PRINCIPAL FIBRE BUNDLES WITH THE 1-DIMENSIONAL TOROIDAL GROUP

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In this paper, we study the principal fibre bundles whose structure group is the 1-dimensional toroidal group T^1, mainly from the differential geometrical point of view. Therefore all the spaces considered here are differentiable manifolds (of class C^∞) with the second countability axiom and all the maps are supposed to be of class C^∞.

First we define the additive group structure in the set $P(M, T^1)$ of all principal fibre bundles over M with group T^1. It is possible to study the group $P(M, T^1)$ at least from three points of view; (1) the differential geometrical point of view, (2) the sheaf theoretical point of view and (3) the homotopy theoretical point of view. The method of (2) is the simplest and gives the *best result*; namely an isomorphism between $P(M, T^1)$ and $H^2(M, Z)$ ([14]). The method of (1) gives only a homomorphism of $P(M, T^1)$ onto the Betti part of $H^2(M, Z)$. The deficiency comes from the use of differential forms which makes it difficult to obtain any information about the torsion part. However this method has some advantage for the applications, because the homomorphism is closely related to the curvature form of a connection in a bundle $P \in P(M, T^1)$. The method of (3) can be applied only to the case where M is simply connected. Since the second point of view is well known, we only sketch the outline and in this paper we put stress on the first point of view.

In § 6, 7, 8 and 10, some applications of the first method are discussed and making use of the third method we give in § 9 a very geometrical description of $P(M, T^1)$ under the assumption that M is a simply connected homogeneous space with the compact isotropic subgroup.

Most of results in this paper are *mutatis mutandis* valid in the case the group is the k-dimensional toroidal group, and such a case can be usually reduced to the case of group T^1.

We remark also that the second method can be applied to the case where the structure group is not abelian ([6], [8]). The case where M is a complex analytic manifold and the group is the complex torus has been studies by Blanchard [2].

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1. Principal fibre bundles with group T^1

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Let P and P' be principal fibre bundles over an n-dimensional manifold M with the 1-dimensional toroidal group T^1. (T^1 is the additive group of real numbers modulo 1, written multiplicatively in this section). We shall define the sum $P+P'$, which will be also a principal fibre bundle over M with group T^1. Let $\Delta(P\times P')$ be the set of all elements $(u,u')\in P\times P'$ such that $\pi(u)=\pi'(u')$, where π and π' are respectively the projections of P and P' onto M. We say that two elements (u_1,u_1') and (u_2,u_2') of $\Delta(P\times P')$ are equivalent if there exists an element $s\in T^1$ such that

$$u_1 s = u_2 \text{ and } u_1' s^{-1} = u_2'.$$

We denote by $P+P'$ the quotient space of $\Delta(P\times P')$ by this equivalence relation. The projection from $\Delta(P\times P')$ onto M induces a projection from $P+P'$ onto M, which we shall denote by π''. The action of T^1 on $\Delta(P\times P')$ defined by

$$(u,u')s=(us,u') \qquad s\in T^1,\ (u,u')\in\Delta(P\times P'),$$

preserves the equivalence relation; hence it defines the action of T^1 on $P+P'$. (Note the fact that T^1 is abelian). Now we shall show that T^1 acts simply transitively on $\pi''^{-1}(x)$ for each $x\in M$. Let u_1'',u_2'' be arbitrary elements of $P+P'$ and let $(u_1,u_1'),(u_2,u_2')$ be representatives for u_1'',u_2'' respectively. Then there are elements $s,s'\in T^1$ such that

$$u_2=u_1 s \quad\text{and}\quad u_2'=u_1's',$$

if $\pi''(u_1'')=\pi''(u_2'')$. Since (u_2,u_2') and (u_2s',u_1') are equivalent, we have that $u_2''=u_1''ss'$. Hence T^1 is transitive on $\pi''^{-1}(x)$. Now suppose $u''s=u''$ for some element $u''\in P+P'$ and $s\in T^1$. Let (u,u') be a representative for u''. Then (u,u') and (us,u') are equivalent. Hence s is the unit of T^1. We shall prove that the set of all principal fibre bundles over M with group T^1 forms an additive group under the above defined addition operation. It is evident that the correspondence

$$\Delta(P\times P')\ni(u,u')\leftrightarrow(u',u)\in\Delta(P'\times P)$$

gives rise to the bundle isomorphism between $P+P'$ and $P'+P$. (Note the fact that (us,u') and $(u,u's)$ are equivalent in $\Delta(P\times P')$). Hence the addition operation is commutative. Let P_0 be the product bundle $M\times T^1$. Then the mapping

$$P\ni u\to(u,(\pi(u),1))\in\Delta(P\times P_0)$$

induces a bundle isomorphism of P onto $P+P_0$. (1 is the unit of T^1 and $(\pi(u),1)\in P_0$). This shows the existence of the unit element. Let $-P$ be a space homeomorphic to P and let $-u$ be the element of $-P$ corresponding to an element u of P. Then we define the projection $-\pi$ of $-P$ onto M by

$$-\pi(-u)=\pi(u).$$

The action of T^1 on $-P$ is defined by

$$(-u)s=-(us^{-1}).$$

Then $-P$ is a principal fibre bundle over M with group T^1. We shall show that $P+(-P)=P_0$. Let $(u_1,-u_2)$ be any element of $\Delta(P\times(-P))$. Then there exists a unique element s of T^1 such that

$$u_1 = u_2 s.$$

The mapping

$$\Delta(P \times (-P)) \ni (u_1, -u_2) \rightarrow (\pi(u_1), s) \in M \times T^1 = P_0$$

induces a bundle isomorphism of $P + (-P)$ onto P_0. This shows the existence of the inverse elements. Finally we shall show the associativity. Let P, P' and P'' be arbitrary principal fibre bundles over M with group T^1 and let $\Delta(P \times P' \times P'')$ be the set of all elements $(u, u', u'') \in P \times P' \times P''$ such that $\pi(u) = \pi'(u') = \pi''(u'')$. We say that (u, u', u'') and $(us, u's^{-1}s', u''s'^{-1})$ are equivalent for any $s, s' \in T^1$. The quotient space of $\Delta(P \times P' \times P'')$ by this equivalence relation is naturally isomorphic to the spaces $P + (P' + P'')$ and $(P + P') + P''$. Furthermore the action of T^1 on $\Delta(P \times P' \times P'')$ given by

$$(u, u', u'')s = (us, u', u'')$$

defines the action of T^1 on the quotient space, which turns out to be a bundle isomorphic to $P + (P' + P'')$ and $(P + P') + P''$. (The detail will be omitted).

