ON ENTIRE FUNCTIONS OF INFINITE ORDER

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1. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of infinite order and let $M(r)$, $M'(r)$, $M(r, f^{(p)})$, $\mu(r)$, $\nu(r) \equiv \nu(r, f)$ and $\nu(r, f^{(p)})$ have their usual meanings. It is known [3, (80-1)] that

$$\lim_{r\to\infty} \frac{\log \mu(r)}{\nu(r)} = 0. \tag{1. 1}$$

A result better than (1. 1) viz.,

$$\lim_{r\to\infty} \frac{\log M(r)}{\nu(r)} = 0, \tag{1. 2}$$

for every entire function of infinite order has been proved by Shah [4, (113-4)]. Later on Shah and Khanna [5, (47-8)] proved that for an entire function of infinite order

$$\lim_{r\to\infty} \frac{\log \{rM'(r)\}}{\nu(r, f)} = 0, \tag{1. 3}$$

—— a result better than (1. 2) since [6, (116)]

$$rM'(r) > M(r)\,\frac{\log M(r)}{\log r},\ r \geqq r_0 = r_0(f).$$

Clunie [1] has gone still further to prove that if s is any function of ν such that $s(\nu) = o\left(\frac{\nu}{\log \nu}\right)$, then

$$\lim_{r\to\infty} \frac{\log \{r^s M(r, f^{(s)})\}}{\nu(r, f)} = 0. \tag{1. 4}$$

In § 2 of this note I give an alternative proof of (1. 4). In § 3 I prove still another result better than (1. 1).

2. We have

$$r^s M(r, f^{(s)}) \leqq \sum_{n=s}^{\infty} n(n-1)\dots(n-s+1)\,|a_n|\, r^n$$

in the notation of G.Valiron, [7, (30)] for $n \geqq p$

$$\begin{aligned} n(n-1)\dots(n-s+1)\,|a_n|\, r^n &\leqq n(n-1)\dots(n-s+1) e^{-G_n} r^n \\ &\leqq n(n-1)\dots(n-s+1)\mu(r)\left(\frac{r}{R_p}\right)^{n-p+1} \end{aligned}$$

$$r^s M(r, f^{(s)}) \leqq \sum_{n=s}^{p-1} n(n-1)\ldots.(n-s+1)\mu(r)$$

$$+ \sum_{n=p}^{\infty} n(n-1)\ldots.(n-s+1)\mu(r)\left(\frac{r}{R_p}\right)^{n-p+1}$$

$$< \mu(r)p^{s+1} + \mu(r)p^{2s}\left[\frac{r}{R_p - r} + \frac{r^2}{(R_p - r)^2} + \ldots. + \frac{r^{s+1}}{(R_p - r)^{s+1}}\right].$$

Take now

$$p = \nu\left(r + \frac{1}{r\nu^2(r)}\right) + 1,$$

so that

$$R_p - r \geqq \frac{1}{r\nu^2(r)}$$

and we have

$$r^s M(r, f^{(s)}) \leqq \mu(r)p^{s+1} + \mu(r)p^{2s}[r^2\nu^2(r) + \ldots. + \{r\nu(r)\}^{2s+2}]$$

$$< \mu(r)\left\{\nu\left(r + \frac{1}{r\nu^2(r)}\right) r\nu(r)\right\}^{2s+2},$$

$$\log\left\{ r^s M(r, f^{(s)}) \right\} < \log \mu(r) + (2s+2)\left\{\log \nu\left(r + \frac{1}{r\nu^2(r)}\right) + \log r + \log \nu(r)\right\}. \tag{2. 1}$$

Since $f(z)$ is of infinite order

$$\overline{\lim_{r \to \infty}} \frac{\log \nu(r)}{\log r} = \infty.$$

Hence

$$\overline{\lim_{n \to \infty}} \frac{\log \nu(R_n)}{\log R_n} = \infty.$$

There will exist therefore a sequence $n_q = N$ such that

$$\frac{\log \nu(R_n)}{\log R_n} < \frac{\log \nu(R_N)}{\log R_N} \quad \text{for } n = 1, 2, \ldots., N-1.$$

