in $\mathfrak{H}(\sigma)$. And we say that a vector-function $\mathbf{x}$ on X is *continuous in a point* σ_0, if for any $\varepsilon > 0$, there exist a neighborhood V of σ_0 and a $\mathbf{y} \in \Lambda$ such that $\|\mathbf{x}(\sigma) - \mathbf{y}(\sigma)\| < \varepsilon$ for any $\sigma \in V$.

DEFINITION 4.2.[9] We say that a vector-function $\mathbf{x}$ is *squarely-summable* (with respect to Λ and $\hat{\mu}$), if $\int_{\mathrm{X}} < \mathbf{x}(\sigma), \mathbf{x}(\sigma) > d\hat{\mu}(\sigma) < +\infty$ and for any $\varepsilon > 0$, there exists a continuous $\mathbf{y}$ such that $\left(\int_{\mathrm{X}} \|\mathbf{x}(\sigma) - \mathbf{y}(\sigma)\|^2 \, d\hat{\mu}\,(\sigma)\right)^{1/2} < \varepsilon$.
The set of these vector-functions, modulo the null-set, and defined the norm by

$$\|\mathbf{x}\| = \left(\int_{\mathrm{X}} < \mathbf{x}(\sigma),\ \mathbf{x}(\sigma) > d\hat{\mu}(\sigma)\right)^{1/2}, \tag{4.5}$$

will be denoted by L^2_Λ. Then we can easily see that L^2_Λ forms a Hilbert space with the above norm.

5. Proof of Theorem 4.1.

The aim of our proof is to construct a Radon measure $\hat{\mu}_\alpha$ on each compact (or locally compact) set Γ_α, which gives the required formula with respect to the elements of $(\mathfrak{H}_0)_F$; and then to extend these measures $\hat{\mu}_\alpha$ to a measure $\hat{\mu}$ on the whole space $\mathbf{X}$ and to all elements of $\mathfrak{H}$. Here Γ_α denotes the subset of $\mathbf{X}$, which corresponds to the defining system Z_α. In the part of the finite class, this problem is reduced to the theorem 3.1 of [15], because $Z_0\mathfrak{H}_0$ is a maximal Hilbert algebra of the finite class. Therefore it is sufficient to consider only the part of the infinite class. We will reduce the proof to the case of the finite class: this is an analogous procedure to the construction of the $\natural$-operation in a W^*-algebra of the infinite class, discussed in [14].

Let $F(\sigma)$ be a continuous function on $\mathbf{X}'$, then it corresponds to an operator $U_F \in \mathbf{R}^\natural$, because $\mathbf{X}'$ is isomorphic to the Boolean space of $(\mathbf{R}^\natural)'$; moreover we can take a $U_x \in (\mathbf{R}^s_0)_F$ such that $F(\sigma) = \sigma(U_x)$ on Γ_α by the definition of the $\natural$-operation. As shown in Lemma 1 of [13], $Z_\alpha\,\mathbf{R}^s$ is isomorphic to the $\mathbf{R}^s_{(U_{e_\alpha})}$, where $\mathbf{R}^s_{(U_{e_\alpha})}$ is the all $A \in \mathbf{R}^s$ such that $AU_{e_\alpha} = U_{e_\alpha}\cdot A = A$, and this is a W^*-algebra of the finite class on the Hilbert space $U_{e_\alpha}\mathfrak{H}$. Now we shall show

9) Godement considered only the Radon measure $\hat{\mu}$ on X, but our measure $\hat{\mu}$ is not necessarily a Radon measure. So we need this formally generalized definition.

LEMMA 5.1. *A necessary and sufficient condition that a* U_x *should be in* $\mathbf{R}_{(U^s e_\alpha)}$ *is that* $\mathbf{x}$ *should be in* $U_{e_\alpha}V_{e_\alpha}\mathfrak{H}$.

PROOF. Assume that $\mathbf{x}\in U_{e_\alpha}V_{e_\alpha}\mathfrak{H}$, then $U_{e_\alpha}V_{e_\alpha}\mathbf{x}=\mathbf{x}$, or $U_{e_\alpha}V_{e_\alpha}U_x, =U_x$, this implies $U_{e\alpha}U_x=U_x$ and $U_xU_{e\alpha}=U_{e_\alpha}U_{U_{e_\alpha}}V_{e_\alpha}x=U_{e_\alpha}U_x=U_x$, that is, $U_x\in\mathbf{R}_{(U e_\alpha)}$. Conversely if $U_x\in\mathbf{R}^s{}_{(U e_\alpha)}$, then clearly we obtain $U_{e_\alpha}\mathbf{x}=\mathbf{x}$, and $V_{e_\alpha}\mathbf{x}=\mathbf{x}$; this complete the proof.

