INDUCTIVE LIMIT AND INFINITE DIRECT PRODUCT OF OPERATOR ALGEBRAS

ZIRÔ TAKEDA

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Introduction. The first hurdle for the definition of the direct product of operator algebras was the construction of the product space on which the product algebra acts as an operator algebra. This task was accomplished by F. J. Murray and J. von Neumann in the prominent papers [6] [8]. However by the recent progress of the theory of operator algebras, for some problems we can now proceed without the explicit usage of underlying Hilbert spaces. Thus T. Turumaru [14] has succeeded to define the direct product of two C^*-algebras without regard to the product space and Y. Misonou [5] [13] [14] has proved that the algebraical structure of the direct product of two W^*-algebras is uniquely determined only by the component algebras themselves and is independent from Hilbert spaces they act. The purpose of this paper is to extend these results to infinite direct product of operator algebras.

We introduce in § 1 a notion of inductive limit of C^*-algebras and get a dual relation: the state space of an inductive limit of C^*-algebras A_γ $(\gamma \in \Gamma)$ is the projective limit of state spaces of A_γ $(\gamma \in \Gamma)$. After a brief résumé of the finite direct product in § 2, using the concept of inductive limit we define the infinite direct product of C^*-algebras in § 3. As seen in the construction of the infinite product of measure spaces [3], this infinite product has been already popular for commutative algebras in some implicit forms. For this product the associative law holds in the satisfactory manner.

Next, in the recent literatures [1] [2] [9], it is known that the σ-weak topology[1] plays a striking rôle in the theory of W^*-algebras and a state is σ-weakly continuous if and only if it is normal[2] in the sense of Dixmier. Hence employing normal states instead of ordinary states and regarding the dual relation stated above we define the inductive limit of W^*-algebras in §4. We take a W^*-algebra A as the inductive limit of W^*-algebras A_γ $(\gamma \in \Gamma)$ if the set of normal states of A coincides with the projective limit of normal states on A_γ $(\gamma \in \Gamma)$. Then this W^*-inductive limit is uniquely determined. In § 5, relaxing the condition of W^*-inductive limit, we consider a limit of W^* -algebras on a fixed Hilbert space which is named a direct limit. An approximately finite factor [7] gives an example of the direct limit. The relation between the inductive limit and the direct limit is given by the following fact: a direct limit of W^*-algebras A_γ $(\gamma \in \Gamma)$ is always a normal homomorphic image of the W^*-inductive limit of A_γ $(\gamma \in \Gamma)$. In § 6, parallel to the direct product of C^*-algebras we define an infinite direct product of W^*-algebra.

1) This terminology is due to Griffin [2]. Dixmier [1] called it *la topologie ultrafaible*.
2) A state σ is called normal [1] if $x_\alpha \uparrow x$ implies $\sigma(x_\alpha) \uparrow \sigma(x)$.

This product is determined freely from the underlying Hilbert spaces of component algebras and does not coincide with the infinite direct product defined by von Neumann [8]. The latter is a homomorphic image of the former. At last we remark that the example of factors contructed in [8] is reduced to the infinite direct product of normal traces or normal pure states in our sense.

A B^*-algebra is a Banach algebra possessing a $*$-operation such as $\|x^*x\| = \|x\|^2$. It is always represented faithfully as a uniformly closed self-adjoint operator algebra of bounded operators (shortly a C^*-algebra) on a Hilbert space. Theorem A in below concerning the representation of a B^*-algebra is most frequently used and offers the foundation of the reasoning in this paper.

When a B^*-algebra A is represented faithfully as a C^*-algebra on a Hilbert space H (we call it a C^*-representation of A on H), a state σ of A which permits the expression

$$\sigma(a) = \sum_{i=1}^{\infty} < a^{\#}\varphi_i, \varphi_i > \qquad \text{for} \quad a \in A$$

where φ_i are elements in H such as $\sum_{i=1}^{\infty} \|\varphi_i\|^2 = 1$ and $a^{\#}$ is the representative operator of $a \in A$, is called a *distinguished state of A with respect to this representation.*

Then the set S of the distinguished states with respect to a C^*-representation of A has the following properties [12]

(i) S is weakly dense in the state space Ω of A,

(ii) S is strongly closed in the conjugate space of A,

(iii) S is convex,

(iv) if $\rho \in S$ and $\sigma < \rho$ then $\sigma \in S$, where $\sigma < \rho$ means that the representation of A constructed with σ by the usual method is unitarily equivalent to a restriction on an invariant subspace of the representation constructed by ρ.

A subset in the state space Ω which satisfies (i)–(iv) is called a *basic subset.* Then we get

THEOREM A. *For every basic subset S in the state space of a B^*-algebra A, there exists a C^*-representation of A for which the set of distinguished states coincides with S. If the set of distinguished states with respect to a C^*-representation of A is contained in the set defined by another representation of A, there is a normal homomorphism of the weak closure of the latter representation onto that of the former representation. When these sets are identical, the weak closures of these representations are normally isomorphic*[3] *each other* [10], [11].

3) In this paper, an algebraic isomorphism between two $*$-algebras means an isomorphism preserving $*$-operation.

1. Inductive limit of C^*-algebras.

After I. E. Segal, a C^*-algebra means a uniformly closed self-adjoint operator algebra on a certain Hilbert space and a W^*-algebra means a weakly closed self-adjoint operator algebra. Though there is no distinction between B^*-algebras and C^*-algebras from the algebraic aspect since every B^*-algebra is faithfully represented as a C^*-algebra, we use in the following the terminology of C^*-algebras instead of B^*-algebras to be symmetrical with W^*-algebras. A *principal isomorphism* $f_{\beta\alpha}$ of a C^*-algebra A_α with the identity 1_α into another C^*-algebra A_β with the identity 1_β means an isomorphism which satisfies

(1) $$f_{\beta\alpha}(1_\alpha) = 1_\beta.$$

Now, we introduce a new concept of inductive limit for C^*-algebras.

Definition 1. Let Γ be an increasingly directed set and A_γ be a C^*-algebra having an identity 1_γ associated with γ in Γ. If there exists a C^*-algebra A with the identity 1 and a principal isomorphism f_γ of A_γ into A for every $\gamma \in \Gamma$ such that

(2) $$f_\alpha(A_\alpha) \subset f_\beta(A_\beta) \qquad \text{if } \alpha < \beta \ (\alpha, \beta \in \Gamma)$$

and that the join of $f_\gamma(A_\gamma)$ $(\gamma \in \Gamma)$ is uniformly dense in A, A is called *the C^*-inductive limit of* A_γ, and is denoted by $A = C^*\text{-}\lim_\Gamma A_\gamma$.

THEOREM 1. *Let $(A_\gamma,\ \gamma \in \Gamma)$ be a family of C^*-algebras where Γ denotes an increasingly directed set. If, for every α, β with $\alpha < \beta$, there exists a principal isomorphism $f_{\beta\alpha}$ of A_α into A_β satisfying*

(3) $$f_{\gamma\alpha} = f_{\gamma\beta} \cdot f_{\beta\alpha} \qquad \textit{if } \alpha < \beta < \gamma,$$

then there exists the C^*-inductive limit of A_γ.

PROOF. Let K be the collection of all pairs (γ, x_γ) where γ varies over Γ and x_γ takes all elements in A_γ associated with γ. We introduce an equivalence relation in K by defining $(\alpha, x_\alpha) \sim (\beta, x_\beta)$ if and only if there exists (γ, x_γ) in K such that $\alpha < \gamma, \beta < \gamma$ and

$$f_{\gamma\alpha}(x_\alpha) = f_{\gamma\beta}(x_\beta) = x_\gamma.$$

By this equivalence relation, K is divided into equivalence classes. Put $\{x_\gamma\}$ the equivalence class which contains (γ, x_γ). Then we can define the algebraic operations and the $*$-operation in the set A^0 of all equivalence classes as follows,

addition: $\{x_\alpha\} + \{y_\beta\} = \{x_\gamma + y_\gamma\}$,
where $\alpha < \gamma$, $\beta < \gamma$ and $x_\gamma = f_{\gamma\alpha}(x_\alpha)$, $y_\gamma = f_{\gamma\beta}(y_\beta)$,

scalar multiplication: $\lambda\{x_\alpha\} = \{\lambda x_\alpha\}$, where λ is a complex number,

product; $\{x_\alpha\}\{y_\beta\} = \{x_\gamma y_\gamma\}$
where γ, x_γ and y_γ are taken similarly as above,

$*$-operation: $\{x_\alpha\}^* = \{x_\alpha^*\}$.

