CESÀRO SUMMABILITY OF FOURIER SERIES

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Let $\varphi(t)$ be an even periodic function with Fourier series

(1) $$\varphi(t) \sim \sum_{n=0}^{\infty} a_n \cos nt, \quad a_0 = 0.$$

The α-th integral of $\varphi(t)$ is defined by

(2) $$\varphi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \varphi(u)(t-u)^{\alpha-1}\, du \qquad (\alpha > 0).$$

G. Sunouchi [1] has proved the following theorem[1];

THEOREM 1. *Let* $\Delta = \gamma/\beta \geqq 1$. *If*

(3) $$\varphi_\beta(t) = o(t^\gamma) \quad (t \to 0),$$

and further if

(4) $$\int_0^t |d(u^\Delta \varphi(u))| = O(t), \quad 0 < t < \eta,$$

then the Fourier series of $\varphi(t)$ *converges to zero at* $t = 0$.

Concerning this theorem M. Kinukawa [4] has proved the following theorem ;

THEOREM 2. *Let* $\Delta \geqq 1$, $-1 < \alpha < 1$. *If*

(5) $$\gamma = \Delta - \frac{2\alpha(\Delta - 1)}{1+\alpha},$$

(6) $$\int_0^t \varphi(u)\, du = o(t^\gamma)$$

and

(4) $$\int_0^t |d(u^\Delta \varphi(u))| = O(t), \quad 0 < t < \eta,$$

then the Fourier series of $\varphi(t)$ *is summable* (C, α) *to zero at* $t = 0$.

The object of this paper is to generalize the above theorems.

THEOREM[2]. *If*

(7) $$\varphi_\beta(t) = o(t^\gamma), \quad \gamma > \beta > 0,$$

and

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1) An Alternative proof was given by Prof. S. Izumi [5].

2) This theorem was proposed by Prof. G. Sunouchi.

$$\int_0^t |d(u^\Delta \varphi(u))| = O(t), \quad 0 < t < \eta, \tag{4}$$

then the Fourier series of $\varphi(t)$ *is summable* (C, α) *to zero at* $t = 0$, *where* $\alpha = \dfrac{\Delta\beta - \gamma}{\Delta + \gamma - \beta - 1}$ *and* $\Delta \geqq \dfrac{\gamma}{\beta}$.

If we put $\Delta = \dfrac{\gamma}{\beta}$, we have Theorem 1. And if we put $\beta = 1$, then $\alpha = \dfrac{\Delta - \gamma}{\Delta + \gamma - 2}$, that is, $\gamma = \Delta - \dfrac{2\alpha(\Delta - 1)}{1 + \alpha}$. This is Theorem 2 in the case $0 < \alpha < 1$.

PROOF. We use Bessel summability instead of Cesàro summability. Accordingly, the proof is due to Sunouchi's method [2].

Let $J_\mu(t)$ denote the Bessel function of order μ, and put

$$\alpha_\mu(t) = J_\mu / t^\mu \tag{8}$$

$$V_{1+\mu}(t) = \alpha_{\mu+1/2}(t) \tag{9}$$

then

$$V_{1+\mu}^{(k)}(t) = O(1) \qquad \text{as } t \to 0 \qquad \text{and}$$

$$V_{1+\mu}^{(k)}(t) = O(t^{-(\mu+1)}) \qquad \text{as } t \to \infty, \ \text{for } k = 0, 1, 2, \ldots\ldots \tag{10}$$

We denote by σ_ω^α the α-th Bessel mean of the Fourier series (1). Since the case $\alpha = 0$ is Theorem 1, we may suppose that $\alpha > 0$. Neglecting the constant factor,

$$\sigma_\omega^\alpha = \int_0^\infty \omega\varphi(t)V_{1+\alpha}(\omega t)\, dt = \left(\int_0^{C\omega^{-\rho}} + \int_{C\omega^{-\rho}}^\infty\right)\omega\varphi(t)V_{1+\alpha}(\omega t)dt = I + J, \tag{11}$$

say, where C is a fixed large constant and $\rho = \dfrac{\alpha + 1}{\alpha + \Delta} < 1$.

