LOCAL PROPERTY OF THE SUMMABILITY $|R, \lambda_n, 1|$

KISHI MATSUMOTO

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1. Let $f(t)$ be a summable function, periodic with period 2π, and its Fourier series be

$$\mathfrak{S}[f] = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \equiv \sum_{n=0}^{\infty} A_n(t)$$

We write

$$\phi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\}.$$

It is well known that the absolute convergence of $\mathfrak{S}[f]$ at a point x_0 is not a local property but depends on the behaviour of $f(x)$ in the whole interval $(0, 2\pi)$. Further we can easily show that even the absolute convergence of $\sum A_n(t)/n \log(n+1)$ is not a local property. L.S. Bosanquet and H. Kestelman [4] have shown that the summability $|C, 1|$ of a Fourier series at a given point is not a local property of the generating function. R. Mohanty [2] has further remarked that the summability $|C, 1|$ of the series $\sum A_n(t)/\log(n+1)$ is also not a local property. S. Izumi [3] and R. Mohanty [2] have independently proved that the summability $|R, \log n, 1|$ of a Fourier series at a given point is not a local property. Furthermore we can see that the summability $|R, \log n, 1|$ of $\sum A_n(t)/\log\log(n+1)$ is not a local property of the function.

The main object of the present paper is to treat the local property of the absolute Riesz summabiiity $|R, \lambda_n, 1|$. More precisely we prove the following theorems.

THEOREM 1. *The $|R, \lambda_n, 1|$ summability of the series $\sum A_n(t)l_n$ is not a local property, where*

λ_n	$\exp n^{\Delta}$	$\exp(\log n)^{\Delta}$	$\exp(\log\log n)^{\Delta}$
l_n	$1/n^{\Delta}\log(n+1)$	$1/(\log(n+1))^{\Delta}$	$1/(\log\log(n+1))^{\Delta}$
Δ	$0 < \Delta \leqq 1$	$0 < \Delta$	$0 < \Delta$

that is, if $x < \alpha < \beta < x + 2\pi$, there is a function summable over the interval (α, β) and vanishing in the remainder of the interval $(x, x+2\pi)$, such that $\sum A_n(t)l_n$ is not summable $|R, \lambda_n, 1|$ at $t = x$.

This is a generalization of theorems due to L.S. Bosanquet and H.

Kestelman, S. Izumi, and R. Mohanty, quoted above. For the proof of this theorem we require the following fundamental lemma.

LEMMA 1. *If* $0 < \lambda_1 < \lambda_2 < \dots\dots < \lambda_n \to \infty$, *and* $\sum c_n$ *is summable* $|R, \lambda_n, 1|$, *then the series* $\sum c_n/(\log \lambda_n)^{1/\Delta - 1}$ $(0 < \Delta < 1)$ *is summable* $|R, \exp(\log \lambda_n)^{1/\Delta}, 1|$.

LEMMA 2. *If* $0 < \lambda_1 < \lambda_2 < \dots <$, $\lambda_n \to \infty$, *and the series* $\sum c_n$ *is summable* $|R, \lambda_n, 1|$, *then the series* $\sum c_n/\exp(\log \lambda_n)^{1/\Delta}(\log \lambda_n)^{1/\Delta - 1}$ $(\Delta > 0)$ *is summable* $|R, \exp\{\exp(\log \lambda_n)^{1/\Delta}\}, 1|$.

The second is a generalization of the following result.

MOHANTY'S LEMMA [2]. *If* $0 < \lambda_1 < \lambda_2 < \dots, \lambda_n \to \infty$, *and* $\sum c_n$ *is summable* $|R, \lambda_n, 1|$, *then* $\sum \lambda_n^{-1} c_n$ *is summable* $|R, l_n, 1|$ *where* $l_n = \exp \lambda_n$.

Theorem 1 is the best possible in a sense. In fact we prove the following theorem.