It is easily confirmed that these operations are well defined. Moreover by a Kaplansky's theorem [4; Theorem 6. 4], each $f_{\gamma\beta}$ is an isometric mapping, that is,

(4) $$\| x_\beta \| = \| f_{\gamma\beta}(x_\beta) \| = \| x_\gamma \|.$$

Hence, if we put $\|\{x_\alpha\}\| = \| x_\alpha \|$, $\|\{x_\alpha\}\|$ is uniquely defined and possesses the norm properties, furthermore

(5) $$\| \{x_\alpha\}^*\{x_\alpha\} \| = \|\{x_\alpha^* x_\alpha\}\| = \| x_\alpha^* x_\alpha \| = \| x_\alpha \|^2 = \|\{x_\alpha\}\|^2.$$

Thus the totality A^0 of these equivalence classes constitutes a normed $*$-algebra, which will be called *the algebraic inductive limit of* A_γ. Put A the completion of this normed $*$-algebra. Then A gives the desired C^*-inductive limit of A_γ.

q. e. d.

By the definition of the C^*-inductive limit we get immediately

PROPOSITION 1. *Let a C^*-algebra A be a C^*-inductive limit of* A_γ $(\gamma \in \Gamma)$ *and Γ' be a cofinal subset in Γ, then A is the C^*-inductive limit of* $A_{\gamma'}$ $(\gamma' \in \Gamma')$.

Next proposition asserts the algebraic uniqueness of the C^*-inductive limit.

PROPOSITION 2. *Let A and B be C^*-inductive limits of C^*-algebras A_γ and B_γ* $(\gamma \in \Gamma)$ *respectively. If there is an algebraic isomorphism h_γ between A_γ and B_γ for every $\gamma \in \Gamma$ which satisfies*

(6) $$h_\beta \cdot f_{\beta\alpha} = f'_{\beta\alpha} \cdot h_\alpha \quad \text{if } \alpha < \beta,$$

(*where $f_{\beta\alpha}$ and $f'_{\beta\alpha}$ are the principal isomorphisms of A_α into A_β and the one of B_α into B_β respectively*), *then A and B are algebraically isomorphic.*

Let Ω, Ω_γ be state spaces of A and A_γ respectively. When A is a C^*-inductive limit of A_γ $(\gamma \in \Gamma)$, every state σ of A defines a state σ_α on A_α. Then, for every α, β such as $\alpha < \beta$, we put $f^*_{\alpha\beta}$ the conjugate mapping of the principal isomorphism $f_{\beta\alpha}$ of A_α into A_β which maps Ω_β onto Ω_α. $f^*_{\alpha\beta}$ has the following properties

(7) $$\sigma_\alpha = f^*_{\alpha\beta}(\sigma_\beta),$$

(8) $$f^*_{\alpha\gamma} = f^*_{\alpha\beta} f^*_{\beta\gamma} \quad \text{if } \alpha < \beta < \gamma.$$

Conversely, a system of states $(\sigma_\gamma \in \Omega_\gamma,\ \gamma \in \Gamma)$ which satisfies the condition (7) defines a state on A since every positive bounded linear functional on the algebraic inductive limit A^0 of A_γ is uniquely extended over A. Furthermore as the state σ_κ of A converges to a state σ by the usual weak topology if and only if every $\sigma_{\kappa\alpha}$ converges to σ_α on each A_α, the state space Ω of A is homeomorphic to the closed subspace composed of all systems satisfying (7) in the product space $\prod_{\alpha \in \Gamma} \Omega_\alpha$ of all state spaces Ω_α. Thus we get

THEOREM 2. *If a C^*-algebra A is a C^*-inductive limit of A_γ* $(\gamma \in \Gamma)$, *the state space Ω of A is homeomorphic to the projective limit of the state space Ω_γ of A_γ* $(\gamma \in \Gamma)$.

Thus if A_γ is commutative we can consider the projective limit of bounded positive Radon measures. This is closely related to the result announced by A. L. Shields [11] and the extension of Kolmogoroff's theorem due to I. E. Segal [10].

COROLLARY. *If a commutative C^*-algebra A is a C^*-inductive limit of the commutative C^*-algebras A_γ $(\gamma \in \Gamma)$, the spectrum X of A is the projective limit of spectrums X_γ of A_γ $(\gamma \in \Gamma)$.*

2. Direct product of C^*-algebras.

T. Turumaru has defined the direct product of two C^*-algebras [14]. We shall extend this notion to the direct product of infinitely many C^*-algebras in the next section. For its preparation we give here a brief explanation of Turumaru's product and a theorem connecting direct product and inductive limit of C^*-algebras.

For two C^*-algebras A_1, A_2, put $\mathfrak{L}$ the set of all formal expressions

$$\sum_{i=1}^{n} x_i \times y_i = x_1 \times y_1 + \dots + x_n \times y_n, \tag{9}$$

where $x_i \in A_1$, $y_i \in A_2$, $i = 1, 2, \dots, n$; $n = 1, 2, \dots$.

In $\mathfrak{L}$, we give a relation $\sim$ which obeys to the following rules:

$$\text{(i)} \qquad \sum_{i=1}^{n} x_i \times y_i \sim \sum_{i=1}^{n} x_{p(i)} \times y_{p(i)},$$

where $p(1), \dots, p(n)$ denotes a permutation of the integers $1, 2, \dots, n$.

$$\text{(ii)} \qquad (x_1' + x_1'') \times y_1 + \sum_{i=2}^{n} x_i \times y_i \sim x_1' \times y_1 + x_1'' \times y_1 + \sum_{i=2}^{n} x_i \times y_i,$$

$$\text{(ii)}' \qquad x_1 \times (y_1' + y_2'') + \sum_{i=2}^{n} x_i \times y_i \sim x_1 \times y_1' + x_1 \times y_1'' + \sum_{i=2}^{n} x_i \times y_i ,$$

$$\text{(iii)} \qquad \sum_{i=1}^{n} (\lambda_i x_i) \times y_i \sim \sum_{i=1}^{n} x_i \times (\lambda_i y_i),$$

where λ_i are complex numbers.

Two expressions $\sum_{i=1}^{n} x_i \times y_i$ and $\sum_{j=1}^{m} s_j \times t_j$ in $\mathfrak{L}$ will be termed *equivalent* if one can be transformed into the other by a finite number of successive applications of rules (i)-(iii). $\mathfrak{L}$ is divided into equivalence classes by this equivalence relation. If we define scalar multiplication, product and $*$-operation in $\mathfrak{L}$ as following

$$\text{scalar multiplication: } \lambda\left(\sum_{i=1}^{n} x_i \times y_i\right) = \sum_{i=1}^{n} (\lambda x_i) \times y_i$$

where λ is a complex number,

$$\text{product: } \left(\sum_{i=1}^{n} x_i \times y_i\right) \cdot \left(\sum_{j=1}^{m} s_j \times t_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} x_i s_j \times y_i t_j,$$

$$*\text{-operation: } \left(\sum_{i=1}^{n} x_i \times y_i\right)^* = \sum_{i=1}^{n} x_i^* \times y_i^*,$$

then these operations are invariant under the equivalence relation, hence the set of equivalence classes $A_1 \odot A_2$ becomes a $*$-algebra, which we call the ***algebraic direct product*** of A_1 and A_2.