THEOREM 1. *The set $P(M, T^1)$ of all principal fibre bundles over M with group T^1 forms an additive group. The zero element is given by the trivial bundle.*

REMARK 1. If we adopt the Steenrod's definition of a fibre bundle, then the sum of two bundles P and P' can be defined as follows. Let $f_{ij}: U_i \cap U_j \rightarrow T^1$ (resp. $f'_{ij}: U_i \cap U_j \rightarrow T^1$) be the coordinate transformations of the bundle P (resp. P'), where U_i, U_j are open sets in M. Then the bundle whose coordinate transformations are given by $f_{ij} \cdot f'_{ij}$ is the sum of P and P'.

REMARK 2. The results in this section hold for any principal fibre bundle over M with abelian structure group.

2. From the point of view of the differential geometry

Let P be a principal fibre bundle over M with group T^1. We note the fact that the Lie algebra of T^1 is the real line R with the trivial bracket operation. An infinitesimal connection in P is defined by a real valued linear differential form ω on P with the following properties ([7], [10]):

(ω.1) $\quad \omega(u\bar{s}) = s^{-1}\bar{s} \quad$ for $u \in P$, $\bar{s} \in T_s(T^1)$.

(ω.2) $\quad \omega(\bar{u}s) = \omega(\bar{u}) \quad$ for $\bar{u} \in T(P)$, $s \in T^1$.

REMARK 1. Since T^1 is abelian, the property (ω.2) is equivalent to the usual one: $\omega(\bar{u}s) = s^{-1}\,\omega(\bar{u})s$.

The structure equation of E. Cartan is given by

$$d\omega = \Omega$$

where Ω is the curvature form.

REMARK 2. Since T^1 is abelian, the usual structure equation

$$d\omega = -\frac{1}{2}[\omega, \omega] + \Omega$$

is reduced to the above equation.

Since Ω satisfies the following condition:

$$\Omega(\bar{u}_1\bar{s}_1, \bar{u}_2\bar{s}_2) = \Omega(\bar{u}_1, \bar{u}_2) \qquad \text{for all } \bar{u}_1, \bar{u}_2 \in T_u(P), \quad \bar{s}_1, \bar{s}_2 \in T_s(T^1),$$

there exists a unique 2-form Ω^* on M such that

$$\Omega = \pi^*(\Omega^*),$$

where π^* is the mapping induced from π.
From

$$\pi^*(d\Omega^*) = d\Omega = dd\omega = 0,$$

it follows that $d\Omega^* = 0$.

Let ω' be another form defining a connection in P. Then

$$(\omega - \omega')(\bar{u}\,\bar{s}) = (\omega - \omega')(\bar{u}s + u\bar{s}) = (\omega - \omega')(\bar{u}).$$

Hence there exists a unique differential form Ψ on M such that

$$\pi^*(\Psi) = \omega - \omega'.$$

From

$$\pi^*(d\Psi) = d(\pi^*(\Psi)) = d\omega - d\omega' = \pi^*(\Omega^*) - \pi^*(\Omega'^*),$$

we obtain

$$d\Psi = \Omega^* - \Omega'^*.$$

(Ω' is the curvature form of the connection defined by ω').

Therefore *the cohomology class of Ω^* is independent from the choice of connections* and is called the *characteristic class* of P. This is a particular case of the theorem of Cartan-Weil [3].

Let P and P' be principal fibre bundles over M with group T^1 and let ω and ω' be connection forms on P and P' respectively. We define a linear differential form $\omega \times \omega'$ on $P \times P'$ as follows:

$$\omega \times \omega' = \varphi^*(\omega) + \varphi'^*(\omega'),$$

where φ and φ' are the natural projections from $P \times P'$ onto P and P' respectively. We denote also by $\omega \times \omega'$ the restriction of $\omega \times \omega'$ on $\Delta(P \times P')$. From the properties (ω. 1 & 2) it follows that $\omega \times \omega'$ induces a linear differential form on $P + P'$; more explicity, there exists a unique differential form ω'' on $P + P'$ such that

$$\mu^*(\omega'') = \omega \times \omega',$$

where μ is the natural projection of $\Delta(P \times P')$ onto $P + P'$. It is easy to see that ω'' defines a connection in $P + P'$ and

$$\Omega''^* = \Omega^* + \Omega'^*.$$

Therefore *the mapping which sends P into its characteristic class is a homomorphism of $P(M, T^1)$ into the second cohomology group $H^2(M, R)$ of M with real coefficients.* We shall study the kernel of this homomorphism. Suppose that the characteristic class of P vanishes. If a form ω defines a connection in P, then the 2-form Ω^* on M is cohomologous to zero, i.e., there exists a 1-form θ on M such that

$$d\theta = \Omega^*.$$

It is easy to see that the form $\pi^*(\theta)$ has the following properties:

$$\pi^*(\theta)(u\bar{s}) = 0 \qquad \text{for any } u \in P,\ \bar{s} \in T(T^1),$$

$$\pi^*(\theta)(\bar{u}s) = \pi^*(\theta)(\bar{u}) \qquad \text{for any } \bar{u} \in T(P),\ s \in T^1.$$