THEOREM 2. *The* $|R, \lambda_n, 1|$ *summability of the series* $\sum A_n(t) l_{n,\epsilon}$ *at* $t = x$, *is a local property where*

λ_n	$\exp n^\Delta$	$\exp(\log n)^\Delta$	$\exp(\log\log n)^\Delta$
$l_{n,\epsilon}$	$1/n^\Delta(\log(n+1))^{1+\epsilon}$	$1/(\log(n+1))^{\Delta+\epsilon}$	$1/(\log\log(n+1))^{\Delta+\epsilon}$
Δ	$0 < \Delta \leqq 1$	$0 < \Delta$	$0 < \Delta$
$g(t)$	$t/\log(1/t)$	$t/\log(1/t)$	$t/\log(1/t)\log\log(1/t)$

More precisely, if

$$\int_0^t |\phi(u)|\, du = O\{g(t)\},$$

the series $\sum A_n(x) l_{n,\epsilon}$ *is summable* $|R, \lambda_n, 1|$.

In this direction the following theorems are known:

IZUMI'S THEOREM [3]. *If the Fourier coefficients are of order* $o(1/\log^2 n)$, *then the* $|R, \log n, 1|$ *summability has local property.*

MOHANTY'S THEOREM [2]. *Let* $\Phi(t) = \int_t^\pi \phi(u)/u\, du$. *If* $\Phi(t)/\{\log(k/t)\}$ $(k > \pi)$ *is of bounded variation in the interval* $(0, \eta)$ *where* $\eta > 0$, *then* $\sum A_n(x)/\log(n+1)$ *is summable* $|R, \log n, 1|$.

2. Proof of Lemma 1. We write

$$C_\lambda(\tau) = \sum_{\lambda_n \leqq \tau} c_n, \quad D_\lambda(\tau) = \sum_{\lambda_n \leqq \tau} c_n/(\log \lambda_n)^{1/\Delta-1}, \quad D_l(t) = \sum_{l_n \leqq t} c_n/(\log \lambda_n)^{1/\Delta-1},$$

where $l_n = \exp(\log \lambda_n)^{1/\Delta-1}$. Then, by the assumption,

$$g(w) = \frac{1}{w}\int_{\lambda_1}^{w} C_\lambda(\tau)\, d\tau$$

is of bounded variation in (λ_1, ∞); and we shall show that,

$$\frac{1}{u}\int_{l_1}^{u} D_l(t)\, dt$$

is also of bounded variation, which becomes, putting $u = \exp(\log w)^{1/\Delta}$, $t = \exp(\log \tau)^{1/\Delta}$,

$$h(w) = \exp\{-(\log w)^{1/\Delta}\}\int_{\lambda_1}^{w} D_\lambda(\tau)\frac{d}{d\tau}\{\exp(\log \tau)^{1/\Delta}\}\, d\tau.$$

Since

$$D_\lambda(\tau) = \int_{\lambda_1}^{\tau} 1/(\log u)^{1/\Delta-1}\, dC_\lambda(u),$$

we have

$$\text{(1)} \quad h(w) = \exp\{-(\log w)^{1/\Delta}\}\int_{\lambda_1}^{w}\frac{d}{d\tau}\{\exp(\log \tau)^{1/\Delta}\}\, d\tau \int_{\lambda_1}^{\tau}\frac{d\,C_\lambda(u)}{(\log u)^{1/\Delta-1}}$$

$$= \exp\{-\log w)^{1/\Delta}\}\int_{\lambda_1}^{w}\frac{d\,C_\lambda(u)}{(\log u)^{1/\Delta-1}}\int_{u}^{w}\frac{d}{d\tau}\exp(\log \tau)^{1/\Delta}\, d\tau$$

$$= \exp\{-\log w)^{1/\Delta}\}\int_{\lambda_1}^{w}\frac{\exp(\log w)^{1/\Delta} - \exp(\log u)^{1/\Delta}}{(\log u)^{1/\Delta-1}}\, dC_\lambda(u)$$

$$= -\exp\{-(\log w)^{1/\Delta}\}\int_{\lambda_1}^{w} C_\lambda(u)\frac{d}{du}\frac{\exp(\log w)^{1/\Delta} - \exp(\log u)^{1/\Delta}}{(\log u)^{1/\Delta-1}}\, du$$

$$= -\exp\{-(\log w)^{1/\Delta}\}\left\{\left[ug(u)\frac{d}{du}\frac{\exp(\log w)^{1/\Delta} - \exp(\log u)^{1/\Delta}}{(\log u)^{1/\Delta-1}}\right]_{\lambda_1}^{w}\right.$$

$$\left. - \int_{\lambda_1}^{w} ug(u)\frac{d^2}{du^2}\frac{\exp(\log w)^{1/\Delta} - \exp(\log u)^{1/\Delta}}{(\log u)^{1/\Delta-1}}\, du\right\}.$$