If we put

$$\theta(t) = t^\Delta \varphi(t), \ \ \Theta(t) = \int_0^t |d\theta(t)|,$$

then we have, by (4)

$$\Theta(t) = O(t), \qquad \theta(t) = O(t). \tag{12}$$

Next we consider the formula [6]

$$\int_0^\infty \frac{J_\nu(a\sqrt{t^2 + z^2})}{(t^2 + z^2)^{\nu/2}} t^{2\mu+1} dt = 2^\mu \Gamma(\mu + 1) \frac{J_{\nu-\mu+1}(az)}{a^{\nu+1} z^{\nu-\mu+1}},$$

where $a > 0$, $\Re\left(\dfrac{\nu}{2} - \dfrac{1}{4}\right) > \Re(\mu) > -1$.

In the above formula, if we put $t^2 + z^2 = \tau^2, \mu = \lambda$, then

$$\int_z^\infty \frac{J_\nu(a\tau)(\tau^2 - z^2)^\lambda}{\tau^{\nu-1}} d\tau = 2^\lambda \Gamma(\lambda + 1) J_{\nu-\lambda-1}(az)/(a^{\lambda+1} z^{\nu-\lambda-1}),$$

where $a > 0$, $\Re\left(\frac{\nu}{2} - \frac{1}{4}\right)$ $\Re(\lambda) > -1$.

This formula is valid when $\lambda = 0$, $\nu > 1/2$. Therefore

$$\int_z^\infty \frac{J_\nu(a\tau)}{\tau^{\nu-1}}\, d\tau = J_{\nu-1}(az)/a\, z^{\nu-1}. \tag{13}$$

Now, by (8) and (9)

$$\Lambda(t) = \int_t^\infty \frac{V_{1+\alpha}(\omega t)}{u^\Delta}\, du = \int_t^\infty \frac{J_{\alpha+1/2}(\omega u)}{u^\Delta(\omega u)^{\alpha+1/2}}\, du$$

$$= \omega^{-(\alpha+1/2)} \int_t^\infty \frac{J_{\alpha+1/2}(\omega u)}{u_{\alpha-1/2} u^{\Delta+1}}\, du.$$

By (10) and (13), integrating by parts we get

$$\omega^{(\alpha+1/2)}\Lambda(t) = \left[-\int_u^\infty \frac{J_{\alpha+1/2}(\omega v)}{v^{\alpha-1/2}} dv \cdot u^{-(\Delta+1)} \right]_t^\infty$$

$$-(\Delta+1) \int_t^\infty \left\{ \int_u^\infty \frac{J_{\alpha+1/2}(\omega v)}{v^{\alpha-1/2}} dv \right\} u^{-(\Delta+2)} du$$

$$= \left[J_{\alpha-1/2}(\omega\, u)\omega^{-1} u^{-(\alpha-1/2)} u^{-(\Delta+1)}) \right]_t^\infty$$

$$-(\Delta+1)\omega^{-1} \int_t^\infty J_{\alpha-1/2}(\omega\, u) u^{-(\alpha-1/2)} u^{-(\Delta+2)}\, du$$

$$= O\left\{ \left[\omega^{-1}(\omega\, u)^{-1/2} u^{-(\alpha+\Delta+1/2)} \right]_t^\infty \right\} + O\left\{ \int_t^\infty \omega^{-3/2} u^{-1/2} u^{-(\Delta+\alpha+3/2)}\, du \right\}$$

$$= O(\omega^{-3/2} t^{-(\Delta+\alpha+1)}) + (O\omega^{-3/2} t^{-(\Delta+\alpha+1)}),$$

for $\omega t > 1$. Thus if $\omega t > 1$, then we have

$$\Lambda(t) = O(\omega^{-(\alpha+2)} t^{-(\Delta+\alpha+1)}). \tag{14}$$

We first estimate J. By integration by parts, we have

$$J = \int_{C\omega^{-\rho}}^\infty \omega\varphi(t) V_{1+\alpha}(\omega\, t) dt = \int_{C\omega^{-\rho}}^\infty \omega\theta(t) \frac{V_{1+\alpha}(\omega\, t)}{t^\Delta}\, dt$$

$$= -\int_{C\omega-\rho}^\infty \omega\theta(t) d\Lambda(t) = -\left[\theta(t)\omega\Lambda(t) \right]_{C\omega^{-\rho}}^\infty + \omega \int_{C\omega^{-\rho}}^\infty \Lambda(t) d\theta(t)$$

$$= J_1 + J_2,$$

say. Then, by (12) and (14)

$$J_1 = O\left[\omega t \omega^{-(\alpha+2)} t^{-(\Delta+\alpha+1)} \right]_{C\omega^{-\rho}}^\infty = O(\omega^{-(\alpha+1)} C^{-(\Delta+\alpha)} \omega^{\rho(\Delta+\alpha)})$$

$$= O(C^{-(\Delta+\alpha)}) \leqq \varepsilon,$$

for large C, since $\rho = \dfrac{1+\alpha}{\Delta+\alpha}$.