Hence the form $\omega - \pi^*(\theta)$ possesses the properties (ω. 1 & 2) and defines a connection in P. Its curvature form vanishes identically. We have proved that, *if the characteristic class of P vanishes, then there exists in P a connection with the vanishing curvature form.* The restricted holonomy group of this connection contains only the unit. Let $\widetilde{M}$ be the universal covering of M and p the projecton of $\widetilde{M}$ onto M. Let $\widetilde{P}$ be the bundle over $\widetilde{M}$ induced from P by p. If a form ω defines in P a connection with the vanishing curvature, then $\bar{p}^*(\omega)$ defines in $\widetilde{P}$ a connection with the vanishing curvature, where $\bar{p}$ is the natural bundle map of $\widetilde{P}$ onto P. Since the base manifold of $\widetilde{P}$ is simply connected, the holonomy group of this connection contains only the unit. Hence the structure group of $\widetilde{P}$ can be reduced to the unit; in other words, $\widetilde{P}$ is the product bundle $\widetilde{M} \times T^1$. We have proved that, *if the characteristic class of P vanishes, then the bundle $\widetilde{P}$ over the universal covering space $\widetilde{M}$ of M, induced from P by the projection $p: \widetilde{M} \to M$, is trivial. In particular, if M is simply connected, the above homomorphism is an isomorphism.*

Let P be again a principal fibre bundle with the vanishing characteristic class and let ω be a connection form on P whose curvature vanishes. We have the natural homomorphism from the fundamental group $\pi_1(M)$ onto the holonomy group. Since the holonomy group is abelian, the commutator subgroup $[\pi_1(M), \pi_1(M)]$ of $\pi_1(M)$ is in the kernel of the homomorphism. On the other hand it is well known that the 1st homology group $H_1(M, Z)$ of M with integer coefficients is isomorphic to $\pi_1(M)/[\pi_1(M), \pi_1(M)]$. Hence *if $H_1(M, Z) = 0$, then the homomorphism of $P(M, T^1)$ into $H^2(M, R)$ is an isomorphism into.*

Now we shall study the image of the homomorphism of $P(M, T^1)$ into $H^2(M, R)$. To this end, we state a result of Weil in [17].

LLMMA OF WEIL. *Every differentiable manifold admits a differentiably simple open covering.*

An open covering $\{U_i\}_{i \in I}$ is, by definition, *differentiably simple,* if

(1) It is locally finite and each U_i is relatively compact.

(2) Let J be a subset of I, if $\bigcap_{j \in J} U_j$ is non-empty, then it has a differentiable retraction.

Observe that the cohomology of the nerve N associated with a differentiably simple open covering of M is isomorphic to the Čech cohomology of M.

Let Ψ be a closed 2-form on M. Then, for each $i \in I$, there exists a

1-form Ψ_i on U_i such that

$$\Psi = d\Psi_i \quad \text{on } U_i.$$

If $U_i \cap U_j$ is non-empty, then $\Psi_i - \Psi_j$ is a closed 1-form on it. Hence there exists a function Ψ_{ij} on $U_i \cap U_j$ such that

$$d\Psi_{ij} = \Psi_i - \Psi_j \text{ and } \Psi_{ij} = -\Psi_{ji}.$$

Suppose $U_i \cap U_j \cap U_k$ is non-empty. Then $f_{ijk} = \Psi_{ij} + \Psi_{jk} + \Psi_{ki}$ is closed, hence it is a constant function on $U_i \cap U_j \cap U_k$. Since (i, j, k) is a 2-simplex in the nerve N, a function $f\colon N \to R$ given by

$$f(i, j, k) = f_{ijk}$$

is a 2-cochain of the nerve N. It can be easily shown that f is a cocycle. According to a theorem of Weil [17], the correspondence

$$\Psi \to f$$

gives rise to an isomorphism between the De Rham cohomology $H^2(M, R)$ and the cohomology $H^2(N, R)$ of the nerve N with real coefficients.

A closed form Ψ is *integral*, if

$$\int_c \Psi = \text{integer}$$

for any finite singular cocycle c with integer coefficients. The set of all elements of $H^2(M, R)$ which contain integral closed forms is the image of the natural homomorphism $H^2(M, Z) \to H^2(M, R)$, which will be denoted by $H^2(M, Z)_b$ (the subscript b stands for "the Betti part"). We define similarly $H^2(N, Z)_b$. Then the isomorphism of $H^2(M, R)$ onto $H^2(N, R)$ induces an isomorphism of $H^2(M, Z)_b$ onto $H^2(N, Z)_b$.

We shall apply the above results to the form Ω^*. Let $\{U_i\}_{i\in I}$ be a differentiably simple open covering of M. Let σ_i be a local cross-section in P defined on U_i. Then $\omega_i = \sigma_i^*(\omega)$ is a 1-form defined on U_i and satisfies the following equation:

$$d\omega_i = \Omega^* \text{ on } U_i.$$

If $U_i \cap U_j$ is non-empty, we define the coordinate transformation function γ_{ij} as follows:

$$\sigma_i(x)\gamma_{ij}(x) = \sigma_j(x) \text{ for } x \in U_i \cap U_j.$$

γ_{ij} is a mapping from $U_i \cap U_j$ into T^1, hence it can be considered as a real valued function modulo 1. We have ([5])

$$\omega_j - \omega_i = d\gamma_{ij} \quad \text{on } U_i \cap U_j.$$

From the fundamental property of coordinate transformations:

$$\gamma_{ij} + \gamma_{jk} + \gamma_{ki} \equiv 0 \ (\text{mod. } 1),$$

it follows that the cocycle of the nerve N corresponding to the closed form Ω^* is integral. Therefore *the characteristic class of P belongs to $H^2(M, Z)_b$.*

We shall prove the converse. Given an arbitrary element of $H^2(M, Z)_b$, let Ψ be a 2-form which represents it. According to the method of Weil, we define a 1-form Ψ_i on each U_i and a function Ψ_{ij} on each non-empty

$U_i \cap U_j$. If we choose properly Ψ_{ij}, then we have that

$$f_{ijk} = \Psi_{ij} + \Psi_{jk} + \Psi_{ki} = \text{integer} \qquad \text{on } U_i \cap U_j \cap U_k.$$

In fact, if the f_{ijk} 's are not integers, the 2-cochain f defined by

$$f(i, j, k) = f_{ijk}$$

is cohomologous to an integral 2-cochain f' of the nerve N; that is, there exists a 1-cochain g of the nerve N such that

$$f' = f + \partial g.$$

Then we have to only take $\Psi_{ij} + g(i, j)$ in place of Ψ_{ij}.