Now

$$\frac{d}{du}\frac{\exp(\log w)^{1/\Delta} - \exp(\log u)^{1/\Delta}}{(\log u)^{1/\Delta-1}}$$

$$= -\frac{1}{\Delta}\frac{1}{u}\exp(\log u)^{1/\Delta} + \left(1 - \frac{1}{\Delta}\right)\frac{1}{u}\frac{\exp(\log w)^{1/\Delta} - \exp(\log u)^{1/\Delta}}{(\log u)^{1/\Delta}}.$$

If we substitute this into the first term of the right side of (1), then it becomes

$$-\exp\{-(\log w)^{1/\Delta}\} wg(w)(-1/\Delta)w^{-1}\exp(\log w)^{1/\Delta} = (1/\Delta)\, g(w),$$

which is of bounded variation in (λ_1, ∞), by the assumption. Let us now estimate the second term of the right side of (1). We have

$$\begin{aligned}
&\frac{d^2}{du^2}\{(\exp(\log w)^{1/\Delta} - \exp(\log u)^{1/\Delta})/\log u)^{1/\Delta-1}\}\\
&= -\frac{1}{\Delta}\left(1-\frac{1}{\Delta}\right)\exp(\log w)^{1/\Delta}/(\log u)^{1/\Delta+1}\, u^2\\
&\quad -\left(1-\frac{1}{\Delta}\right)\exp(\log w)^{1/\Delta}/(\log u)^{1/\Delta}\, u^2\\
&\quad +\exp(\log u)^{1/\Delta}/\Delta\, u^2 - \exp(\log u)^{1/\Delta}(\log u)^{1/\Delta-1}/\Delta^2\, u^2\\
&\quad -\left(1-\frac{1}{\Delta}\right)\exp(\log u)^{1/\Delta}/\Delta u^2 \log u\\
&\quad +\left(1-\frac{1}{\Delta}\right)\exp(\log u)^{1/\Delta}/\Delta u^2(\log u)^{1/\Delta+1}\\
&= K_1(u) + K_2(u) + \dots + K_7(u),
\end{aligned}$$

say, and we put

$$h_i(w) = -\exp\{-(\log w)^{1/\Delta}\}\int_{\lambda_1}^{w} ug(u)\, K_i(u)\, du \qquad (i = 1, 2, \dots, 7).$$

We have first

$$h_2(w) = -\left(1-\frac{1}{\Delta}\right)\int_{\lambda_1}^{w}\frac{g(u)}{u(\log u)^{1/\Delta}}\, du,$$

which is of bounded variation, since, using $g(w) = O(1)$ in (λ_1, ∞),

$$\begin{aligned}
\int_{\lambda_1}^{\infty}\left|\frac{d}{dw}h_2(w)\right| dw &= \int_{\lambda_1}^{\infty}\left|\frac{-(1-(1/\Delta)g(w))}{w(\log w)^{1/\Delta}}\right| dw\\
&= \int_{\lambda_1}^{\infty} O\left\{\frac{1}{w(\log w)^{1/\Delta}}\right\} dw < \infty \qquad (0 < \Delta < 1).
\end{aligned}$$

Similarly $h_1(w)$ is also of bounded variation, since $K_1(u) = K_2(u)/\Delta \log u$. Among $K_i(u)$ $(i = 3, 4, \dots, 7)$, $K_4(u)$ has the greatest absolute value. Let us now estimate $h_4(w)$, i. e.

$$h_4(w) = -\Delta^{-2}\exp\{-(\log w)^{1/\Delta}\}\int_{\lambda_1}^{w}\frac{g(u)\exp(\log u)^{1/\Delta}(\log u)^{1/\Delta-1}}{u}\, du.$$

By partial integration, this becomes

$$h_4(w) = -\frac{\exp\{-(\log w)^{1/\Delta}\}}{\Delta}\left\{\Big[\exp(\log u)^{1/\Delta} g(u)\Big]_{\lambda_1}^{w} - \int_{\lambda_1}^{w}\exp(\log u)^{1/\Delta}g'(u)\, du\right\}.$$

$$h_4'(w) = -\frac{g'(w)}{\Delta} + \frac{1}{\Delta}\frac{d}{dw}\exp\{-(\log w)^{1/\Delta}\}\cdot\int_{\lambda_1}^{w}\exp(\log u)^{1/\Delta}g(u)\,du + \frac{g'(u)}{\Delta}$$

$$= \frac{1}{\Delta}\frac{d}{dw}\exp\{-(\log w)^{1/\Delta}\}\cdot\int_{\lambda_1}^{w}\exp(\log u)^{1/\Delta}g'(u)\,du$$