Next we consider a linear functional $\rho \times \sigma$ on the $*$-algebra $A_1 \odot A_2$, where ρ and σ are the states on A_1 and A_2 respectively, defined by

$$[\rho \times \sigma]\left(\sum_{i=1}^{n} x_i \times y_i\right) = \sum_{i=1}^{n} \rho(x_i)\sigma(y_i). \tag{10}$$

Let $\mathfrak{S} = \{\rho \times \sigma \mid \rho \in \Omega_1, \sigma \in \Omega_2\}$ (Ω_i is the state space of A_i, $i = 1, 2$) be the set of all such linear functionals and put

$$N\left(\sum_{i=1}^{n} x_i \times y_i\right)^2 = \sup \frac{[\rho \times \sigma]\left(\left(\sum_{j=1}^{m} s_j \times t_j\right)^*\left(\sum_{i=1}^{n} x_i \times y_i\right)^*\left(\sum_{i=1}^{n} x_i \times y_i\right)\left(\sum_{j=1}^{m} s_j \times t_j\right)\right)}{[\rho \times \sigma]\left(\left(\sum_{j=1}^{m} s_j \times t_j\right)^*\left(\sum_{j=1}^{m} s_j \times t_j\right)\right)} \tag{11}$$

where $\rho \times \sigma$ varies over $\mathfrak{S}$ and $\sum_{j=1}^{m} s_j \times t_j$ takes every element in $A_1 \odot A_2$.

Then this $N\left(\sum_{i=1}^{n} x_i \times y_i\right)$ defines a cross-norm (in the sense of Schatten) on $A_1 \odot A_2$ and $A_1 \odot A_2$ becomes a non-complete C^*-algebra. Hence completing $A_1 \odot A_2$, we obtain a new C^*-algebra which we call the ***direct product*** of A_1 and A_2 and denote it by $A_1 \times A_2$.

When A_1 and A_2 are C^*-algebras on Hilbert spaces H_1, H_2 respectively, then we can construct the direct product space $H = H_1 \times H_2$ by the method of F. J. Murray-J. von Neumann [6] and $\sum_{i=1}^{n} x_i \times y_i$ may be seen as an operator on H. We can prove its operator bound is exactly equal to $N\left(\sum_{i=1}^{n} x_i \times y_i\right)$. This means the C^*-algebra on the product space H generated by such operators is algebraically isomorphic to $A_1 \times A_2$. That is, if A_1 and A_2 are operator algebras on H_1 and H_2 respectively, then $A_1 \times A_2$ can be seen as an operator algebra on $H_1 \times H_2$, though the algebraic structure of $A_1 \times A_2$ is uniquely determined freely from H_1 and H_2.

THEOREM 3. *If A is the C^*-inductive limit of A_γ $(\gamma \in \Gamma)$ and B is the C^*-inductive limit of B_δ $(\delta \in \Delta)$, then the direct product $A \times B$ is the C^*-inductive limit of $A_\gamma \times B_\delta$ $((\gamma, \delta) \in (\Gamma, \Delta))$ where (Γ, Δ) means the directed set of all pairs (γ, δ), $\gamma \in \Gamma$, $\delta \in \Delta$ ordered by $(\gamma, \delta) < (\gamma', \delta')$ if and only if $\gamma < \gamma'$, and $\delta < \delta'$.*

PROOF. If $(\gamma, \delta) < (\gamma', \delta')$, there exists a principal isomorphism $f_{(\gamma',\delta')(\gamma,\delta)}$ of $A_\gamma \times B_\delta$ into $A_{\gamma'} \times B_{\delta'}$ and the same condition as (3) is satisfied. Thus we can consider the C^*-inductive limit of $A_\gamma \times B_\delta$ $((\gamma, \delta) \in (\Gamma, \Delta))$. Let A^0 and B^0 be the algebraic inductive limits of A_γ and of B_δ respectively then $A^0 \odot B^0$, the set of all elements of the form $\sum_{i=1}^{n} a_i^0 \times b_i^0$ $(a_i^0 \in A^0,\ b_i^0 \in B^0)$ in $A \times B$, is isomorphic to a dense subalgebra of the C^*-inductive limit of $A_\gamma \times B_\delta$.

On the other hand, $A^0 \odot B^0$ is dense in $A \times B$. To prove this, we show $A^0 \odot B^0$ is dense in $A \odot B$. As A^0 and B^0 are dense in A and B respectively, for every $\sum_{i=1}^{n} a_i \times b_i \in A \odot B$, there exist $\alpha \in \Gamma$ and $\beta \in \Delta$, $a_i^0 \in A_\alpha$ and $b_i^0 \in B_\beta$ such as

$$\|a_i - a_i^0\| < \varepsilon, \quad \|b_i - b_i^0\| < \varepsilon, \tag{12}$$

then

$$\begin{aligned}
&\left\|\sum_{i=1}^{n} a_i \times b_i - \sum_{i=1}^{n} a_i^0 \times b_i^0\right\| \\
&\leqq \left\|\sum_{i=1}^{n} a_i \times b_i - a_i^0 \times b_i + a_i^0 \times b_i - a_i^0 \times b_i^0\right\| \\
&\leqq \sum_{i=1}^{n} \|b_i\| \cdot \|a_i - a_i^0\| + \|a_i^0\| \cdot \|b_i - b_i^0\| \\
&\leqq \varepsilon \sum_{i=1}^{n} \|b_i\| + \varepsilon \sum_{i=1}^{n} (\|a_i\| + \varepsilon).
\end{aligned} \tag{13}$$

This shows $A^0 \odot B^0$ is dense in $A \odot B$. Thus, since $A^0 \odot B^0$ is isometrically isomorphic to a dense subalgebra of $C^*\text{-}\lim_{(\Gamma,\Delta)} A_\gamma \times B_\delta$ and dense in $A \times B$, these two C^*-algebras must be algebraically isomorphic.

3. Infinite direct product of C^*-algebras.

Using the concept of the C^*-inductive limit we can define the infinite direct product of C^*-algebras.

Definition 2. Let A_i $(i \in I)$ be a collection of C^*-algebras, where the set of indices I may have an arbitrary cardinal. For every finite subset $\gamma = (i_1, \dots, i_n)$ of I we associate the C^*-algebra $A_\gamma = A_{i_1} \times A_{i_2} \times \dots \times A_{i_n}$. Then for the directed set Γ composed of all finite subsets γ of I ordered by the inclusion relation, we can construct the C^*-inductive limit of A_γ. We define this C^*-inductive limit as *the infinite direct product of* A_i $(i \in I)$, and denote it by $\times_I A_i$.

Corresponding to Theorem 3 we get

THEOREM 4. *$\times_I A_i$ is algebraically isomorphic to $\times_{I'} A_{i'} \times \times_{I''} A_{i''}$ where I', I'' are disjoint subset in I such as $I' \cup I'' = I$.*

PROOF. By Γ', Γ'' denote the directed sets of all finite sets in I' and I'' respectively. Then by Theorem 3, $\times_I A_{i'} \times \times_{I''} A_{i'}$ is the C^*-inductive limit of

$$A_{(\gamma', \gamma'')} = A_{\gamma'} \times A_{\gamma''} \tag{13}$$

where $\gamma' = (i'_1, \dots, i'_k) \in \Gamma'$, $\gamma'' = (i''_1, \dots, i''_l) \in \Gamma''$.

On the other hand the family of sets $(\gamma', \gamma'') = (i'_1, \dots, i'_k; i''_1, \dots, i''_l)$ forms a cofinal subset in Γ. Thus by Proposition 1 $C^*\text{-}\lim_{(\Gamma', \Gamma'')} A_{(\gamma', \gamma'')}$ is algebraically isomorphic to $\times_I A_i$. Hence $\times_I A_i$ is algebraically isomorphic to $\times_{I'} A_{i'} \times \times_{I''} A_{i''}$. q. e. d.