If we consider the Ψ_{ij}'s as functions with values in T^1 and take them as the coordinate transformations, then we obtain a principal fibre bundle P over M with group T^1. From

$$d\Psi_{ij} = \Psi_i - \Psi_j,$$

it follows that the set of Ψ_i's defines a connection in P, whose curvature form is $\pi^*(\Psi)$. More precisely, there exists a connection form ω on P such that

$$\Psi_i = \sigma_i^*(\omega) \quad \text{on } U_i,$$

where σ_i is the natural cross-section on U_i associated with the coordinate transformations Ψ_{ij}. $(\sigma_i(x)\Psi_{ij}(x) = \sigma_j(x),\ x \in U_i \cap U_j)$.

We have obtained the following

THEOREM 2. *Let $P(M, T^1)$ be the additive group of all principal fibre bundles P over M with group T^1. The mapping which sends P into its characteristic class is a homomorphism of $P(M, T^1)$ onto $H^2(M, Z)_b$. If P is in the kernel of this homomorphism, then there exists a connection in P whose restricted holonomy group is trivial. If the first homology group $H_1(M, Z)$ of M with integer coefficients is zero, then the homomorphism is an isomorphism.*

3. From the point of view of the theory of sheaves

Since the msthod based on the theory of sheaves (faisceaux) is well known ([6], [8], [14]), we shall only sketch the outline of the method and compare with the geometrical method in the section 2.

Let F (resp. F' and F'') be the sheaf of germs of all local mappings of class C^∞ from M into R (resp. Z and T^1). Let $\{U_i\}_{i \in I}$ be a differentiably simple open covering of M and $\gamma_{ij}: U_i \cap U_j \to T^1$ the coordinate transformations of a bundle P. Then the set of γ_{ij}'s defines a cochain γ of M with coefficients in the sheaf F''. It can be easily shown that γ is a cocycle and the mapping $P \to \gamma$ gives rise to an *isomorphism of $P(M, T^1)$ onto $H^1(M, F'')$.* From the natural exact sequence

$$0 \to Z \to R \to T^1 \to 0$$

we obtain an exact sequence

$$H^1(M, F) \to H^1(M, F'') \to H^2(M, F') \to H^2(M, F) \to \dots.$$

Since $H^1(M, F) = H^2(M, F) = 0$ and $H^2(M, F') = H^2(M, Z)$, we have the isomorphism

$$H^1(M, F'') = H^2(M, Z).$$

Finally we obtain an isomorphism of $P(M, T^1)$ onto $H^2(M, Z)$. It is easy to see that the composed homomorphism

$$P(M, T^1) \to H^2(M, Z) \to H^2(M, Z)_b$$

is nothing but the homomorphism defined in the section 2.

4. From the point of view of the homotopy theory

Consider the exact homotopy sequence of a bundle P:

$$\pi_2(P) \to \pi_2(M) \xrightarrow{\Delta} \pi_1(T^1) \to \pi_1(P).$$

Given a bundle P, we have a homomorphism $\Delta : \pi_2(M) \to \pi_1(T^1)$ ($\cong Z$). We shall prove that, under a certain condition (which is satisfied if M is simply conneced), the above defined mapping $P(M, T^1) \to \mathrm{Hom}(\pi_2(M), Z)$ is an isomorphism.

Let α be an arbitrary element of $\pi_2(M)$ and $f: I \times I \to M$ a differentiable mapping which gives α. Then f maps the boundary of I^2 into a single point x_0. Let $\bar{f}$ be a differentiable mapping of I^2 into P with the following properties:

(1) $$\pi \circ \bar{f}(\tau, t) = f(\tau, t) \text{ for any } \tau, t \in I,$$

(2) $$\bar{f}(\tau, 0) = \bar{f}(0, t) = \bar{f}(1, t) = u_0 \quad \text{for any } \tau,\ t \in I,$$

where u_0 is a point of P such that $\pi(u_0) = x_0$.
Put

$$\Delta f(\tau) = \bar{f}(\tau, 1).$$

Then Δf is a mapping of I into the fibre of P over x_0 and defines $\Delta\alpha \in \pi_1(T^1)$. Let β_0 be the generator of $\pi_1(T^1)$. Then the integer m defined by $\Delta\alpha = m\beta_0$ is the winding number of the closed curve Δf along the circle T^1.

Let h be the natural homomorphism of $\pi_2(M)$ into $H_2(M, Z)$. Consider the following integral of the characteristic class:

$$\int_{h(\alpha)} \Omega^* = \int_f \Omega^* = \int_{\pi(\bar{f})} \Omega^* = \int_{\bar{f}} \pi^*(\Omega^*) = \int_{\bar{f}} \Omega.$$

Applying the Stokes formula, we obtain

$$\int_{\bar{f}} \Omega = \int_{\bar{f}} d\omega = \int_{\partial \bar{f}} \omega = \int_{\sigma_1} \omega + \int_{\sigma_2} \omega - \int_{\sigma_3} \omega - \int_{\sigma_4} \omega$$

where

$$\sigma_1(t) = \bar{f}(0, t) \quad t \in I$$
$$\sigma_2(\tau) = \bar{f}(\tau, 1) \quad \tau \in I$$
$$\sigma_3(t) = \bar{f}(1, t) \quad t \in I$$
$$\sigma_4(\tau) = \bar{f}(\tau, 0) \quad \tau \in I.$$