Since by the assumption

$$\int_{\lambda_1}^{\infty}|g'(w)|\,dw < \infty,$$

we have

$$\int_{\lambda_1}^{\infty}h_4'(w)\,dw \leqq \int_{\lambda_1}^{\infty}\left|\frac{1}{\Delta}\frac{d}{dw}\exp\{-(\log w)^{1/\Delta}\}\right|\int_{\lambda_1}^{w}\exp(\log u)^{1/\Delta}|g'(u)|\,du\,dw$$

$$= \int_{\lambda_1}^{\infty}\exp(\log u)^{1/\Delta}|g'(u)|\int_{u}^{\infty}\left|\frac{1}{\Delta}\frac{d}{dw}\exp\{-(\log w)^{1/\Delta}\}\right|dw\,du$$

$$= -\int_{\lambda_1}^{\infty}\exp(\log u)^{1/\Delta}|g'(u)|\int_{u}^{\infty}\frac{1}{\Delta}\frac{d}{dw}\exp\{-(\log w)^{1/\Delta}\}\,dw\,du$$

$$= \frac{1}{\Delta}\int_{\lambda_1}^{\infty}|g'(u)|\,du < \infty,$$

which shows that $h_4(w)$ is of bounded variation in the interval (λ_1, ∞). Concerning K_3, K_5, K_6 and K_7, we can prove similarly that they are of bounded variation in the interval (λ_1, ∞). Hence all $h_i(w)$ are of bounded variation, and then $h(w)$ is also, which is the required.

3. **Proof of Lemma 2.** If $0 < \Delta < 1$, then this Lemma is obtained by repeated use of Lemma 1 and Mohanty's lemma which was quoted above. In the case $\Delta \geqq 1$, we can prove the lemma quite similarly to the proof of Lemma 1, hence we omit it. The reason that the restriction $0 < \Delta < 1$ is required in the proof of Lemma 1 lies in that the integral

$$\int_{\lambda_1}^{\infty}\frac{dw}{w(\log w)^{1/\Delta}}$$

converges for $0 < \Delta < 1$. On the other hand that Δ is not restricted in Lemma 2 follows from that the corresponding integral used in the proof becomes

$$\int_{\lambda_1}^{\infty}\frac{dw}{w\exp(\log w)^{1/\Delta}},$$

which is convergent for all $\Delta > 0$.

4. We shall define k_n, h_n, depending on λ_n, as follows:

λ_n	$\exp n^{\Delta}$	$\exp(\log n)^{\Delta}$	$\exp(\log\log n)^{\Delta}$
k_n	$1/n^{1-\Delta}$	$1/n(\log n)^{1-\Delta}$	$1/n\log n(\log\log n)^{1-\Delta}$
h_n	$1/n\log n$	$1/n\log n$	$1/n\log n\log\log n$

Then we have

LEMMA 3. *If* $\sum c_n$ *is* $|R, \lambda_n, 1|$ *summable then* $\sum c_n k_n$ *is summable* $|R, \exp n, 1|$.

LEMMA 4. *If* $\sum A_n(x)\, l_n$ *is summable* $|R, \lambda_n, 1|$, *then* $\sum A_n(x) h_n$ *is summable* $|R, \exp n, 1|$.

These results are easily obtained by Lemma 1, 2 and Mohanty's lemma. For example, if $\lambda_n = \exp n^{\Delta}$, then

$$\exp(\log\lambda_n)^{1/\Delta} = \exp n, \quad k_n = 1/(\log\lambda_n)^{1/\Delta - 1} = 1/n^{1-\Delta},$$

and hence by Lemma 1 we get Lemma 3, in the case $\lambda_n = \exp n^{\Delta}$.