The above lemma shows that the elements $\mathbf{x}$ corresponding to $U_x\in\mathbf{R}^s{}_{(U e_\alpha)}$ form the maximal Hilbert algebra $(U_{e_\alpha}V_{e_\alpha}\mathfrak{H})_0$ in $U_{e_\alpha}V_{e_\alpha}\mathfrak{H}$. So that we can apply the results of [15] to the part of $U_{e_\alpha}V_{e_\alpha}\mathfrak{H}$; as the $\natural$-operation is defined in $(U_{e_\alpha}V_{e_\alpha}\mathfrak{H})_0$, for each $\sigma(U_x{}^\natural)$, $\mathbf{x}\in(U_{e_\alpha}V_{e_\alpha}\mathfrak{H})$, if we take a real $\sigma(U_y{}^\natural)$, $\mathbf{y}\in(U_{e_\alpha}V_{e_\alpha}\mathfrak{H})$, such that $\sigma(U^\natural{}_x)=\sigma(U^\natural_x)\sigma(U^\natural_y)$ and put

$$I_\alpha(\sigma(U^\natural{}_x))=<\mathbf{x}^\natural,\mathbf{y}^\natural>, \tag{5.1}$$

where $\mathbf{x}^\natural$ is the projection of $\mathbf{x}$ to $(U_eV_{e_\alpha}\mathfrak{H})^\natural$ ([15; Theorem 2.1]), we obtain a Radon measure $\hat{\mu}_\alpha$ on Γ_α such that

$$I_\alpha(\sigma(U^\natural_x))=\int_{\Gamma_\alpha}\sigma(U^\natural_x)d\hat{\mu}_\alpha(\sigma)=<\mathbf{x}^\natural,\mathbf{y}\ >, \tag{5.2}$$

or by the same reasons to [15], we obtain

LEMMA 5.2. *For* $\mathbf{x},\ \mathbf{y}\in(U_{e_\alpha}V_{e_\alpha}\mathfrak{H})_0$ *and* $U_F\in Z_\alpha\mathbf{R}^\natural$,

$$<\mathbf{x},U_F\mathbf{y}>=\int_{\Gamma_\alpha}<\mathbf{x}(\sigma),\ \mathbf{y}(\sigma)>\overline{F(\sigma)}\,d\hat{\mu}_\alpha(\sigma). \tag{5.3}$$

But there exists an $\mathbf{x}^\natural$ not necessarily in $U_{e_\alpha}V_{e_\alpha}\mathfrak{H}$, but $\sigma(U)^\natural_x=\sigma(U^\natural_{x1})$ on Γ_α. Therefore we will assume that if $U_x\in\mathbf{R}^s{}_{(P)}$, $P\precsim U_{e_\alpha}$, then a U_y, required in (5.1), should be taken in the same $\mathbf{R}^s{}_{(P)}$. In this assumption, we have

LEMMA 5.3. *The positive linear functional* I_α *on the space of the continuous functions of* Γ_α *is well-defined by the relation* (5.1).

PROOF. First, let E-finite projections P and P_1 be equivalent with the partially isometric operator W; let U_x and U_{x1} be contained in P and P_1 respectively, and assume $U^\natural_x=U^\natural_{x1}$. Then, if we denote the $\mathbf{x}^\natural$ and $\mathbf{x}^\natural_1$ the images of the $\natural$-operation defined in $PSPS\mathfrak{H}$ and $P_1SP_1S\mathfrak{H}$ resp., we have $\mathbf{x}^\natural=W^*SW^*S\mathbf{x}^\natural_1$. In fact by [13; Lemma 1], we have $U_{x\natural}=PU^\natural_x$ and $U_{x_1}{}^\natural=P_1U^\natural_{x1}=P_1U^\natural_x$. Therefore, $W^*U_{x_1\natural}\ W=W^*P_1WU_{x\natural}=PU_{x\natural}=U_{x\natural}$. Evidently, this implies $\mathbf{x}^\natural=W^*SW^*S\mathbf{x}_1{}^\natural$. As W^*SW^*S is partially isometric, and as it is sufficient to consider the above case by the definition of the $\natural$-operation [14; § 2], we obtain the proof.