From the 2nd property of $\bar{f}$, we get immediately

$$\int_{\sigma_1}\omega=\int_{\sigma_3}\omega=\int_{\sigma_4}\omega=0,\qquad \int_{\sigma_2}\omega=\int_{\Delta f}\omega.$$

From (ω.1), it follows that the integral of ω around the fibre of P is equal to 1. Hence the integral of ω over Δf is equal to the winding number m. We have proved the

THEOREM 3. *Let $h:\pi_2(M)\to H_2(M,Z)$ be the natural homomorphism and let m be an integer given by $\Delta\alpha=m\beta_0$, where β_0 is the generator of $\pi_1(T^1)$ and Δ is the boundary operator of the exact homotopy sequence of a bundle P. Then*

$$\int_{h(\alpha)}\Omega^*=m,$$

where Ω^* is the characteristic class of P.

If M is simply connected, then h is an isomrophism of $\pi_2(M)$ onto $H_2(M,Z)$. From the theorems 2 and 3 we obtain the

COROLLARY. *If M is simply connected, then $P(M,T^1)$ is isomorphic to Hom $(\pi_2(M),Z)$. The isomorphism is given by $P\to\Delta$ (the boundary operator of the homotopy sequence of P).*

5. Properties of $P(M,T^1)$

Let f be a mapping of a manifold M' into a manifold M and let $P\in P(M,T^1)$. The induced bundle $f^{-1}(P)$ over M' is defined as follows.

$$f^{-1}(P)=\{(x',u);f(x')=\pi(u)\}.$$

The projection π' and the action of the group T^1 are defined by

$$\pi'(x',u)=x',\ (x'u)s=(x',us)\qquad \text{for } s\in T^1.$$

The following theorem is easy to prove.

THEOREM 4. *Let f be a mapping of M' into M, Then the mapping $P\to f^{-1}(P)$ is a homomorphism of $P(M,T^1)$ into $P(M',T^1)$.*

Let G_m be the cyclic subgroup of T^1 of order m. Since T^1 acts on P on the right, G_m acts on P on the right. The quotient space P/G_m is a principal fibre bundle over M with group T^1/G_m. Since T^1/G_m is isomorphic to T^1, it can be considered as a principal fibre bundle with group T^1. More precisely, the bundle structure in P/G_m is defined as follows.

Let $[u]$ denote the element of P/G_m which is represented by $u\in P$. We define the action of T^1 on P/G_m by

$$[u]s=[us']\qquad \text{for any } s\in T^1 \text{ and } [u]\in P/G_m;$$

where s' is an element of T^1 such that

$$s=s'^m.$$

This defintion is independent from the choice of representative u and m-th root s'. In fact, if g is any element of G_m,

$$[ug]s = [ugs'] = [us'g] = [us'] = [u]s.$$

If $s''^m = s$, then $(s'^{-1}s'')^m$ is the unit, hence $s'^{-1}s''$ is in G_m, Therefore

$$[us''] = [us's'^{-1}s''] = [us'].$$

We shall prove that T^1 acts simply transitively on each fibre of P/G_m. Given two elements $[u]$ and $[u']$ in the same fibre, let s' be an an element of T^1 such that $u' = us'$. Then $[u'] = [u]s'^m$. Suppose $[u]s = [us'] = [u]$, Then s' is in G_m, hence $s = s'^m$ is the unit element.

THEOREM 5. *Let P be a principal fibre bundle over M with group T^1 and let G_m be the cyclic subgroup of T^1 of order m. We consider P/G_m as a principal fibre bundle over M with group T^1 as defined above. Then*

$$P/G_m \cong m \cdot P \ (= P + \ldots\ldots + P, \ m \text{ times}).$$

PROOF. From the definition given in §1, it follows by induction that $m \cdot P$ can be defined directly as follows. Let $\Delta(P \times \ldots\ldots \times P)$ be the set of all points $(u_1, \ldots\ldots u_m) \in P \times \ldots\ldots \times P$ such that $\pi(u_1) = \ldots\ldots = \pi(u_m)$. Two elements $(u_1, \ldots\ldots, u_m)$ and $(u_1s_1, \ldots\ldots, u_ms_m)$ of $\Delta(P \times \ldots\ldots \times P)$ are equivalent if and only if $s_1s_2\ldots\ldots s_m$ is the unit element. The quotient space of $\Delta(P \times \ldots\ldots \times P)$ by this equivalence relation is $m \cdot P$. We denote by $[(u_1, \ldots\ldots, u_m)]$ the equivalence class of $(u_1, \ldots\ldots, u_m)$. The action of T^1 on $m \cdot P$ is given by

$$[(u_1, \ldots\ldots u_m)]s = [(u_1s, u_2, \ldots\ldots u_m)].$$

We shall define a mapping f of P/G_m into $m \cdot P$ by

$$f([u]) = [(u, u, \ldots\ldots u)].$$

First of all, f is independent from the choice of representative u. If $g \in G_m$, then

$$f([ug]) = [(ug, ug, \ldots\ldots, ug)] = [(u, u, \ldots\ldots, u)]$$

because g^m is the unit element.
Let s be any element of T^1 and s' an element such that $s'm = s$. Then

$$\begin{aligned} f([u]s) = f([us']) &= [(us', \ldots\ldots, us')] = [(us, u, \ldots\ldots, u)] \\ &= [(u, \ldots\ldots, u)]s = (f([u]))s. \end{aligned}$$

Therefor f is a bundle isomorphism of P/G_m onto $m \cdot P$. Q. E. D.

COROLLARY. *If P is simply connected and m is an integer greater than* 1, *there does not exist a principal fibre bundle P' such that $P = m \cdot P'$.*

Proof. If such a bundle P$'$ does exist, then $P \cong P'/G_m$. Hence P' is a covering space of P. Since P' is connected, this can happen only in the case where $m = 1$.