$$J_2 = O\left\{ \int_{C\omega^{-\rho}}^{\infty} \omega^{-(\alpha+1)} t^{-(\Delta+\alpha+1)} |d\theta(t)| \right\}$$

$$= O\left\{ \omega^{-(\alpha+1)} \left[\Theta(t) t^{-(\Delta+\alpha+1)} \right]_{C\omega^{-\rho}}^{\infty} + \int_{C\omega^{-\rho}}^{\infty} \omega^{-(\alpha+1)} \Theta(t) t^{-(\Delta+\alpha+2)} \, dt \right\}$$

$$= O\left\{ \omega^{-(\alpha+1)} (C\omega^{-\rho})^{-(\Delta+\alpha)} + \omega^{-(\alpha+1)} \left[t^{-(\Delta+\alpha)} \right]_{C\omega^{-\rho}}^{\infty} \right\}$$

$$= O\{\omega^{-(\alpha+1)} C^{-(\Delta+\alpha)} \omega^{\rho(\Delta+\alpha)}\}$$

$$= O(C^{-(\Delta+\alpha)}) \leqq \varepsilon.$$

Thus

$$J = J_1 + J_2 \leqq \varepsilon. \tag{15}$$

Now there is an integer $k > 1$ such that $k-1 < \beta \leqq k$. We suppose that $k-1 < \beta < k$, for the case $\beta = k$ can be easily deduced by the following argument. By integration by parts k-times, we have

$$I = \sum_{h=1}^{k} (-1)^{h-1} \left[\omega^h \varphi_h(t) V_{1+\alpha}^{(h-1)}(\omega t) \right]_0^{C\omega^{-\rho}} + (-1)^k \omega^{k+1} \int_0^{C\omega^{-\rho}} \varphi_k(t) V_{1+\alpha}^{(k)}(\omega t) \, dt$$

$$= \sum_{h=1}^{k} (-1)^{h-1} I_h + (-1)^k I_{k+1}, \text{ say.}$$

Since $\varphi(t) = O(t^{1-\Delta})$ by (12) and $\varphi_\beta(t) = o(t^\gamma)$, we have, by convexity theorem due to G. Sunouchi [3],

$$\varphi_h(t) = o(t^{\{(\beta-h)(1-\Delta)+h\gamma\}/\beta}), \quad \textit{for } h = 1, 2, \ldots. k-1,$$
$$\varphi_k(t) = o(t^{\gamma-\beta+k}). \tag{16}$$

Therefore, if $\beta > 1$

$$I_h = \left[\omega^h \varphi_h(t) V_{1+\alpha}^{(h-1)}(\omega t) \right]_0^{C\omega^{-\rho}}$$

$$= o\{\omega^{h-(1+\alpha)} \omega^{-\rho\{(\beta-h)(1-\Delta)+h\gamma\}/\beta} \omega^{\rho(1+\alpha)} C^{\{(\beta-h)(1-\Delta)+h\gamma\}/\beta} C^{-(1+\alpha)}\}$$
$$+ \omega^h \lim_{t\to 0} V_{1+\alpha}^{h-1}(\omega t) t^{\{(\beta-h)(1-\Delta)+h\gamma\}/\beta}.$$

Now, if the condition (7) holds then the Fourier series of $\varphi(t)$ is summable $\left(C, \dfrac{\beta}{1+\gamma-\beta}\right)$ to zero at $t = 0$. Therefore if

$\dfrac{\beta}{1+\gamma-\beta} > \dfrac{\Delta\beta-\gamma}{\Delta+\gamma-\beta-1}$, that is $\dfrac{\gamma+\beta+1}{\beta} > \Delta$, then our theorem has the meaning. Hence we may suppose $\dfrac{\gamma+\beta+1}{\beta} > \Delta$.