THE BOSANQUET-KESTELMAN LEMMA [4]. *Suppose* $f_n(x)$ *to be measurable in* (α, β), where $\beta - \alpha \leqq \infty$, *for* $n = 1, 2, \ldots$. *Then a necessary and sufficient condition that, for every function* $\phi(x)$, *summable in* (α, β) *the function* $f_n(x)\phi(x)$ *are summable in* (α, β) *and*

$$\sum_{n=1}^{\infty}\left|\int_{\alpha}^{\beta} f_n(x)\ \phi(x)\ dx\right| < \infty,$$

is that $\sum |f_n(x)|$ *is essentially bounded in* (α, β).

PROOF OF THEOREM 1. We assume for a moment that, for any function summable in the interval $(x+\alpha, x+\beta)$ and vanishing in the remainder of the interval $(x, x+2\pi)$, the series $\sum A_n(t) l_n$ is summable $|R, \lambda_n, 1|$ at $t = x$. Then by Lemma 4, $\sum A_n(x) h_n$ is summable $|R, \exp n, 1|$ or what is the same thing, it is absolutely convergent. Hence

$$\sum\left|\frac{A_n(x)}{n\log n}\right| = \sum\left|\int_{\alpha}^{\beta}\frac{\phi(t)\cos nt}{n\log n}dt\right| < \infty$$

and

$$\sum\left|\frac{A_n(x)}{n\log n\ \log\log n}\right| = \sum\left|\int_{\alpha}^{\beta}\frac{\phi(t)\ \cos nt}{n\log n\ \log\log n}dt\right| < \infty.$$

By Bosanquet and Kestelman's lemma, we have

$$\sum_{n=2}^{\infty}\left|\frac{\cos nt}{n\log n}\right| < M_1, \qquad \sum_{n=2}^{\infty}\left|\frac{\cos nt}{n\log n\log\ \log n}\right| < M_2.$$

almost everywhere in (α, β) where M_1, M_2 are absolute constants. On the other hand if $0 < t < 2\pi$ $(t \neq \pi)$,

$$\sum_{n=2}^{\infty} \frac{|\cos nt|}{n \log n} \geqq \sum \frac{\cos^2 nt}{n \log n} = \frac{1}{2} \sum \frac{1 + \cos 2nt}{n \log n}$$

$$\geqq \frac{1}{2} \sum \frac{1}{n \log n} - \frac{1}{2} \sum \left| \frac{\cos 2nt}{n \log n} \right|$$

$$\geqq \frac{1}{2} \sum \frac{1}{n \log n} - O(1) = \infty,$$

and

$$\sum_{n=2}^{\infty} \frac{|\cos nt|}{n \log n \log \log n} = \infty.$$

These are contradictions which arise from the assumption that the summability $|R, \lambda_n, 1|$ of $\sum A_n(t) l_n$ is a local property.

Thus Theorem 1 is proved.

5. **Proof of Theorem 2.** THE CASE $\lambda_n = \exp n^\Delta$. We shall prove that, if

$$\int_0^t |\phi(u)|\, du = O\left(\frac{1}{\log(1/t)}\right) \tag{2}$$

then $\displaystyle\sum \frac{A_n(x)}{n^\Delta (\log n)^{1+\epsilon}}$ is summable $|R, \exp n^\Delta, 1|$.

We begin to prove the following estimation.

$$\sum \frac{\exp n^\Delta \cos nt}{n^\Delta (\log n)^{1+\epsilon}} = O\left\{\frac{\exp w^\Delta \cdot w^{1-2\Delta}}{(\log w)^{1+\epsilon}}\right\} \tag{3}$$

For, since $|\cos nt| \leqq 1$, we have

$$\sum_{n \leqq w} \frac{\exp n^\Delta}{n^\Delta (\log n)^{1+\epsilon}}$$

$$= \sum \frac{\Delta \exp n \cdot n^{\Delta - 1}}{\Delta\, n^{2\Delta - 1} (\log n)^{1+\epsilon}}$$

$$= O\left\{\int_e^w \frac{\Delta \exp x^\Delta \cdot x^{\Delta-1}}{\Delta x^{2\Delta-1} (\log x)^{1+\epsilon}}\, dx\right\}$$

$$= O\left\{\left[\frac{\exp x^\Delta}{x^{2\Delta-1} (\log x)^{1+\epsilon}}\right]_e^w - \int_e^w \exp x^\Delta \left(\frac{1}{x^{2\Delta} (\log x)^{1+\epsilon}} - \frac{1+\varepsilon}{x^{2\Delta} (\log x)^{2+\epsilon}}\right) dx\right\}$$

$$= O\{\exp w^\Delta \cdot w^{1-2\Delta} (\log w)^{-1-\epsilon}\}$$

which gives (3).