This theorem gives the associative law of a restricted form for the infinite direct product. But the associative law is true for this product in the complete form as shown in below.

THEOREM 5. *$\times_I A_i$ is algebraically isomorphic to $\times_J (\times_{I_j} A_{i_j})$ where I_j $(j \in J)$ are mutually disjoint subsets in I such that $\cup_J I_j = I$ and $\times_{I_j} A_j$ means the infinite product of $A_{i_j} (i_j \in I_j)$.*

PROOF. Let $\odot_{I_j} A_{i_j}$ be the algebraic infinite direct product of A_{i_j} $(i_j \in I_j)$, that is, the subset in $\times_{I_j} A_{i_j}$ of all form such as

$$\sum_{p=1}^{m} a_{i_{j_1} p} \times a_{i_{j_2} p} \times \dots \times a_{i_{j_n} p} \tag{14}$$

where $a_{i_{j_k} p} \in A_{i_{j_k}}$ and $(i_{j_1}, i_{j_2}, \dots, i_{j_n})$ is an arbitrary finite set in I_j, n is a positive integer and put $\odot_J(\odot_{I_j}A_{i_j})$ the algebraic infinite product of $\odot_{I_j}A_{i_j}$ $(j \in J)$. Then $\odot_J(\odot_{I_j}A_{i_j})$ is isometrically isomorphic to the algebraic infinite direct product $\odot_I A_i$ and they are dense in $\times_J(\times_{I_j}A_{i_j})$ and $\times_I A_i$ respectively. Hence $\times_I A_i$ is isomorphic to $\times_J(\times_{I_j}A_{i_j})$. q. e. d.

By Theorem 2 the set of states of $\times_I A_i$ is the totality of the projective limits of states σ_γ on A_γ, $\gamma \in \Gamma$. If a state σ of $\times_I A_i$ is the projective limit of states of the form

(15) $$\sigma_\gamma = \sigma_{i_1} \times \sigma_{i_2} \times \ldots \times \sigma_{i_n} \text{ for } \gamma = (i_1, \ldots, i_n)$$

where σ_{i_k} is a state of A_{i_k}, then σ is called the product state of σ_i and is denoted by $\times_I \sigma_i$.

When each A_i $(i \in I)$ is commutative, the spectrum of infinite direct product is given by the next theorem.

THEOREM 6. *The spectrum X of the infinite direct product of commutative C^*-algebras A_i $(i \in I)$ with the identity 1_i is homeomorphic to the product space $\prod_{i \in I} X_i$ of the spectrum X_i of every component algebra A_i.*

PROOF. As every algebra A_i has the identity 1_i, each X_i and its product space $X = \prod_{i \in I} X_i$ are compact Hausdorff spaces. By a theorem of Turumaru [14], the direct product $A_{i_1} \times \ldots \times A_{i_n}$ is algebraically isomorphic to the C^*-algebra $C\left(\prod_{k=1}^{n} X_{i_k}\right)$ of all continuous complex valued functions on the product space $\prod_{k=1}^{n} X_{i_k}$. Then by the definition of infinite direct product and the Weierstrass-Stone Theorem the C^*-inductive limit of $A_\gamma, \gamma = (i_1, \ldots, i_n) \in \Gamma$ is algebraically isomorphic to $C(X)$. That is, the spectrum of the infinite product of A_i is homeomorphic to X. q. e. d.

Furthermore if a measure $d\mu_i$ is given on each spectrum X_i such that $d\mu_i(X_i) = 1$ or equivalently if a state μ_i is associated to each commutative C^*-algebra A_i, we can consider the infinite product $\times_I \mu_i$ or equivalently a measure on X. This is nothing but the infinite product of measures $d\mu_i$ [3].

4. Inductive limit of W^*-algebras.

By the definition, a W^*-algebra is a weakly closed self-adjoint operator algebra on a Hilbert space but as we treat in this section the properties of W^*-algebras which do not depend on the underlying Hilbert space, we do not specify the space except the cases especially need it.

Now we assume that A_γ $(\gamma \in \Gamma)$ is a family of W^*-algebras, Γ being an increasingly directed set, and assume that there exists a normal principal isomorphism f_γ of A_γ into a certain W^*-algebra A on a Hilbert space H [1] for every $\gamma \in \Gamma$ which satisfies

$$(16) \qquad f_\alpha(A_\alpha) \subset f_\beta(A_\beta) \quad \text{if } \alpha < \beta.$$

Furthermore if the join of $f_\gamma(A_\gamma)$ $(\gamma \in \Gamma)$ is weakly dense in A, then to employ A as the inductive limit of W^*-algebras A seems to be rather natural comparing with the C^*-inductive limit, but unfortunately we cannot get the algebraical uniqueness of A in this case, that is, the algebraic structure of A is not independent from the Hilbert space H and so a proposition corresponding to Proposition 2 does not hold. Hence it compels us to employ a more restricted algebra as the W^*-inductive limit of A_γ. However, since the above limit algebra A is useful in practical applications, we name it *the direct limit of* A_γ *on the Hilbert space* H and discuss it in the next section.

While, every normal state σ_α on A_α can be extended to a normal state σ_β on A_β if $\alpha < \beta$ since $f_\alpha(A_\alpha) \subset f_\beta(A_\beta)$. Hence we can consider the projective limit σ of normal states σ_γ of A_γ $(\gamma \in \Gamma)$, that is, the state σ of the C^*-inductive limit A^u which induces a normal state σ_γ on each A_γ. Then refering Theorem 2 we put the next definition.

Definition 3. When a W^*-algebra A is a direct limit of A_γ $(\gamma \in \Gamma)$ and the set of states of $A^u = C^*\text{-}\lim_\Gamma A_\gamma$ which are induced by all normal states of A coincides with the totality of the projective limit of normal states of A_γ $(\gamma \in \Gamma)$, A is called the W^*-inductive limit of $A_\gamma (\gamma \in \Gamma)$ and is denoted by $A = W^*\text{-}\lim_\Gamma A_\gamma$.

To assure the existence and the uniqueness of W^*-inductive limit, we need two lemmas.

LEMMA 1. *Let σ be a state of A^u defined as the projective limit of normal states σ_γ of A_γ $(\gamma \in \Gamma)$. Then in the representation $A^{u\#}_\sigma$ of A^u on a Hilbert space H_σ constructed by the state σ, the representation $A^\#_{\alpha\sigma}$ of A_α considered as a subalgebra of $A^{u\#}_\sigma$ forms a weakly closed subalgebra.*

PROOF. To simplify the statement, we assume that each A_γ is a subalgebra of A^u and put A^0 the algebraic inductive limit of A_γ $(\gamma \in \Gamma)$ (this is nothing but the join of all A_γ under the above assumption). By the definition of H_σ there exists a mapping from A^u into H_σ. Denote by a^θ_γ the image of $a_\gamma \in A_\gamma$ by this mapping, then

$$(17) \qquad \| a^\theta_\gamma \| = [\sigma_\gamma(a^*_\gamma a_\gamma)]^{\frac{1}{2}}.$$

Then the image $A^{0\theta}$ of A^0 by this mapping is dense in H_σ since A^0 is uniformly dense in A^u.

We denote the representative operator on H_σ of $a_\alpha \in A_\alpha$ by $a^\#_\alpha$. We suppose that the representative operator $a^\#_{\alpha l}$ of a directed family $a_{\alpha l}$ in A_α $(l \in L)$ converges weakly to m satisfying

(18) $\qquad \|a^{\#}_{\alpha l}\| \leqq \|m\|$ for all $l \in L$.

Then we can assume without loss of generality

(19) $\qquad \|a_{\alpha l}\| \leqq M \quad (l \in L)$ $\qquad$ for a constant M.