Summarizing the mentioned above, we see

LEMMA 5.4. *Let* $\mathbf{x},\mathbf{y}\in(\mathfrak{H}_0)_F$, *and let* F *be a continuous function on* $\mathbf{X}$, *then we have a unique Radon measure* $\hat{\mu}_\alpha$ *on* Γ_α *such that*

(5.4) $$< Z_\alpha, \ U_F\mathbf{y} > = \int_{\Gamma_\alpha} < \mathbf{x}(\sigma), \mathbf{y}(\sigma) > \overline{F(\sigma)} d\hat{\mu}_\alpha(\sigma).$$

Now we shall extend the Radon measure $\hat{\mu}_\alpha$ constructed above on each Γ_α to a regular measure $\hat{\mu}$ on the whole space $\mathbf{X}$. But this will be done by the well-known extension theorem of measures[10]; let $\mathbf{S}$ be a ring, generated by the compact sets contained in some Γ_α, that is, $S \in \mathbf{S}$ if and only if S is the following form: $S = \bigcup_{\iota=1}^{n} S_{\alpha,\iota}$, $S_{\alpha,\iota} \subseteqq \Gamma_\alpha$ and compact; if we put

(5.5) $$\hat{\mu}(S) = \sum_{\iota=1}^{n} \hat{\mu}_\alpha(S_{\alpha,\iota}),$$

then $\hat{\mu}$ becomes a σ-finite measure on $\mathbf{S}$, so that we have a *unique σ-finite measure* $\hat{\mu}$ on the σ-finite ring $\overline{\mathbf{S}}$, generated by $\mathbf{S}$.

Let $\mathbf{x}$ be a bounded element in $\mathfrak{H}$, then it is evident that the set of $\alpha \in A$ such that $Z_\alpha \mathbf{x} \neq 0$ is at most countable, say α_i; $\mathbf{x} = \sum_{\iota=1}^{\infty} Z_{\alpha_\iota}\mathbf{x}$. Furthermore, let U_x be E-finite, and denote $\varphi_{\Gamma_\alpha}(\sigma)$ the characteristic function of Γ_α, then we can easily see that $\varphi_{\Gamma_\alpha}(\sigma)\sigma(U_x) = \sigma(Z_\alpha U_x) = \sigma(U_{Z_\alpha x})$, so that $\alpha \in A$ such that $\varphi_{\Gamma_\alpha}(\sigma)\sigma(U_x) \not\equiv 0$ are at most countable. Hence $\sigma(U_x)$ becomes measurable for the above $\hat{\mu}$, and

(5.6) $$\int_{X} \sigma(U_y{}^* U_x) \overline{F(\sigma)} d\hat{\mu}(\sigma) = \sum_{\iota=1}^{\infty} \int_{\Gamma_{\alpha_\iota}} \sigma(U_y{}^* U_x) \overline{F(\sigma)} d\hat{\mu}_{\alpha_\iota}(\sigma)$$

$$= \sum_{\iota=1}^{\infty} < Z_{\alpha_\iota}\mathbf{x}, \ U_F\mathbf{y} > = < \mathbf{x}, U_F\mathbf{y} > .$$

Thus we obtain

LEMMA 5.5. *For bounded $\mathbf{x}, \mathbf{y} \in \mathfrak{H}$, which give the E-finite operators U_x and U_y, we have*

(5.7) $$< \mathbf{x}, U_F\mathbf{y} > = \int_{X} < \mathbf{x}(\sigma), \ \mathbf{y}(\sigma) > \overline{F(\sigma)} d\hat{\mu}(\sigma).$$

But there exist bounded elements of $\mathfrak{H}$, which give the not necessarily E-finite operators. First we note

LEMMA 5.6. *Any s.a. idempotent element* $\mathbf{e} \in \mathfrak{H}$ *is the (strong) limit of the elements of* $(\mathfrak{H}_0)_F$

PROOF. It is sufficient to consider in the part of the infinite class. As U_e is a bounded projection, it is finite in the sense of W^*-algebras (Lemma 2.1); U_e is decomposed in the following form by [14; Theorem 1]:

10) See Halmos, Measure Theory, (1950). Especially Theorem A of p. 54.