6. Applications to bundles with group $U(m)$.

Let P be a principal fibre bundle over an n-dimensional manifold M with group $U(m)$. ($U(m)$ stands for the unitary group in m variables). Since $SU(m) = \{s \in U(m)\,;\ \det s = 1\}$ is a normal subgroup of $U(m)$, $P/SU(m)$ is a principal fibre bundle over M with group $U(m)/SU(m)$ (which is isomorphic to T^1). The Lie algebra of $U(m)$ consists of all skew-hermitian matrices in m variables. Hence a connection in the bundle P is is given by a matrix differential form $\omega = (\omega^i{}_j)_{i,j=1,\dots,m}$ such that

(1) Each $\omega^i{}_j$ is a complex valued linear differential form,

(2) $\omega = (\omega^i{}_j)$ is skew-hermitian,

(3) $\omega(u\bar{s}) = s^{-1}\bar{s}$ for any $\bar{s} \in T_s(U(m)),\ u \in P$,

(4) $\omega(\bar{u}s) = s^{-1}\omega(\bar{u})s$ for any $s \in U(m),\ \bar{u} \in T(P)$.

The structure equation of E. Cartan is given by

$$d\omega = -\frac{1}{2}[\omega, \omega] + \Omega$$

or, more explicitly,

$$d\omega^i_j = -\sum \omega^i{}_a \wedge \omega^a{}_j + \Omega^i{}_j.$$

Consider the trace $\sum \omega^i{}_j$, which is a complex valued linear differential form on P. Then there exists a unique linear differential form ζ on $P/SU(m)$ such that

$$\nu^*(\zeta) = \sum \omega^i{}_i,$$

where ν is the natural projection from P onto $P/SU(m)$. Then ζ defines a connection in the bundle $P/SU(m)$.

REMARK 1. This is a particular case of a general theorem. If P is a principal fibre bundle over M with group G and if H is a closed normal subgroup of G, then P/H is a principal fibre bundle over M with group G/H. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H respectively and let λ be the natural projection of $\mathfrak{g}$ onto $\mathfrak{g}/\mathfrak{h}$. If ω is a $\mathfrak{g}$-valued linear differential form on P defining a connection in P, then there exists a unique $\mathfrak{g}/\mathfrak{h}$-valued linear differential form ζ on P/H such that

$$\nu^*(\zeta) = \lambda \circ \omega,$$

where ν is the natural projection of P onto P/H. And ζ defines a connection in P/H.

The curvature form of the connection in $P/SU(m)$ defined by ζ is $d\zeta$ and there exists a unique 2-form ρ on M such that

$$\pi^*(\rho) = d\zeta,$$

where π is the projection of $P/SU(m)$ onto M. We call ρ the *generalized Ricci curvature form* of the connection in P defined by ω. We call the 2-dimensional *Chern class* the cohomology class of $\rho/2\pi\sqrt{-1}$.

REMARK 2. If P is the bundle of unitary frames over a Hermitian space M and ω defines a connection in P, then the generalized Ricci curvature is nothing but the classical Ricci curvature and the Chern class coincides with the classical one.

The generalized Ricci curvature vanishes identically if and only if the restricted holonomy group of the connection in $P/SU(m)$ contains only the unit element ([1], [10]). The natural homomorphism of $U(m)$ onto $U(m)/SU(m)$

maps the (restricted) holonomy group of the connection in P *onto* the (restricted) holonomy group of the connection in $P/SU(m)$ ([10]). Hence the restricted holonomy group of the connection in P is contained in $SU(m)$, if and only if the restriceted holonomy group of the connection in $P/SU(m)$ is trivial. Thus we have generalized the theorem of Iwamoto [9] and Lichnerowicz [11], [12].

THEOREM 6. *The generalized Ricci curvature of a connection in a principal fibre bundle P over M with group $U(m)$ vanishes identically if and only if the restricted holonomy group is contained in $SU(m)$.*

Now we shall prove the following

THEOREM 7. *The 2-dimensional Chern class of a principal fibre bundle P over M with group $U(m)$ is zero if and only if there exists a connection in P whose restricted holonomy group is contained in $SU(m)$.*

PROOF. The sufficiency is an immediate consequence of the theorem 6. Suppose that the Chern class is zero. Let ω be a connection form on P. Then the Ricci curvature form is cohomologous to zero. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of $U(m)$ and $SU(m)$ respectively and let $\mathfrak{c}$ be the center of $\mathfrak{g}$. Then

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{c}.$$

According to the above decomposition of the Lie algebra $\mathfrak{g}$, we decompose the $\mathfrak{g}$-valued linear differential form ω into the direct sum of an $\mathfrak{h}$-valued form ω_1 and a $\mathfrak{c}$-valued form ω_2:

$$\omega = \omega_1 + \omega_2.$$

Now we shall modify the second component of ω in order to obtain a required connection. Since the characteristic class of $P/SU(m)$ is zero, there exists a connection in $P/SU(m)$ whose restricted holonomy group is trivial (Theorem 2). Let ω' be the form on $P/SU(m)$ which defines such a connection. Then $\nu^*(\omega')$ is a form on P, where ν is the natural projection of P onto $P/SU(m)$. It is easy to see that the form $\omega_1 + \nu^*(\omega')$ defines a required connection in P. Q. E. D.

7. Bundle S^{2n+1} over the complex n-dimensional projective space

Consider the complex $(n+1)$-dimensional Euclidean space C^{n+1}, whose coordinate system is given by $z^0, z^1, \dots, z^n$. The set of points $z = (z^0, z^1, \dots, z^n)$ satisfying

$$|z|^2 = z^0\bar{z}^0 + z^1\bar{z}^1 + \dots + z^n\bar{z}^n = 1$$

forms the $(2n+1)$-dimensional sphere S^{2n+1}. We shall identify two points z and z' of S^{2n+1} if there exists a complex number $e^{2\pi r i}$ such that

$$z' = e^{2\pi r i}z.$$

Then the set of equivalence classes is the complex n-dimensional projective space, which we shall denote by M in this section. The sphere S^{2n+1} is a principal fibre bundle over M with group $\{e^{2\pi ri}; r \in R\}$, which is isomorphic to T^1. We shall study the relationship between this principal fibre bundle and the second cohomology group of M.