Furthermore

(4) $$\sum_{n \leq w} \frac{\exp n^\Delta \cos nt}{n^\Delta (\log n)^{1+\epsilon}} = O\left\{\frac{\exp w^\Delta}{w^\Delta (\log w)^{1+\epsilon} t}\right\}$$

For, denoting by $D_n(t)$ the Dirichlet kernel, and using Abel's transformation we obtain

$$\sum_{n \leq w} \frac{\exp n^\Delta \cos nt}{n^\Delta (\log n)^{1+\epsilon}}$$
$$= \sum_{n=2}^{[w]-1} \left\{\delta\left(\frac{\exp n^\Delta}{n^\Delta (\log n)^{1+\epsilon}}\right) D_n(t)\right\} + \frac{\exp [w]^\Delta D[w](t)}{[w]^\Delta (\log [w])^{1+\epsilon}} - \frac{\exp 2^\Delta D_1(t)}{2^\Delta (\log 2)^{1+\epsilon}}$$
$$= O\{\exp w^\Delta \cdot w^{-\Delta} (\log w)^{-1-\epsilon} t^{-1}\},$$

where $\delta \alpha_n = \alpha_n - \alpha_{n+1}$ and the monotonity of the sequence $\{\exp n^\Delta \cdot n^{-\Delta} (\log n)^{-1-\epsilon}\}$ is used, which follows from

$$\frac{d}{dx} \frac{\exp x^\Delta}{x^\Delta (\log x)^{1+\epsilon}} = \frac{\exp x^\Delta}{x (\log x)^{1+\epsilon}} \left(\Delta - \frac{\Delta}{x^\Delta} + \frac{1+\varepsilon}{x^\Delta (\log x)}\right) > 0 \qquad (x > 1).$$

Thus (4) is proved.

Let us now consider the series $\sum_{=2}^{\infty} A_n(x)\, n^{-\Delta} (\log n)^{-1-\epsilon}$, which is summable $|R, \exp n^\Delta, 1|$ if

$$I = \Delta \int_2^\infty \frac{w^{\Delta-1}}{\exp w^\Delta} \left| \sum_{n \leq w} \exp n^\Delta \cdot \frac{A_n(x)}{n^\Delta (\log n)^{1+\epsilon}} \right| dw < \infty.$$

We have

$$\sum_{n \leq w} \exp n^\Delta \cdot \frac{A_n(x)}{n^\Delta (\log n)^{1+\epsilon}} = \frac{2}{\pi} \int_0^\pi \phi(t) \left\{\sum_{n \leq w} \exp n^\Delta \frac{\cos nt}{n^\Delta (\log n)^{1+\epsilon}}\right\} dt$$
$$= \frac{2}{\pi} \left(\int_0^{w^{-1+\Delta}} + \int_{w^{-1+\Delta}}^\pi\right) = I_1 + I_2,$$

say. By (2) and (3) we have

$$I_1 = O\left\{\frac{\exp w^\Delta}{w^{2\Delta-1} (\log w)^{1+\epsilon}} \int_0^{w^{-1+\Delta}} |\phi(t)|\, dt\right\} = O\left\{\frac{\exp w^\Delta}{w^\Delta (\log w)^{2+\epsilon}}\right\}.$$

From (4) we have

$$I_2 = O\left\{\frac{\exp w^\Delta}{w^\Delta (\log w)^{1+\epsilon}} \int_{w^{-1+\Delta}}^\pi \frac{|\phi(t)|}{t}\, dt\right\}$$
$$= O\left\{\frac{\exp w^\Delta}{w^\Delta (\log w)^{1+\epsilon}} \left(\left[\frac{1}{\log (1/t)}\right]_{w^{-1+\Delta}}^\pi + \int_{w^{-1+\Delta}}^\pi \frac{dt}{t \log(1/t)}\right)\right\}$$
$$= O\{\exp w^\Delta \cdot \log \log w \cdot w^{-\Delta} (\log w)^{-1-\epsilon}\}.$$

Hence we obtain

$$I \leqq \Delta \int_2^\infty \frac{w^{\Delta-1}}{\exp w^\Delta}(|I_1| + |I_2|)\,dw = O\left\{\int_2^w \frac{\log\log w}{w(\log w)^{1+\epsilon}}\,dw\right\} < \infty,$$

which completes the proof.