$$(5.8) \qquad U_e = \sum_{i=1}^{\infty} Q_i U_e = \sum_{i=1}^{\infty} U_{Q_i e},$$

where Q_i are the central projections such that $I = \sum_{i=1}^{\infty} \oplus Q_i$, and every $U_{Q_i e}$ is E-finite. The (5.8) means that $\mathbf{e} = \sum_{i=1}^{\infty} Q_i \mathbf{e}$, because $\mathbf{e} = U_e \mathbf{e} = \sum_{i=1}^{\infty} Q_i U_e \mathbf{e}$ $= \sum_{i=1}^{\infty} Q_i \mathbf{e}$, and the fact that $U_{Q_i e}$ is E-finite, implies $Q_i \mathbf{e} \in (\mathfrak{H}_0)_F$ Thus the proof is completed.

Combining the above lemma and Lemma 1.3, we see that every bounded element $\mathbf{x}$ is the (strong) limit of some elements $\mathbf{y}_n \in (\mathfrak{H}_0)_E$, that is, $(\mathfrak{H}_0)_F$ is dense in $\mathfrak{H}$.

Consider now the linear subspace $\mathfrak{M}$ of $\mathfrak{H}$, constructed by the elements of the form:

$$(5.9) \qquad \mathbf{x} = U_{F_1}\mathbf{x}_1 + \ldots\ldots + U_{F_n}\mathbf{x}_n, \quad \mathbf{x}_i \in (\mathfrak{B}_0)_F, \quad F_i \in \mathbf{L}(\mathbf{X}),$$

then by (5.7), we can associate to such an $\mathbf{x}$ a vector-function

$$(5.10) \qquad \mathbf{x}(\sigma) = F_1(\sigma)\mathbf{x}_1(\sigma) + \ldots + F_n(\sigma)\mathbf{x}_n(\sigma).$$

Evidently, this vector-function is continuous and has a compact support on $\mathbf{X}$; by the formula (5.7) and the fact that the correspondence $F \to U_F$ is multiplicative, we have

$$(5.11) \qquad < \mathbf{x}, \mathbf{y} > = \int_{\mathbf{X}} < \mathbf{x}(\sigma), \mathbf{y}(\sigma) > d\hat{\mu}(\sigma)$$

for any $\mathbf{x}, \mathbf{y} \in \mathfrak{M}$.

On the other hand, any continuous vector-function on $\mathbf{X}$ is the uniform limit of the vector-functions of the form (5.10) on every compact set of $\mathbf{X}$.[11] Therefore the vector-functions (5.10) are dense in L^2_Λ; so that we have the isomorphism between the closure of $\mathfrak{M}$ in $\mathfrak{H}$ and the space L^2_Λ. It remains to prove that $\mathfrak{M}$ is dense in $\mathfrak{H}$ and that this isomorphism gives the transformation $\mathbf{x}$ to $\mathbf{x}(\sigma)$ for $\mathbf{x} \in (\mathfrak{H}_0)_F$, defined previously in §4.

If $F(\sigma) \in \mathbf{L}(\mathbf{X})$ converges to 1 uniformly on every compact set, then

$$\lim < U_F\mathbf{x}, \ \mathbf{y} > = \lim \int_{\mathbf{X}} < \mathbf{x}(\sigma), \ \mathbf{y}(\sigma) > F(\sigma) d\hat{\mu}(\sigma)$$

$$= \int_{\mathbf{X}} < \mathbf{x}(\sigma), \mathbf{y}(\sigma) > d\hat{\mu}(\sigma) = < \mathbf{x}, \mathbf{y} > \qquad \text{for } \mathbf{x}, \mathbf{y} \in (\mathfrak{H}_0)_F.$$

Therefore, $\mathbf{x} \in (\mathfrak{H}_0)_F$ is the weak limit of the elements of the forms $U_F\mathbf{x}$; $(\mathfrak{H}_0)_F$ is dense in $\mathfrak{H}$ (Lemma 5.6) and $\mathfrak{M}$ is the linear subspace of $\mathfrak{H}$, so we obtain the first part of the above statesments. So that the vector-function

11) See [2;Chap. III prop. 6]. This proposition does not depend on the measure.

$F(\sigma)\mathbf{x}(\sigma)$ converges uniformly to $\mathbf{x}(\sigma)$ on $\mathbf{X}$, therefore the above isomorphism transforms $\mathbf{x}$ to $\mathbf{x}(\sigma)$. Thus the proof is completed.