As the unit sphere of a W^*-algebra is weakly compact and the weak topology in the unit sphere is purely algebraic [1], a sub-family $a_{\alpha l'}\,(l' \in L')$ of $a_{\alpha l}$ $(l \in L)$ converges weakly to a definite operator $a_\alpha \in A_\alpha$ independently from the underlying Hilbert space for A_α. Since, if $\alpha < \beta$, A_α is weakly closed subalgebra of A_β and σ_β is a normal state of A_β, $\sigma_\beta\,(a^*_\beta a_{\alpha l'} a_\beta)$ converges to $\sigma_\beta\,(a^*_\beta a_\alpha a_\beta)$ for every $a_\beta \in A_\beta$. On the other hand, the representation of A_β by the state σ_β is unitarily equivalent to the restriction of the representation $\{A^{\#}_{\beta\sigma}, H_\sigma\}$ on a subspace $H_{\sigma\beta}$ in H_σ. Thus the above fact implies $< a^{\#}_{\alpha l}\psi, \psi >$ converges to $< a^{\#}_{\alpha}\psi, \psi >$ for every element $\psi \in H_{\sigma\beta}$. Since $H_{\sigma\gamma} \subset H_{\sigma\beta}$ if $\gamma < \beta$ and $A^{0\theta} \subset \bigcup_\beta H_\beta$ where β runs over all indices such as $\alpha < \beta$, by the denseness of $A^{0\theta}$ in H_σ, $a^{\#}_{\alpha l'}$ converges to $a^{\#}_{\alpha}$ weakly on H_σ. Thus m must be coincident with $a^{\#}_{\alpha}$. This shows the weak closedness of the representation of A^α.

LEMMA 2. *Put N the set of all states of A^u defined as the projective limit of normal states of A_γ. Then N constitutes a basic subset in the state space of A^u.*

PROOF. We assume that A_γ are subalgebras of A^u as in the preceding lemma.

(i) *N is weakly dense in the state space Ω of A^u.* As N is a subset in the unit sphere of the conjugate space of A^u and the algebraic inductive limit A^0 of A_γ is uniformly dense in A^u, we prove N is dense in Ω by the topology $\sigma(\Omega, A^0)$[4]. For any finite elements $x_1, x_2, \ldots, x_n$ in A^0, there is a $\alpha \in \Gamma$ such that every $x_i\ (i = 1, \ldots, n)$ are contained in A_α. Then the restriction ρ_γ on A_γ of a state ρ in Ω gives a state of A_α and by the weak denseness of the normal states in the state space Ω_γ of every A_γ, there exists a normal state σ_α in the weak neighborhood $V(\rho_\alpha, x_1, x_2, \ldots, x_n\,;\, \varepsilon)$ defined by $x_1, x_2, \ldots, x_n$ and $\varepsilon > 0$. There exists a projective limit σ of $\{\sigma_\gamma, \sigma_\gamma \in N_\gamma\}$, where $\sigma_\gamma = \sigma_\alpha$ for $\gamma = \alpha$. Then clearly σ is in the weak neighborhood $V(\rho, x_1, x_2, \ldots, x_n\,;\, \varepsilon)$ of ρ in Ω. Thus N is weakly dense in Ω.

(ii) *N is closed in the norm topology.* Let $\{\sigma_n\}$ be a Cauchy sequence in N in the norm topology. Denote by $\sigma_{n\gamma}$ the restriction of σ_n on A_γ, then $\{\sigma_{n\gamma}\}$ is a Cauchy sequence in N_γ since $\|\sigma_n - \sigma_m\| \geqq \|\sigma_{n\gamma} - \sigma_{m\gamma}\|$. As each N_γ is closed in the norm topology $\sigma_{n\gamma}$ converges to a normal state σ_γ. Then the projective limit of $\{\sigma_\gamma,\ \gamma \in \Gamma\}$ is clearly the limit of $\{\sigma_n\}$.

(iii) *N is a convex set in Ω.* This is obvious by the definition of N.

(iv) *If the representation $\{A^{u\#}_\rho, H_\rho\}$ of A^u by a state ρ is unitarily equivalent to the restriction on an invariant subspace of the representation $\{A^{u\#}_\sigma, H_\sigma\}$ of A^u by a state $\sigma \in N$, then $\rho \in N$.* By Lemma 1 the representation of A_α

4) $\sigma(\Omega, A^0)$ is the weakest topology by which functions $a^0(\rho)$ on $\Omega\ (a^0 \in A^0)$ are continuous.

on H_σ is weakly closed, hence the representation of A_α on H_ρ is weakly closed too. This means the restriction of ρ on A_α is a normal state, that is, $\rho \in N$.

Thus N is a basic subset in Ω. q. e. d.

Then, corresponding to Theorem 1, we get

THEOREM 7. *Let A_γ ($\gamma \in \Gamma$) be a family of W^*-algebras, Γ being an increasingly directed set. If there exists a normal principal isomorphism $f_{\beta\alpha}$ of A_α into A_β for every pair of indices α, β svch as $\alpha < \beta$ satisfying*

(20) $$f_{\gamma\alpha} = f_{\gamma\beta} \cdot f_{\gamma\alpha} \text{ if } \alpha < \beta < \gamma,$$

then there exists the W^*-inductive limit of A_γ.

PROOF. By Lemma 2, the set N of all states of the C^*-inductive limit A^u defined as the projective limit of normal states of A_γ forms a basic subset in the state space of A^u. Hence by Theorem A, we can represent A^u as a uniformly closed self-adjoint operator algebra on a certain Hilbert space for which the set of distinguished states coincides with N, and then each A_γ is represented as a weakly closed subalgebra. The weak closure A of this representation is a W^*-algebra whose normal states are the σ-weakly continuous extensions of states in N. This means A is the W^*-inductive limit of A_γ.

q. e. d.

Of course Proposition 1 remains valid for the W^*-inductive limit and Proposition 2 is slightly modified as follows.

PROPOSITION 3. *Let A and B be W^*-inductive limits of W^*-algebras A_γ and B_γ ($\gamma \in \Gamma$) respectively. If there is an isomorphism h_γ between A_γ and B_γ for every $\gamma \in \Gamma$ which satisfies*

(21) $$h_\beta \cdot {}_{\beta\alpha} = f'_{\beta\alpha} \cdot h_\alpha \quad \text{if } \alpha < \beta$$

(where $f_{\beta\alpha}$ and $f'_{\beta\alpha}$ are the normal principal isomorphism of A_α into A_β and the one of B_α into B_β respectively). Then A and B are algebraically (hence normally) isomorphic.

PROOF. By proposition 2, the C^*-inductive limits A^u of A_γ and B^u of B_γ are algebraically isomorphic and the sets of projective limits of normal states of A_γ and B_γ are the same if we neglect the algebraical isomorphism between A^u and B^u. Hence by Theorem A, A and B are normally isomorphic.

q. e. d.

It is favorable to conclude, corresponding to Theorem 3 for the C^*-inductive limit, that if A and B are the W^*-inductive limits of A_γ ($\gamma \in \Gamma$) and B_δ ($\delta \in \Delta$) respectively, the W^*-direct product[5] $A \otimes B$ is the W^*-inductive limit of the W^*-direct product $A_\gamma \otimes B_\delta$ $(\gamma, \delta) \in (\Gamma, \Delta)$). But the present author can prove only that $A \otimes B$ is a normal homomorphic image of W^*-lim $_{(\Gamma,\Delta)}$

5) For two W^*-algebras A, B acting on Hilbert spaces H, K respectively W^*-direct prduct $A \otimes B$ is the weak closure of the C^*-algebra $A \times B$ on $H \times K$.

$A_\gamma \otimes B_\delta$ and whether these W^*-algebras are isomorphic each other or not is undecidable for him.

For a latter use we notice here the following facts concerning the W^*-inductive limit of factors.