6. Proof of Theorem 2. THE CASE $\lambda_n = \exp(\log n)^\Delta$. We shall prove that, if

$$\text{(5)} \qquad \int_0^t |\phi(u)|\,du = O\left(\frac{t}{\log(1/t)}\right)$$

then $\sum \frac{A_n(x)}{(\log n)^{\Delta+\epsilon}}$ $(\varepsilon > 0)$ is summable $|R, \exp(\log w)^\Delta, 1|$. We need the following estimations:

$$\text{(6)} \qquad \sum_{n\leqq w} \frac{\exp(\log n)^\Delta \cos nt}{(\log n)^{\Delta+\epsilon}} = O\{\exp(\log w)^\Delta\, w\,(\log w)^{1-2\Delta-\epsilon}\}.$$

$$\text{(7)} \qquad \sum_{n\leqq w} \frac{\exp(\log n)^\Delta \cos nt}{(\log n)^{\Delta+\epsilon}} = O\left\{\frac{\exp(\log w)^\Delta}{t(\log w)^{\Delta+\epsilon}}\right\}.$$

These are easily proved, so we omit their proof.

The series $\sum_{n=2}^\infty A_n(x)(\log n)^{-\Delta-\epsilon}$ is summable $|R, \exp(\log w)^\Delta, 1|$ if

$$I = \int_2^\infty \frac{(\log w)^{\Delta-1}}{w\exp(\log w)^\Delta}\left|\sum_{n\leqq w}\exp(\log n)^\Delta\frac{A_n(x)}{(\log n)^{\Delta+\epsilon}}\right|dw < \infty.$$

We have

$$\sum_{n\leqq w}\frac{\exp(\log n)^\Delta A_n(x)}{(\log n)^{\Delta+\epsilon}} = \frac{2}{\pi}\int_0^\pi \phi(t)\left\{\sum_{n\leqq w}\frac{\exp(\log n)^\Delta \cos nt}{(\log n)^{\Delta+\epsilon}}\right\}dt$$

$$= \frac{2}{\pi}\left\{\int_0^{(w\log w)^{-1}} + \int_{(w\log w)^{-1}}^\pi\right\} = I_1 + I_2,$$

say. By (5) and (6) we have

$$I_1 = O\left\{\exp(\log w)^\Delta \cdot w(\log w)^{1-2\Delta-\epsilon}\int_0^{(w\log w)^{-1}}|\phi(t)|\,dt\right\}$$

$$= O\{\exp(\log w)^\Delta \cdot (\log w)^{-2\Delta-1-\epsilon}\}.$$

Using both (5) and (7), we have, putting $\int_0^t |\phi(u)|\,du = \phi^*(t)$,

$$I_2 = O\{\exp(\log w)^\Delta \cdot (\log w)^{-\Delta-\epsilon}\}\int_{(w\log w)^{-1}}^\pi |\phi(t)|t^{-1}\,dt$$

$$= O\{\exp(\log w)^\Delta \cdot (\log w)^{-\Delta-\epsilon}\}\left\{\Big[\phi^*(t)t^{-1}\Big]_{(w\log w)^{-1}}^\pi + \int_{(w\log w)^{-1}}^\pi \phi^*(t)t^{-2}\,dt\right\}$$

$$= O\{\exp(\log w)^\Delta \cdot \log\log w(\log w)^{-\Delta-\epsilon}\}$$

Thus

$$I \leqq \int_2^\infty \frac{(\log w)^{\Delta-1}}{w \exp(\log w)^\Delta} (|I_1| + |I_2|)\, dw = O\left\{\int_2^\infty \frac{\log\log w}{w(\log w)^{1+\epsilon}}\, dw\right\} < \infty$$

which proves the theorem.

REMARK. When $\Delta = 1$, $|R, \exp(\log w)^\Delta, 1| = |C, 1|$, and for this case M. T. Cheng [1] has proved the following theorem.