If $\beta > 1$ we have $\dfrac{\gamma+\beta-1}{\beta-1} > \dfrac{\gamma+\beta+1}{\beta}$. Thus we have $(\beta-1)(1-\Delta) + \gamma > 0$.

Since $(\beta-h)(1-\Delta)+h\gamma>(\beta-1)(1-\Delta)+\gamma>0$ and $V_{1+\alpha}^{(h-1)}(\omega t)=O(1)$ as $t\to 0$ the second term is zero. Since $\rho=(1+\alpha)/(\Delta+\alpha)=(\beta+1)/(\Delta+\gamma)$ the ω's exponent of the first term is

$$h-(1+\alpha)-\frac{\rho}{\beta}\{((\beta-h)(1-\Delta)+h\gamma-\beta(1+\alpha)\}$$

$$=h-(1+\alpha)-\frac{\rho}{\beta}\{-\beta(\alpha+\Delta)-h(1-\Delta-\gamma)\}$$

$$=h-(1+\alpha)+\frac{\rho}{\beta}\frac{\beta(1+\alpha)}{\rho}-\frac{\rho}{\beta}h(\Delta+\gamma-1)$$

$$=\frac{h}{\beta}\left\{\beta-\frac{(\beta+1)(\Delta+\gamma-1)}{\Delta+\gamma}\right\}=\frac{h}{\beta(\Delta+\gamma)}(\beta+1-\Delta-\gamma)<0,$$

$(h=1,2,3,\ldots.k-1)$. If $\beta<1$

$$I_1=\Big[\omega\varphi_1(t)V_{1+\alpha}(\omega t)\Big]_0^{C\omega^{-\rho}}=O\{\omega^{1-(1+\alpha)}\,\omega^{-\rho\{(\beta-1)(1-\Delta)+\gamma\}/\beta}\,\omega^{\rho(1+\alpha)}\}$$

$$\cdot\, C^{\{(\beta-1)(1-\Delta)+\gamma\}/\beta}C^{-(1+\alpha)}-\lim_{t\to 0}\omega t^{\{(\beta-1)(1-\Delta)+\gamma\}/\beta}\,V_1{}^{+\alpha}(\omega t).$$

Since $(\beta-1)(1-\Delta)+\gamma>0$ and $V_{1+\alpha}(\omega t)=O(1)$ as $t\to 0$, the second term is zero. About the ω's expont of the first term we have

$$(\beta+1-\Delta-\gamma)/\beta(\Delta+\gamma)<0,$$

by similar calculation. In this case another terms of I_h disappear for $h=2,3,\ldots.k-1$. Thus we have

(17) $$I_h=o(1),\ \text{ as } \omega\to\infty \qquad \text{for } h=1,2,\ldots.k-1.$$

Concerning I_k,

$$I_k=\Big[\omega^k\varphi_k(t)V_{1+\alpha}^{(k-1)}(\omega\, t)\Big]_0^{C\omega^{-\rho}}$$

$$=o\{\omega^k\omega^{-\rho(k+\gamma-\beta)}\,\omega^{-(1+\alpha)}\,\omega^{\rho(1+\alpha)}\}-\lim_{t\to 0}\omega^k t^{(k+\gamma-\beta)}\,V_{1+\alpha}^{(k-1)}(\omega t)$$

$$=o\{\omega^{k(1-\rho)-\rho(\gamma-\beta)-(1-\rho)(1+\alpha)}\}$$

The exponent of ω is

$$\frac{k(\Delta-1)}{\Delta+\alpha}-\frac{1+\alpha}{\Delta+\alpha}(\gamma-\beta)-\frac{\Delta-1}{\Delta+\alpha}(1+\alpha)$$

$$=\frac{k(\Delta-1)}{\Delta+\alpha}-\frac{1+\alpha}{\Delta+\alpha}(\gamma+\Delta-\beta-1)$$

$$=\frac{\Delta-1}{\Delta+\alpha}(k-\beta-1)=(1-\rho)(k-\beta-1)<0,$$

for $1+\alpha=(\Delta-1)(\beta+1)/(\gamma+\Delta-\beta-1)$. Therefore

(18) $$I_k=o(1),\qquad \text{as } \omega\to\infty.$$

Concerning I_{k+1}, we split it up into four parts,

$$I_{k+1}=\omega^{k+1}\int_0^{C\omega^{-\rho}}\varphi_k(t)V_{1+\alpha}^{(k)}(\omega t)dt=\omega^{k+1}\int_0^{C\omega^{-\rho}}V_{1+\alpha}^{(k)}(\omega t)\,dt$$