PROPOSITION 4. *Let A be the W^*-inductive limit of finite factors A_γ ($\gamma \in \Gamma$) and τ_γ be the trace of A_γ. If a normal trace τ of A is the projective limit of τ_γ, that is, if the restriction of τ on each A_γ coincides with τ_γ, then the representation of A by τ is a finite factor.*

PROOF. Since τ is normal, the representation $A^{\#}_\tau$ of A by τ is weakly closed and each A_γ is represented faithfully in it. Moreover, $A^{\#}_\tau$ is a finite W^*-algebra since it has a faithful trace induced by τ. To simplify the notation, we denote it by τ again. If there exists another normal trace τ_1 on $A^{\#}_\tau$, since τ and τ_1 are both normal, they define the trace on the C^*-inductive limit A^u of A_γ and these traces are different each other. The latter fact means that they do not coincide on one A_γ at least. This is a contradiction since each finite factor has a unique trace. Hence $A^{\#}_\tau$ is a factor. q. e. d.

PROPOSITION 5. *Let A be the W^*-inductive limit of factors A_γ of type I and π be a state of A which is the projective limit of normal pure state π_γ ($\gamma \in \Gamma$) then π is a normal pure state of A. Hence the representation of A by π is a factor of type I.*

PROOF. If π is not pure, π is again not pure on the C^*-inductive limit A^u of A hence it permits the expression

$$\pi = c\rho + (1-c)\sigma \tag{22}$$

where $0 < c < 1$ and ρ, σ are different states of A^u. Let ρ_γ, σ_γ be the restrictions of ρ and σ on A_γ respectively, then $\pi_\gamma = \rho_\gamma = \sigma_\gamma$ since π_γ is pure on A_γ. This follows $\rho = \sigma$ on A^u. This contradiction shows that π is a pure state. π is clearly normal by definition of the W^*-inductive limit.

5. Direct limit of W^*-algebras.

Though the inductive limit of W^*-algebras is algebraically unique, it is strongly restricted and is hard to investigate its fundamental properties as seen in the preceding section. Moreover it can not include the typical example of a limit of algebras such as the approximately finite factor. For, an approximately finite factor A is the direct limit of a sequence of factors $A_1, A_2, \ldots, A_n, \ldots$ with the following condition (*) on a separable Hilbert space :

(*) *A_n is a factor of type I_{p_n} where p_n is a positive integer and is a divisor of p_{n+1}.*

If an approximately finite factor A would be the W^*-inductive limit of

A_n, every state of the C^*-inductive limit A^u of A_n must be uniquely extended to a normal state of A, since every state of A_n is normal. Thus every pure state of A^u is extended to a normal state of A, that is, we get a pure and normal state of A but such state does not exist for any W^*-algebra of not type I [1, Corollary 6]. This shows A is not the W^*-inductive limit of A_n.

The next theorem gives a relation between the W^*-inductive limit and direct limit.

THEOREM 8. *If a W^*-algebra A acting on a Hilbert space H is the direct limit of W^*-algebras A_γ $(\gamma \in \Gamma)$, then A is a normally homomorphic image of the W^*-inductive limit A^w of A_γ $(\gamma \in \Gamma)$.*

PROOF. Let A^u be the C^*-inductive limit of A_γ, then it is represented as a subalgebra of A on the Hilbert space H. The distinguished states of A^u with respect to this representation are contained in the set of all projective limits of normal states on A $(\gamma \in \Gamma)$. Hence by Theorem A, A is a normal homomorphic image of A^w.

THEOREM 9. *Let W^*-algebras A and B be direct limits of A_γ and B_γ $(\gamma \in \Gamma)$ on Hilbert spaces H and K respectively, h_γ $(\gamma \in \Gamma)$ be an isomorphism between A_γ and B_γ which satisfies the condition* (21) *in Proposition* 3. *Then A and B are algebraically isomorphic if and only if the set of states induced on the C^*-inductive limit A^u of A_γ by the normal states of A is identical with the set of states similarly defined on the C^*-inductive limit B^u neglecting the isomorphism between A^u and B^u.*

By Theorem A, this theorem is obvious. An analogy to Theorem 2 holds for the direct limit as follow

THEOREM 10. *If A and B are direct limits of A_γ $(\gamma \in \Gamma)$ and B_δ $(\delta \in \Delta)$ on Hilbert spaces H and K respectively, then W^*-direct product $A \otimes B$ on the product space $H \otimes K$ is the direct limit of $A_\gamma \otimes B_\delta$ $((\gamma, \delta) \in (\Gamma, \Delta))$ on the Hilbert space $H \times K$.*

PROOF. Since every $A_\gamma \otimes B_\delta$ $(\gamma, \delta) \in (\Gamma, \Delta)$ are W^*-algebras on $H \times K$, there exists the direct limit of $A_\gamma \otimes B_\delta$ on $H \times K$. By the definition of the direct limit, the join of $A_\gamma \otimes B_\delta$ or the join of $A_\gamma \odot B_\delta$ $(\gamma, \delta) \in (\Gamma, \Delta)$ is weakly dense in the direct limit of $A_\gamma \otimes B_\delta$. The latter join is nothing but the algebraic direct product $A^0 \odot B^0$ of the algebraic inductive limits A^0 and B^0 of A_γ and B_δ respectively. On the other hand $A \otimes B$ is a W^*-algebra on the Hilbert space $H \times K$ and $A^0 \odot B^0$ is weakly dense in it, as shown by Misonou in the proof of a lemma in [5]. Hence $A \otimes B$ coincides with the direct limit of $A_\gamma \otimes B_\delta$ on $H \times K$. q. e. d.

COROLLARY (*Misonou*) [5]. *The W^*-direct product of two approximately finite factors is again an approximately finite factor.*

6. **Infinite direct product of W^*-algebras.**

We define the infinite direct product of W^*-algebras using the concept of W^*-inductive limit similarly as we have done for the infinite direct product of C^*-algebras.

Definition 4. Let A_i $(i \in I)$ be a collection of W^*-algebras where I is a set of indices of arbitrary cardinal. For every finite subset $\gamma = \{i_1, i_2, \ldots, i_n\}$ of I we associate the W^*-algebras $A_\gamma = A_{i_1} \otimes A_{i_2} \otimes \ldots \otimes A_{i_n}$. Then for the direct set Γ of all finite subsets γ of I, there exists the W^*-inductive limit of A_γ which is called *the infinite direct product* of A_i $(i \in I)$ and is denoted by $\bigotimes_I A_i$.

Clearly the infinite direct product $\bigotimes_I A_i$ is uniquely determined freely from Hilbert spaces H_i on each of which A_i acts, but since the question concerning to the W^*-direct product of W^*-inductive limits stated in §4 is not solved, the associative law for this product is not certain even in a restricted form. That is, we can only say that if I' and I'' are disjoint subsets in I such that $I' \cup I'' = I$, there is a normal homomorphism of $\bigotimes_I A_i$ onto $\bigotimes_{I'} A_{i'} \otimes \bigotimes_{I''} A_{i''}$ but cannot conclude the isomorphism of these algebras.

In the ordinary measure theory we treat only the infinite product of measures with the total mass 1, hence in our product the most interesting from the stand point of the non-commutative integration theory is not the product $\bigotimes_I A_i$ itself but the infinite product of normal states σ_i on A_i. As in §3, the state on $\times_I A_i$ defined by σ_i $(i \in I)$ is denoted by $\times_I \sigma_i$ and its extension to a normal state on $\bigotimes_I A_i$ is denoted by $\bigotimes_I \sigma_i$. Then the representation of $\bigotimes_I A_i$ by the normal state $\bigotimes_I \sigma_i$ is a W^*-algebra which is called *the restricted infinite direct product* $\bigotimes_I (A_i, \sigma_i)$ *of W^*-algebras A_i with normal states σ_i*. The restricted infinite direct product $\bigotimes_I (A_i, \sigma_i)$ is nothing but the weak closure of the representation of the C^*-algebra $\times_I A_i$ by the state $\times_I \sigma_i$. Thus *the associative law holds for the restricted infinite direct product.*

The next two propositions follow immediately from Proposition 4 and 5.