It is well known that, if M is the complex n-dimensional projective space,

$$H^2(M, Z) = Z.$$

Furthermore M is simply connected; hence the group of all principal fibre bundle over M with group T^1 is isomorphic to the additive group of integers. Let P be the principal fibre bundle over M with group T^1 which corresponds to $1 \in Z$. Suppose the bundle S^{2n+1} corresponds to an integer k. Since S^{2n+1} is simply connected, we have only two possibilities, namely $k = 1$ or -1. (See the theorem 5 and the corollary).

THEOREM 8. *The group of all principal fibre bundles over the complex n-dimensional projective space M with group T^1 is isomorphic to the additive group of integers Z. The bundle S^{2n+1} over M corresponds to $1 \in Z$ for a proper orientation of M.*

REMARK 1. This theorem has been proved in the case where $n = 1$ by Steenrod [15].

The complex projective space M, considered as a homogeneous space $U(n+1)/U(n) \times U(1)$, admits an invariant Kaehlerian metric, which is unique up to a constant factor. We shall define the canonical Kaehlerian metric on M as follows ([4]),

$$ds^2 = \sum dz^i d\bar{z}^i - (\sum z^i d\bar{z}^i)(\sum \bar{z}^i dz^i).$$

Then the fundamental 2-form of the Kaehlerian space M is given by

$$\Phi = \sum dz^i \wedge d\bar{z}^i.$$

On the other hand, the linear differential form ω on S^{2n+1} defined by

$$\omega = \sum z^i dz^i$$

gives a connection in the bundle S^{2n+1} over M. Its curvature form is therefore given by

$$\Omega = d\omega = \sum d\bar{z}^i \wedge dz^i.$$

It is known that, if ρ is the Ricci curvature form of the canonical Kaehlerian connection on M, then ([4])

$$\rho = (n+1)\Phi.$$

The following theorem follows from the theorem 2, 5, 6 and 8.

THEOREM 9. *Let M be the complex n-dimensional projective space with the canonical Kaehlerian metric. Let P be the bundle of unitary frames over M. Then*

(1) *The principal fibre bundle S^{2n+1} over M has the characteristic class represented by the 2-form $-\Phi$, where Φ is the fundamental 2-form of the Kaehlerian space M.*

(2) *The principal fibre bundle $P/SU(n)$ over M is isomorphic to the bundle*

$-(n+1)S^{2n+1}$, *i.e.*, *$P/SU(n)$ corresponds to the integer* $-(n+1)$.

REMARK 2. In this section, we have identified the group $\{e^{2\pi ri}; r \in R\}$ with the group T^1. Note that $\Phi/2\pi\sqrt{-1}$ and $\rho/2\pi\sqrt{-1}$ are integral cocycles of M.

8. Submanifolds of a Hermitian space

Let M be a complex n-dimensional Hermitian space and M' a complex k-dimensional regular submanifold of M. M' is also a Hermitian space with the metric induced naturally from that of M. Let P be the bundle of unitary frames over M. It is a principal fibre bundle with group $U(n)$. We consider the set P' of all unitary frames $(x; e_1, \ldots, e_n) \in P$ such that

(1) $x \in M'$,

(2) the first k vectors are tangent to M' at x,

(3) the last $(n-k)$ vectors are normal to M' at x.

Then P' is a principal fibre bundle over M' with group $U(k) \times U(n-k)$.

The injection map of $U(k) \times U(n-k)$ into $U(n)$ induces an isomorphism of $U(k) \times U(n-k)/SU(n) \cap (U(k) \times U(n-k))$ onto $U(n)/SU(n)$. Hence the injection map $j: P' \to P|M'$ induces an isomorphism

$$j^*: P'/SU(n) \cap (U(k) \times U(n-k)) \cong (P|M')/SU(n).$$

Let P^T(resp. P_N) be the bundle of tangent (resp. normal) unitary frames over M'. Then P_T (resp. P_N) is a principal fibre bundle with group $U(k)$ (resp. $U(n-k)$). We shall show that

$$P_T/SU(k) + P_N/SU(n-k) \cong P'/SU(n) \cap (U(k) \times U(n-k)).$$

First of all, we note the fact that

$$P_T \cong P'/(1) \times U(n-k), \qquad P_N \cong P'/U(k) \times \{1\}.$$

Let $\Delta(P_T \times P_N)$ be the set of all points $(v, w) \in P_T \times P_N$ such that $\pi_T(v) = \pi_N(w)$, where π_T and π_N are the projections of the bundles P_T and P_N onto M' respectively. A tangent frame v at x and a normal frame w at x give a unique frame u *of* M at x, if $x \in M'$. The mapping

$$\eta: \Delta(P^T \times P_N) \ni (v, w) \to u \in P'$$

induces an isomorphism

$$\eta^*: P_T/SU(k) + P_N/SU(n-k) \cong P'/SU(n) \cap (U(k) \times U(n-k)).$$

In fact, if $s \in U(k)$ and $s' \in U(n-k)$, then

$$\eta(vs, ws') = (u(s \times s'))$$

and $s \times s' \in SU(k) \times SU(n-k) \subset SU(n) \cap (U(k) \times U(n-k))$. Hence the mapping η^* is well defined and it is easy to see that η^* is a bundle map. The composed mapping $\eta^* \circ j^{*-1}$ gives an isomorphism of $(P|M')/SU(\mathrm{n})$ onto $P_T/SU(k) + P_N/SU(n-k)$.