$$\cdot \int_0^t \varphi_\beta(u)(t-u)^{k-\beta-1}\,du$$

$$= \int_0^{C\omega^{-\rho}} \omega^{k+1}\varphi_\beta(u)\,du \int_u^{C\omega^{-\rho}} V_{1+\alpha}^{(k)}(\omega t)(t-u)^{k-\beta-1}dt$$

$$= \int_0^{\omega^{-1}} du \int_u^{u+\omega^{-1}} dt + \int_{\omega^{-1}}^{C\omega^{-\rho}} du \int_u^{u+\omega^{-1}} dt + \int_0^{C\omega^{-\rho}-\omega^{-1}} du \int_{u+\omega^{-1}}^{C\omega^{-\rho}} dt - \int_0^{C\omega^{-\rho}} du \int_{C\omega^{-\rho}}^{u+\omega^{-1}} dt$$

$$= K_1 + K_2 + K_3 - K_4,$$

say. Since $V_{1+\alpha}^{(k)}(t) = O(1)$ for $0 \leqq t \leqq 1$,

$$K_1 = \omega^{k+1} \int_0^{\omega^{-1}} \varphi_\beta(u)\,du \int_u^{u+\omega^{-1}} V_{1+\alpha}^{(k)}(\omega t)(t-u)^{k-\beta-1}dt$$

$$= O\left\{ \omega^{k+1} \int_0^{\omega^{-1}} \varphi_\beta(u)\,du \int_u^{u+\omega^{-1}} (t-u)^{k-\beta-1}\,dt \right\}$$

$$= o\left\{ \omega^{k+1} \int_0^{\omega^{-1}} u^\gamma \Big[(t-u)^{k-\beta} \Big]_u^{u+\omega^{-1}} du \right\}$$

$$= o\left\{ \omega^{k+1} \int_0^{\omega^{-1}} u^\gamma \omega^{-(k-\beta)}\,du \right\} = o\left\{ \omega^{\beta+1} \Big[u^{\gamma+1} \Big]_0^{\omega^{-1}} \right\}$$

(19) $$= o(\omega^{\beta-\gamma}) = o(1), \quad \text{for } \gamma > \beta.$$

$$K_2 = \omega^{k+1} \int_{\omega^{-1}}^{C\omega^{-\rho}} \varphi_\beta(u)\,du \int_u^{u+\omega^{-1}} V_{1+\alpha}^{(k)}(\omega t)(t-u)^{k-\beta-1}\,dt$$

$$= o\left\{ \omega^{k+1} \int_{\omega^{-1}}^{C\omega^{-\rho}} u^\gamma\,du \int_u^{u+\omega^{-1}} (\omega t)^{-(1+\alpha)}(t-u)^{k-\beta-1}\,dt \right\}$$

$$= o\left\{ \omega^{k-\alpha} \int_{\omega^{-1}}^{C\omega^{-\rho}} u^\gamma u^{-(1+\alpha)} \int_u^{u+\omega^{-1}} (t-u)^{k-\beta-1}\,dt \right\}$$

$$= o\left\{ \omega^{k-\alpha} \int_{\omega^{-1}}^{C\omega^{-\rho}} u^{\gamma-(1+\alpha)}\,du \Big[(t-u)^{k-\beta} \Big]_u^{u+\omega^{-1}} \right\}$$

$$= o\left\{ \omega^{k-\alpha}\omega^{-(k-\beta)} \Big[u^{\gamma-\alpha} \Big]_{\omega^{-1}}^{C\omega^{-\rho}} \right\} = o(\omega^{\beta-\alpha}\omega^{-\rho(\gamma-\alpha)}),$$

for $\gamma - \alpha = \gamma - \dfrac{\Delta\beta - \gamma}{\Delta + \gamma - \beta - 1} = \dfrac{(\gamma-\beta)(\Delta+\gamma)}{\Delta + \gamma - \beta - 1} > 0.$