PROPOSITION 6. *Let τ_i be a normal trace on each W^*-algebra A_i $(i \in I)$. Then the infinite direct product $\bigotimes_I \tau_i$ is a normal trace of $\bigotimes_I A_i$. Moreover if each A_i is a factor, then the restricted infinite direct product $\bigotimes_I (A_i, \tau_i)$ is a finite factor.*

PROPOSITION 7. *Let π_i be a pure normal state on each W^*-algebras A_i $(i \in I)$ of type I. Then the infinite direct product $\bigotimes_I \pi_i$ is a pure, normal state* *of* $\bigotimes_I A_i$ *and the restricted infinite direct product* $\bigotimes_I (A_i, \pi_i)$ *is a factor of type I.*

Now we compare our product with the infinite direct product of W^*-algebras defined by J. von Neumann [8]. For this purpose we introduce briefly the direct products of Hilbert spaces and of operator algebras defined by J. von Neumann.

Let I be a set of indices with an arbitrary cardinal, and let for each $i \in I$ a Hilbert space H_i be given. Then a C-sequence $\times \varphi_i$ is a sequence such that $\varphi_i \in H_i$ for all $i \in I$ and $\prod_I \| \varphi_i \|$ converges in the extended sense. We consider all finite linear aggregates of C-sequences and for every pair of its elements

$$\Phi = \sum_{\nu=1}^{p} \times \varphi_{i,\nu}, \qquad \Psi = \sum_{\mu=1}^{q} \times \psi_{i,\mu} \tag{23}$$

we associate

$$(\Phi, \Psi) = \sum_{\nu=1}^{p} \sum_{\mu=1}^{q} \prod_I (\varphi_{i,\nu}, \psi_{i,\nu}). \tag{24}$$

With this inner product (Φ, Ψ), the linear aggregates of C-sequences becomes an incomplete Hilbert space. Its completion is called the complete direct product of H_i and is denoted by $\bigotimes_I H_i$.

Next we need here a concept of C_0-sequence. A C_0-sequence is a sequence φ_i $(i \in I)$ such that $\varphi_i \in H_i$ for all $i \in I$ and $\sum_{i \in I} | \| \varphi_i \| - 1 |$ converges. Then every C_0-sequence is a C-sequence and conversely every C-sequence with $\times \varphi_i \neq 0$ is a C_0-sequence too. We say two C_0-sequences φ_i $(i \in I)$ and ψ_i $(i \in I)$ are equivalent, in symbols $(\varphi_i, i \in I) \approx (\psi_i ;\ i \in I)$, if and only if $\sum_{i \in I} | (\varphi_i, \psi_i) - 1 |$ converges. This equivalence for C_0-sequences is reflexive, symmetric and transitive, hence the family of C_0-sequences is divided into equivalence classes. We denote by $\mathfrak{E}$ the set of all equivalence classes. For $\mathfrak{C} \in \mathfrak{E}$, $\bigotimes_I^{\mathfrak{C}} H_i$ means the closed linear set determined by all $\times \varphi_i$ where φ_i $(i \in I)$ is any C_0-sequence from $\mathfrak{C}$, and this is called an incomplete direct product of H_i $(i \in I)$. If $\mathfrak{C} \neq \mathfrak{D}$, then $\bigotimes_I^{\mathfrak{D}} H_i$ is orthogonal to $\bigotimes_I^{\mathfrak{C}} H_i$ and $\bigotimes_I H_i$ is the direct sum of $\bigotimes_I^{\mathfrak{C}} H_i$ $(\mathfrak{C} \in \mathfrak{E})$. Especially the following fact is noteworthy: Each equivalence class $\mathfrak{C}$ contains a C_0-sequence φ_i^0 $(i \in I)$ with $\| \varphi_i^0 \| = 1$ for all $i \in I$ and $\bigotimes_I^{\mathfrak{C}} H_i$ is the closed linear set determined by all C_0-sequences φ_i $(i \in I)$ for which $\varphi_i \neq \varphi_i^0$ occurs for a finite number of i's.

We denote the ring of all bounded operators on H_i by B_i and the ring of those on $\bigotimes_I H_i$ by $\mathfrak{B}_\otimes$. Then for every operator $x_{i0} \in B_i$, there corresponds a unique operator $\bar{x}_{i_0} \in \mathfrak{B}_\otimes$ such that for all C-sequences $\times \varphi_i$

$$(25) \qquad \bar{x}_{i_0}\left(\times \varphi_i\right) = \bar{x}_{i_0}\left(\varphi_{i_0} \times \underset{i \neq i_0}{\times} \varphi_i\right) = (x_{i_0}\varphi_{i_0}) \times \underset{i \neq i_0}{\times} \varphi_i.$$

We call $\bar{x}_{i_0}$ the extension of x_{i_0} and denote by $\bar{B}_{i_0}$ the set of extensions of all $x_{i_0} \in B_{i_0}$ and by $B^\otimes$ the W^*-algebra generated by all $\bar{B}_i$ $(i \in I)$. Clearly $B^\otimes \subset \mathfrak{B}_\otimes$ and $B^\otimes \neq \mathfrak{B}_\otimes$ unless I is finite. This $B^\otimes$ is the infinite direct product of B_i $(i \in I)$ defined by J. von Neumann. Though J. von Neumann did not, the infinite direct product is possible for arbitrary W^*-algebras A_i along with the same idea.

Let $\bigotimes_I B_i$ and $B^\otimes$ be the infinite direct products of the ring of all bounded operators B_i on H_i $(i \in I)$ in our sense and in J. von Neumann's sense respectively. Put σ_i a state of B_i defined by $\sigma_i(a_i) = < a_i\varphi_i, \varphi_i >$ where φ_i is a normalized element in H_i, then σ_i is a normal pure state of B_i. Let σ be the infinite product $\bigotimes_I \sigma_i$ of such σ_i $(i \in I)$ and consider the representation of $\bigotimes_I B_i$ by this σ. By the definition of $\bigotimes_I B_i$, the algebraic infinite direct product $\odot_I B_i$, that is, the set of all form in $\bigotimes_I B_i$ such as

$$(26) \qquad (\ldots \times 1_{ip} \times 1_{iq} \times \underbrace{b_{ir} \times b_{is} \times \ldots \times b_{it}}_{\text{finite number}} \times 1_{iu} \times 1_{iv} \times \ldots),$$

$b_{iK} \in B_{iK}$ $(K = r, s, \ldots, t)$, is strongly dense in the C^*-infinite direct product $\times_I B_i$ and σ-weakly dense in $\bigotimes_I B_i$. Hence by the construction of the representative space H_σ, the image of $\odot_I B_i$ in H_σ is strongly dense. Furthermore,

$$(27) \qquad \| (\ldots \times 1_{ip} \times 1_{iq} \times b_{ir} \times b_{is} \times \ldots \times b_{it} \times 1_{iu} \times 1_{iv} \times \ldots)^\theta \| = \| b_{ir}\varphi_{ir} \| \cdot \| b_{is}\varphi_{is} \| \ldots \| b_{it}\varphi_{it} \|$$

(where θ means the mapping from $\otimes_I B_i$ into H_σ). Next we consider the mapping from $(\odot_I B_i)^\theta$ into the incomplete direct product $\bigotimes_I^{\mathfrak{C}} H_i$ determined by a C_0-sequence $\varphi_i (i \in I)$ such as

$$(28) \qquad \begin{aligned} &(\ldots \times 1_{ip} \times 1_{iq} \times b_{ir} \times b_{is} \times \ldots \times b_{it} \times 1_{iu} \times 1_{iv} \times \ldots)^\theta \to \\ &(\ldots, \varphi_{iq}, \varphi_{iq}, b_{ir}\varphi_{ir}, \ldots, b_{it}\varphi_{it}, \varphi_{iu}, \varphi_{iv}, \ldots). \end{aligned}$$

This mapping is linear and norm preserving and the image is strongly dense in $\bigotimes_I^{\mathfrak{C}} H_i$ by the fact noticed in the explanation of von Neumann's product. Thus this mapping can be extended to a linear isometric mapping u from

H_σ onto $\bigotimes_I^{\mathfrak{C}} H_i$. Then by comparing the definitions of $\bigotimes_I B_i$ and $B^\otimes$, the representation of $\bigotimes_I B_i$ on H_σ is unitarliy equivalent to the restriction of $B^\otimes$ on $\bigotimes_I^{\mathfrak{C}} H_i$.