CHENG'S THEOREM [1]. *If the Fourier series* $\sum_{n=0}^{\infty} A_n(x)$ *is multiplied by one of the following factors*

$$1/(\log n)^{1+\frac{1}{2}+\epsilon},\ 1/(\log n)^{1+\frac{1}{2}} (\log_2 n)^{1+\epsilon}, \ldots,$$

then under the condition

$$\int_0^t |\phi(u)|\, du = o(t),$$

the resulting series is summable $|C, 1|$ *at the point* $t = x$.

7. Proof of Theorem 2. THE CASE $\lambda_n = \exp(\log\log n)^\Delta$. We shall prove that, if

$$\int_0^t |\phi(u)|\, du = O\left(t/\log\left(\frac{1}{t}\right)\log\log\left(\frac{1}{t}\right)\right), \tag{8}$$

then $\sum \dfrac{A_n(x)}{(\log\log n)^{\Delta+\epsilon}}$ is summable $|R, \exp(\log\log w)^\Delta, 1|$.

We have

$$\sum_{n\leqq w} \frac{\exp(\log\log n)^\Delta \cos nt}{(\log\log n)^{\Delta+\epsilon}} = O\{\exp(\log\log w)^\Delta \cdot w \cdot \log w\, (\log\log w)^{1-2\Delta-\epsilon}\}. \tag{9}$$

$$\sum_{n\leqq w} \frac{\exp(\log\log n)^\Delta \cos nt}{(\log\log n)^{\Delta+\epsilon}} = O\left\{\frac{\exp(\log\log w)^\Delta}{t\,(\log\log w)^{\Delta+\epsilon}}\right\}. \tag{10}$$

Proof is omitted.

The series $\sum \dfrac{A_n(x)}{(\log\log n)^{\Delta+\epsilon}}$ is summable $|R, \exp(\log\log w)^\Delta, 1|$ if

$$I = \int_2^\infty \frac{(\log\log w)^{\Delta-1}}{\exp(\log\log w)^\Delta \cdot w \log w} \left| \sum_{n\leqq w} \frac{\exp(\log\log n)^\Delta A_n(x)}{(\log\log n)^{\Delta+\epsilon}} \right| dw < \infty.$$

Now

$$\sum_{n\leqq w} \frac{\exp(\log\log n)^\Delta A_n(x)}{(\log\log n)^{\Delta+\epsilon}} = \frac{2}{\pi}\int_0^\pi \phi(t) \left\{\sum_{n\leqq w} \frac{\exp(\log\log n)^\Delta \cos nt}{(\log\log n)^{\Delta+\epsilon}}\right\} dt$$

$$= \frac{2}{\pi}\left(\int_0^\tau + \int_\tau^\pi\right) = I_1 + I_2,$$

say, where $\tau = 1/w \log w \log\log w$. By (8) and (9), we have

$$I_1 = O\left\{ w \log w (\log\log w)^{1-2\Delta-\epsilon} \exp(\log\log w)^\Delta \int_0^\tau |\phi(t)|\, dt \right\}$$

$$= O\{\exp(\log\log w)^\Delta \cdot (\log w)^{-1} (\log\log w)^{-1-2\Delta-\epsilon}\}.$$

Also using (8) and (10), we have

$$I_2 = O\left\{ \frac{\exp(\log\log w)^\Delta}{(\log\log w)^{\Delta+\epsilon}} \int_\tau^\pi \frac{|\phi(t)|}{t}\, dt \right\}$$

$$= O\left\{ \frac{\exp(\log\log w)^\Delta}{(\log\log w)^{\Delta+\epsilon}} \right\} \left\{ \left[\frac{\phi^*(t)}{t} \right]_\tau^\pi - \int_\tau^\pi \frac{\phi^*(t)}{t^2}\, dt \right\}$$

$$= O\{\exp(\log\log w)^\Delta \cdot \log\log\log w (\log\log w)^{-\Delta-\epsilon}\}.$$

Hence we conclude

$$I \leqq \int_2^\infty \frac{(\log\log w)^{\Delta-1} (|I_1| + |I_2|)\, dw}{\exp(\log\log w)^\Delta\, w \log w} = O\left\{ \int_2^\infty \frac{\log\log\log w}{w \log w (\log\log w)^{1+\epsilon}} dw \right\} < \infty,$$

and this completes the proof.

Thus Theorem 2 is proved.

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MATHEMATICAL DEPARTMENT, TÔKYO TORITSU UNIVERSITY.