THEOREM 10. *Let M be a complex n-dimensional Hermitian space and M' a complex k-dimensional regular submanifold of M. If P is the bundle of unitary frames over M and P_T and P_N are the bundles of tangent and normal* *frames over M' respectivaly, then*

$$P_T/SU(k) + P_N/SU(n-k) \cong (P|M')/SU(n).$$

Consider the case where $k = n-1$. Then a cross-section in the bundle $P_N = P_N/SU(1)$ is a normal vector field to M' and conversely, a non-singular normal vector field is a cross-section. Since P_N is a principal fibre bundle, it is trivial if and only if it admits a cross-section. Hence

COROLLARY. *Suppose M' is a complex $(n-1)$-dimensional submanifold. Then there exists a non-singular normal vector field to M' if and only if*

$$P_T/SU(n-1) \cong (P|M')/SU(n).$$

9. The case where M is a homogeneous space

H. C. Wang proved in [16] that every simply connected compact homogeneous complex manifold (C-space) is a fibre decomposition space of a certain homogeneous space with torus as the fibre. Following his idea, we shall determine geometrically the group $P(G/K, T^1)$, where G is a simply connected Lie group and K is a connected compact subgroup of G.

Since G is simply connected and K is connected, the homogeneous space $M = G/K$ is simply connected. Since K is compact, K is locally the direct product of the connected component of the center (which is the k-dimesional toroidal group T^k) and a connected semi-simple closed subgroup $H = [K, K]$ (the commutator subgroup of K). It is well known that $P = G/H$ is a principal fibre bundle over $M = G/K$ with group $T^k = K/H$. Let $T^{k-1}_{(i)}$ be the normal subgroup of T^k defined by

$$T^{k-1}_{(i)} = T \times \dots \times T \times \{e\} \times T \times \dots \times T,$$

where e is the unit element of the i-th component of T^k. Then $P_i = P/T^{k-1}_{(i)}$ is a principal fibre bundle over M with group T^1. We shall prove the following

THEOREM 11. *Let G be a connected and simply connected Lie group and K a connected compact subgroup of G. Then the group $P(G/K, T^1)$ is spanned by $P_1, \dots, P_k$ with integer coefficients, i.e.,*

$$P(G/K, T^1) = Z \cdot P_1 \dotplus \dots \dotplus Z \cdot P_k.$$

PROOF. First of all, we note that (i). $\pi_1(G) = 0$, (ii). $\pi_1(H) =$ finite (Theorem of Weyl), (iii). $\pi_2(G) = 0$ (Theorem of E. Cartan) and (iv). $\pi_2(G/H) =$ finite. The last fact follows immediately from the exact homotopy sequence of the bundle G over G/H with fibre H. Consider now the following exact homotopy sequence of the bundle P:

$$0 = \pi_2(T^k) \xrightarrow{p} \pi_2(G/H) \xrightarrow{\Delta} \pi_2(G/K) \to \pi_1(T^k) \to \pi_1(G/H) = 0.$$

Let φ_i be the natural homomorphism of T^k onto $T^1 = T^k/T^{k-1}_{(i)}$ and φ_i^* the induced homomophism of $\pi_1(T^k)$ onto $\pi_1(T^1)$. Put $\Delta_i = \varphi_i^* \circ \Delta$. Then each Δ_i is the boundary operator for the exact homotopy sequence of the bundle P_i and the homomorphisms $\Delta_1, \dots, \Delta_k$ of $\pi_2(G/K)$ into $\pi_1(T^1) = Z$ are linearly independent, *i.e.*, if

$$m_1 \cdot \Delta_1 + \dots + m_k \cdot \Delta_k = 0 \qquad (m_1, \dots, m_k \text{ are integers}),$$

then $m_1 = \dots = m_k = 0$. This follows from the fact that Δ induces an isomorphism of $\pi_2(G/K)/\pi_2(G/H)$ onto $\pi_1(T^k)$. If we prove that $\Delta_1, \dots, \Delta_k$ span Hom $(\pi_2(G/K), Z)$, *i.e.*,

$$Z \cdot \Delta_1 \dotplus \dots \dotplus Z \cdot \Delta_k = \mathrm{Hom}(\pi_2(G/K), Z),$$

then our theorem follows immediately from the corollary to the theorem 3. Let ξ be an arbitrary homomorphism of $\pi_2(G/K)$ into $Z = \pi_1(T^1)$. Evidently ξ maps every element of finite order into zero. Since $\pi_2(G/H)$ is finite, ξ induces a mapping of $\pi_2(G/K)/\pi_2(G/H)$ into $Z = \pi_1(T^1)$. Therefore ξ is a linear combination of $\Delta_1, \dots, \Delta_k$ with integer coefficients[2]. Q. E. D.

10. Application to surfaces

Let P be the bundle of oriented orthogonal frames over a 2-dimensional oriented Riemannian space M. Then P is a principal fibre bundle with group $SO(2)$, which is isomorphic to T^1. The Riemannian connection on M is given by a matrix differential form

$$\begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}.$$

There exists a unique 2-form Ω^* on M such that $d\omega = \pi^*(\Omega^*)$. If we identify $SO(2)$ with T^1 and consider P as a bundle with group T^1, then the characteristic class of P is represented by $\Omega^*/2\pi$. If is well known that if M is compact, then

$$\int_M \Omega^* = 2\pi \cdot \chi(M),$$

where $\chi(M)$ is the Euler characteristic of M.

On the other hand, the Betti part $H^2(M, Z)_b$ of the second cohomology group of M is isomorphic to Z and a closed 2-form Φ which represents $1 \in Z = H^2(M, Z)_b$ satisfies the following equality:

$$\int_M \Phi = 1.$$

Therefore *the homomorphism of $P(M, Z)$ onto $H^2(M, Z)_b$ maps P into $\chi(M) \in Z$.*

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2). The assumption that G is simply connected and K is connected, can be replaced by the assumpton that $\pi_1(G)$ is finite and G/K is simply connected. The finiteness of $\pi_1(G)$ is essential.

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