Since

$$\beta - \alpha - \rho(\gamma - \alpha) = \frac{1}{\alpha + \Delta}\{\beta\Delta - \gamma - \alpha(\Delta + \gamma - \beta - 1)\} = 0,$$

we have

(20) $$K_2 = o(1) \qquad \text{as } \omega \to \infty.$$

Concerning K_3, if we use integration by parts in the inner integral, then

$$K_3 = \omega^{k+1} \int_C^{C\omega^{-\rho}-\omega^{-1}} \varphi_\beta(u)\, du \int_{u+\omega^{-1}}^{C\omega^{-\rho}} V_{1+\alpha}^{(k)}(\omega t)\,(t-u)^{k-\beta-1}\, dt$$

(21) $$= \omega^{k+1} \int_0^{C\omega^{-\rho}-\omega^{-1}} \varphi_\beta(u)\, du \left\{ \left[\omega^{-1} V_{1+\alpha}^{(k-1)}(\omega t)\,(t-u)^{k-\beta-1} \right]_{u+\omega^{-1}}^{C\omega^{-\rho}} - (k-\beta-1) \int_{u+\omega^{-1}}^{C\omega^{-\rho}} V_{1+\alpha}^{(k-1)}(\omega t) \cdot (t-u)^{k-\beta-2} dt \right\}$$

$$= M_1 - (k-\beta-1) M_2,$$

say. Then

$$M_1 = \omega^{k+1} \int_0^{C\omega^{-\rho}-\omega^{-1}} \varphi_\beta(u)\, du \left\{ \omega^{-1}\, \omega^{-(1+\alpha)}\, \omega^{\rho(1+\alpha)}\, (C\omega^{-\rho} - u)^{k-\beta-1} - \omega^{-1} \omega^{-(1+\alpha)} (u+\omega^{-1})^{-(1+\alpha)} \omega^{-(k-\beta-1)} \right\}$$

(22) $$= N_1 + N_2.$$

$$N_1 = o\left\{ \omega^{k+(\rho-1)(1+\alpha)} \int_0^{C\omega^{-\rho}} u^\gamma (C\omega^{-\rho} - u)^{k-\beta-1} du \right\}$$

$$= o\left\{ \omega^{k+(\rho-1)(1+\alpha)} \left[u^{\gamma+k-\beta} \right]_0^{C\omega^{-\rho}} \right\} = o(\omega^{k+(\rho-1)(1+\alpha)-\rho(\gamma+k-\beta)}).$$

Since the exponent of ω is

$$k - \frac{(\Delta-1)(\alpha+1)}{\Delta+\alpha} - \frac{\alpha+1}{\Delta+\alpha}(\gamma+k-\beta)$$

$$= \frac{1}{\Delta+\alpha}\{k(\Delta-1) - \alpha(\gamma+\Delta-\beta-1) - \gamma - \Delta + \beta + 1\}$$

$$= \frac{1}{\Delta+\alpha}\{k(\Delta-1) - (\beta\Delta-\gamma) - \gamma - \Delta + \beta + 1\}$$

$$= \frac{\Delta-1}{\Delta+\alpha}(k-\beta-1) = (1-\rho)(k-\beta-1) < 0,$$

(23) $$N_1 = o(1) \qquad \text{as } \omega \to \infty$$

$$N_2 = o\left\{ \omega^{k-(1+\alpha)-(k-\beta-1)} \int_0^{C\omega^{-\rho}-\omega^{-1}} u^\gamma (u+\omega^{-1})^{-(1+\alpha)} du \right\}$$

$$= o\left\{ \omega^{\beta-\alpha} \int_0^{C\omega^{-\rho}} u^{\gamma-(1+\alpha)}\, du \right\}$$

(24) $$= o(\omega^{\beta-\alpha}\omega^{-(\gamma-\alpha)\rho}) = o(1) \qquad \text{as } \omega \to \infty.$$