6. Some remarks.

Firstly, as mentioned in §2, our $\natural$-operation depends on the choice of the generalised unit element, or the defining system of the $\natural$-operation. We discuss here these circumstances.

Let E' be such another, and assume that E' be E-finite. Then any E'-finite operator A is also E-finite ; we obtain by (2.2)

$$\sigma_{E'}(A) = \sigma_E(E')^{-1}\sigma_E(A), \tag{6.1}$$

where σ_E and $\sigma_{E'}$ is the one, defined by the same character of $\mathbf{R}^\natural$, as described in §§3 and 4. It should be noted that $\sigma_E(E')^{-1}$ may be infinite for some σ, but $\sigma_E(E')^{-1}\sigma_E(A)$ is well-defined, because A is contained in some E'-finite projection (see [14; §3]) ; and it is clear that the dual space $\mathbf{X}$ is uniquely determined for any generalized unit element (§4). By the relation (6.1) and the uniqueness of the Radon measure and the extension of the σ-finite measure, we obtain the following formula for any $\mathbf{x}, \mathbf{y}$, which define the E'-finite operators U_x and U_y, respectively:

$$< \mathbf{x}, \mathbf{y} > = \int_{\mathrm{X}} < \mathbf{x}(\sigma), \mathbf{y}(\sigma) > \sigma(E')d\hat{\mu}(\sigma). \tag{6.2}$$

But this formula gives the isomorphism of $\mathfrak{H}$ and L^2_Λ for the measure $\sigma(E')d\hat{\mu}(\sigma)$, thus we obtain

THEOREM 6.1. *If we take a fixed generalized unit element E of $\mathbf{R}^s$ as the basis, then for any another generalized unit element, which is E-finite, our Plancherel formula becomes in the following form:*

$$< \mathbf{x}, \mathbf{y} > = \int_{\mathrm{X}} < \mathbf{x}(\sigma),\ \mathbf{y}(\sigma) > a(\sigma)d\hat{\mu}(\sigma), \tag{6.3}$$

where $a(\sigma)$ is a (generalized) continuous function on $\mathbf{X}$, which depends only on the generalized unit element; the dual locally compact space $\mathbf{X}$ and the measure $\hat{\mu}$ are unique.

But arbitrary two generalized unit elements are not necessarily comparable; in this case we do not know the unicity of the measure $\hat{\mu}$ in the above sense.

Next, let us assume that $\mathbf{R}^s$ be *of the finite class*, then the decomposition obtained in Theorem 4.1 gives the irreducible one of the W^*-algebras $\mathbf{R}^s$ or $\mathbf{R}^d$, proved in [15]. Because $U_s \in \mathbf{R}^s$, $V_s \in \mathbf{R}^d$, we obtain the unitary operators $U_s(\sigma)$ and $V_s(\sigma)$. We can easily verify that the system $\{\mathfrak{H}(\sigma), U_s(\sigma), V_s(\sigma), S\}$ becomes the d.u.r. of the given group G. Moreover, if we assume that G is *separable*, then the Hilbert space $\mathfrak{H}$ becomes separable, and $\mathbf{R}^s$ is also separable in the weak topology. Denote the W^*-algebras generated by $U_s(\sigma)$ and $V_s(\sigma)$, by $\mathbf{R}^s(\sigma)$ and $\mathbf{R}^a(\sigma)$, respectively, and consider a decomposable operator $A \sim T_A(\sigma)$, permutable with $\mathbf{R}^s(\sigma)$, then $A \in \mathbf{R}^a$, that is, $U_A(\sigma) = T_A(\sigma)$, a.e. and $U_A(\sigma) \in \overline{\mathbf{R}^a(\sigma)}$, where $\overline{\mathbf{R}^a(\sigma)}$ denotes the W^*-algebra generated by the image of the elements in $\mathbf{R}^a$. This fact implies $\mathbf{R}^s(\sigma)' = \overline{\mathbf{R}^a(\sigma)}$, a.e. Similarly $\mathbf{R}^a(\sigma)' = \overline{\mathbf{R}^s(\sigma)}$, a.e. By these facts and the Theorem 5.2 of [15], we obtain

THEOREM 6.2. *Assume that $\mathbf{R}^s$ be of the finite class, and G be separable. Then the double unitary representations, obtained by the decomposition in Theorem* 4.1, *is irreducible a.e.*

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MATHEMATICAL INSITITUTE, TÔHOKU UNIVERSITY, SENDAI.