THEOREM 11. *$B^\otimes$ is a normally homomorphic image of $\otimes_I B_i$*

PROOF. Let $\mathfrak{C}$ be the set of all incomplete infinite direct product in $\bigotimes_I H_i$ and from each incomplete infinite product in $\mathfrak{C}$, we pick up a C^0-sequence $\varphi_{i,\mathfrak{C}}$ $(i \in I)$ such as $\|\varphi_{i,\mathfrak{C}}\| = 1$ and make the infinite direct product $\bigotimes_I \sigma_{i,\mathfrak{C}}$ where $\sigma_{i,\mathfrak{C}}$ is a state of B_i such as

$$(29) \qquad \sigma_{i,\mathfrak{C}}(a_i) = \langle a_i\varphi_{i,\mathfrak{C}}, \varphi_{i,\mathfrak{C}} \rangle \qquad \text{for } a_i \in B_i.$$

Clearly $\bigotimes_I \sigma_{i,\mathfrak{C}}$ is a normal state of $\bigotimes_I B_i$. Hence every normal state of $B^\otimes$ can be seen as that of $\bigotimes_I B_i$. Then by Theorem A, $B^\otimes$ is a normally homomorphic image of $\bigotimes_I B_i$ q. e. d.

Moreover $\bigotimes_I B_i$ is not isomorphic to $B^\otimes$ in general. We show this by an example given by J. von Neumann in [8].

Let H_i $(i = 1, 2, \ldots.)$ be a countable family of two dimensional Euclidean spaces and B_i be the ring of all bounded operators on H_i. Then there is a normal trace τ_i on each B_i. The representation of $\bigotimes_I B_i$ by $\bigotimes_I \tau_i$ is a finite factor by Proposition 6, and this is clearly not of type I_p. Thus it must be of type II_1. Thus $\bigotimes_I B_i$ is not the algebra of type I. On the other hand, $B^\otimes$ is of type I [8]. Thus $\bigotimes_I B_i$ is not isomorphic to $B^\otimes$.

Further, if we represent faithfully each B_i by τ_i as a factor of type I_2 on the Hilbert space H_{τ_i} and make a W^*-algebra $B_T^\otimes$ generated by B_i on $\bigotimes_I H_{\tau_i}$ by the von Neumann's method, then $B_T^\otimes$ is not of type I since $B_T^\otimes$ contains the part of type II_1 which is unitarily equivalent to the representation of $\bigotimes_I B_i$ by $\bigotimes_I \tau_i$. Thus we have shown that *the product $B^\otimes$ is not independent from the spaces H_i $(i \in I)$*.

Finally we give a remark concerning the ring $C^{\#'}$ given by J. von Neumann [8]. Let H_1 and H_2 be two 2-dimensional Euclidean spaces and $(\varphi_{11}, \varphi_{12})$ and $(\varphi_{21}, \varphi_{22})$ be complete normalized orthogonal systems in H_1 and H_2 respectively then $(\varphi_{11} \times \varphi_{21}, \varphi_{11} \times \varphi_{22}, \varphi_{12} \times \varphi_{21}, \varphi_{12} \times \varphi_{22})$ is a complete normalized orthogonal system in $H_1 \times H_2$. Let

$$(30) \qquad g = \sqrt{\frac{1+\alpha}{2}}\varphi_{11} \times \varphi_{21} + \sqrt{\frac{1-\alpha}{2}}\varphi_{12} \times \varphi_{22} \qquad (\text{where } 0 \leqq \alpha \leqq 1)$$

be an element in $H_1 \times H_2$, then

$$(31)\quad \begin{aligned}<g,g> &= \\ &< \sqrt{\frac{1+\alpha}{2}}\varphi_{11}\times\varphi_{21} + \sqrt{\frac{1-\alpha}{2}}\varphi_{12}\times\varphi_{22}\ , \\ &\qquad \sqrt{\frac{1+\alpha}{2}}\varphi_{11}\times\varphi_{12} + \sqrt{\frac{1-\alpha}{2}}\varphi_{12}\times\varphi_{22} > \\ &= \frac{1+\alpha}{2} + \frac{1-\alpha}{2} = 1.\end{aligned}$$

Let B_1 be the ring of all bounded operators on H_1. Denote by $a \times 1$, the extension of a bounded operator $a \in B_1$ on $H_1 \times H_2$, then since every bounded operator $a \in B_1$ can be represented by a 2×2-matrix with respect to $(\varphi_{11}, \varphi_{12})$, i. e.

$$(32)\qquad a = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22}\end{pmatrix},$$

$$(33)\quad \begin{aligned}&< (a\times 1)\, g, g > \\ &= <\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22}\end{pmatrix}\left(\sqrt{\frac{1+\alpha}{2}}\varphi_{11}\right)\times\varphi_{21} + \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22}\end{pmatrix}\left(\sqrt{\frac{1-\alpha}{2}}\varphi_{12}\right)\times\varphi_{22}, \\ &\qquad \sqrt{\frac{1+\alpha}{2}}\varphi_{11}\times\varphi_{21} + \sqrt{\frac{1-\alpha}{2}}\varphi_{11}\times\varphi_{22} > \\ &= <c_{11}\sqrt{\frac{1+\alpha}{2}}\varphi_{11}\times\varphi_{21} + c_{12}\sqrt{\frac{1+\alpha}{2}}\varphi_{12}\times\varphi_{21} \\ &\qquad + c_{21}\sqrt{\frac{1-\alpha}{2}}\varphi_{11}\times\varphi_{22} + c_{22}\sqrt{\frac{1-\alpha}{2}}\varphi_{12}\times\varphi_{22}, \\ &\qquad \sqrt{\frac{1+\alpha}{2}}\varphi_{11}\times\varphi_{21} + \sqrt{\frac{1-\alpha}{2}}\varphi_{12}\times\varphi_{22} > \\ &= <c_{11}\sqrt{\frac{1+\alpha}{2}}\varphi_{11}\times\varphi_{21}, \sqrt{\frac{1+\alpha}{2}}\varphi_{11}\times\varphi_{21} > \\ &\qquad + < c_{22}\sqrt{\frac{1-\alpha}{2}}\varphi_{12}\times\varphi_{22}, \sqrt{\frac{1-\alpha}{2}}\varphi_{12}\times\varphi_{22} > \\ &= \frac{1+\alpha}{2}c_{11} + \frac{1-\alpha}{2}c_{22}.\end{aligned}$$

Hence if $\alpha = 0$, $< (a \times 1)\ g,\ g >$ gives the trace of B_1 and if $\alpha = 1$, gives a pure state.

Thus by the construction of $C^{\#\prime}$ (c. f. [8]) if $\alpha_1 = \alpha_2 = \dots = 1$, $C^{\#\prime}$ is a factor of type I_∞ on the incomplete direct product $\otimes^{\mathfrak{D}}_{n=1,2,\cdots}(H_{(n,1)} \otimes H_{(n,2)})$ determined by the C^0-sequence $g^0_{(n)}$ and if $\alpha_1 = \alpha_2 = \dots = 0$, $C^{\#\prime}$ is a factor of type II_1 on the incomplete direct product by Proposition 7 and 6 respectively. If we put $\alpha_i = 1$ for infinitely many but not all i's and $\alpha_i = 0$ for other indices, we get an factor of type II_∞ since the restricted infinite direct product is associative.

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MATHEMATICAL INSTITUTE, TÔHOKU UNIVERSITY, SENDAI