From (23) and (24) we have

(26) $$M_1 = o(1) \qquad \text{as } \omega \to \infty$$

$$M_2 = \omega^k \int_0^{C\omega^{-\rho}-\omega^{-1}} \varphi_\beta(u)\, du \int_{u+\omega^{-1}}^{C\omega^{-\rho}} V_{1+\alpha}^{(k-1)}(\omega t)\,(t-u)^{k-\beta-2} dt$$

$$= o\left\{ \omega^k \int_0^{C\omega^{-\rho}-\omega^{-1}} u^\gamma du \int_{u+\omega^{-1}}^{C\omega^{-\rho}} \omega^{-(1+\alpha)} t^{-(1+\alpha)} (t-u)^{k-\beta-2} dt \right\}$$

$$= o\left\{ \omega^{k-(1+\alpha)} \int_0^{C\omega^{-\rho}-\omega^{-1}} u^\gamma u^{-(1+\alpha)} du \int_{u+\omega^{-1}}^{C\omega^{-\rho}} (t-u)^{k-\beta-2} dt \right\}$$

$$= o\left\{ \omega^{k-(1+\alpha)} \int_0^{C\omega^{-\rho}-\omega^{-1}} u^{\gamma-(1+\alpha)} \Big[(t-u)^{k-\beta-1} \Big]_{u+\omega^{-1}}^{C\omega^{-\rho}} du \right\}$$

$$= o\left\{ \omega^{k-(1+\alpha)} \int_0^{C\omega^{-\rho}} u^{\gamma-(1+\alpha)} \omega^{-(k-\beta-1)} du \right\}$$

$$= o\left\{ \omega^{k-(1+\alpha)-(k-\beta-1)} \Big[u^{\gamma-\alpha} \Big]_0^{C\omega^{-\rho}} \right\}$$

(27) $\quad = o(\omega^{\beta-\alpha}\omega^{-\rho(\gamma-\alpha)}) = o(1) \qquad$ as $\omega \to \infty$.

From (21), (26) and (27) we have

(28) $\quad K_3 = o(1) \qquad$ as $\omega \to \infty$.

$$K_4 = \omega^{k+1} \int_{C\omega^{-\rho}-\omega^{-1}}^{C\omega^{-\rho}} \varphi_\beta(u)\, du \int_{C\omega^{-\rho}}^{u+\omega^{-1}} V_{1+\alpha}^{(k)}(\omega t)\,(t-u)^{k-\beta-1} dt$$

$$= O\left\{ \omega^{k+1} \int_{C\omega^{-\rho}-\omega^{-1}}^{C\omega^{-\rho}} \varphi_\beta(u) du \int_{C\omega^{-\rho}}^{u+\omega^{-1}} (\omega t)^{-(1+\alpha)} (t-u)^{k-\beta-1} dt \right\}$$

$$= O\left\{ \omega^{k+1-(1+\alpha)} \int_{C\omega^{-\rho}-\omega^{-1}}^{C\omega^{-\rho}} \varphi_\beta(u) \omega^{\rho(1+\alpha)} du \int_{C\omega^{-\rho}}^{u+\omega^{-1}} (t-u)^{k-\beta-1} dt \right\}$$

$$= o\left\{ \omega^{k-\alpha+\rho(1+\alpha)} \int_{C\omega^{-\rho}-\omega^{-1}}^{C\omega^{-\rho}} u^\gamma \Big[(t-u)^{k-\beta} \Big]_{C\omega^{-\rho}}^{u+\omega^{-1}} du \right\}$$

$$= o\left\{ \omega^{k-\alpha+\rho(1+\alpha)} \omega^{-(k-\beta)} \Big[u^{\gamma+1} \Big]_{C\omega^{-\rho}-\omega^{-1}}^{C\omega^{-\rho}} \right\}$$

$$= o(\omega^{k-\alpha+\rho(1+\alpha)-(k-\beta)} \omega^{-\rho(\gamma+1)}) = o(\omega^{\beta-\alpha-\rho(\gamma-\alpha)}).$$

Since the exponent of ω is

$$\beta - \alpha - \rho(\gamma-\alpha)$$

$$= \frac{1}{\Delta+\alpha}\{\beta(\Delta+\alpha) - \alpha(\Delta+\alpha) - (\alpha+1)(\gamma-\alpha)\}$$

$$= \frac{1}{\Delta+\alpha}\{\beta\Delta - \gamma - \alpha(\gamma+\Delta-\beta-1)\} = 0,$$

(29) $\quad K_4 = o(1) \qquad$ as $\omega \to \infty$.

Summing up (19), (20), (28) and (29) we have

$$(30) \qquad I_{k+1} = o(1) \qquad \text{as } \omega \to \infty$$

From (11), (15), (17), (18) and (30) we have

$$\sigma_\omega^\alpha = o(1) \qquad \text{as } \omega \to \infty$$

which is required.

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