Further, for $R_n \leqq r < R_{n+1}$ and therefore

$$\frac{\log \nu(r)}{\log r} < \frac{\log \nu(R_N)}{\log R_N} \qquad \text{for } R_1 \leqq r < R_N.$$

Given an arbitrarily large K there exists a sequence $r_1, r_2, \ldots.$ such that

$$\frac{\log \nu(r_n)}{\log r_n} > K, \qquad (n = 1, 2, \ldots.).$$

Hence for every sufficiently large R_N

$$\frac{\log \nu(R_N)}{\log R_N} > K.$$

Also for $x = R_N$ we have

$$\log \mu(x, f) = O(1) + \int_{R_1}^{x} \frac{\nu(r, f)}{r}\, dr$$

$$< O(1) + \int_{R_1}^{x} r^{\frac{\log \nu(x,f)}{\log x} - 1} dr$$

$$< O(1) + \frac{\log x}{\log \nu(x, f)} x^{\frac{\log \nu(x,f)}{\log x}}$$

$$= O(1) + \frac{\log x}{\log \nu(x, f)} \nu(x, f)$$

and so for a sufficiently large $x = R_N$

$$\frac{\log \mu(x, f)}{\nu(x, f)} < o(1) + \frac{1}{K}.$$

Now let E denote the sequence of all positive integers $N = n_q$ $(q \geq n')$ such that

$$\text{(i)} \qquad \frac{\log \nu(R_N)}{\log R_N} > K$$

$$\text{(ii)} \qquad \frac{\log \mu(R_N)}{\nu(R_N)} < \frac{1}{K} + o(1).$$

Either [Case A] there is a subsequence of integers l_t say $(t = 1, 2, 3, \ldots.)$ tending to infinity such that

$$R_{N+1} > R_N + \frac{1}{N^2} \qquad (N = l_t) \tag{2. 2}$$

in which case

$$\nu\left(R_N + \frac{1}{R_N \nu^2(R_N)}\right) = \nu(R_N),$$

$$\log \frac{\{R_N^s M(R_N, f^{(s)})\}}{\nu(R_N)} < \frac{\log \mu(R_N)}{\nu(R_N)} + \frac{2s+2}{\nu(R_N)}\left\{2 \log \nu(R_N)\right\} + \frac{2s+2}{\nu(R_N)} \log R_N$$

$$< O\left(\frac{1}{K}\right) + o(1) \qquad \text{for } q \geq n'' > n' \tag{2. 3}$$

since $s = o\left(\frac{\nu}{\log \nu}\right)$, or [Case B] for all large N, say $N > \mathfrak{N}$, of E

$$R_{N+1} \leq R_N + \frac{1}{N^2} \tag{2. 4}$$

in which case either $R_{N+1} = R_N$ and then $N + 1 \subset E$, or $R_{N+1} > R_N$,

$$\frac{\log \nu(R_{N+1})}{\log R_{N+1}} \geq \frac{\log (N+1)}{\log \left(R_N + \frac{1}{N^2}\right)}$$

$$= \frac{\log N + \log\left(1 + \frac{1}{N}\right)}{\log R_N + \log\left(1 + \frac{1}{N^2 R_N}\right)}$$

$$\geqq \frac{\log N + \log\left(1+\frac{1}{N}\right)}{\log R_N + \left(\frac{1}{N^2 R_N}\right)}$$

$$= \frac{\log N}{\log R_N} \frac{1 + \dfrac{\log\left(1+\frac{1}{N}\right)}{\log N}}{1 + \dfrac{1}{N^2 R_N \log R_N}}$$

$$\geqq \frac{\log N}{\log R_N} \qquad \text{for } N \geqq n''',$$

$$> K,$$

$$\frac{\log \mu(R_{N+1})}{\nu(R_{N+1})} \leqq \frac{1}{N+1}\left\{\log \mu(R_N) + \int_{R_N}^{R_{N+1}} \frac{\nu(x)}{x}\, dx\right\}$$

since $\nu(R_{N+1}) \geqq N+1$. Now, from (2. 4)

$$N \log \frac{R_{N+1}}{R_N} < \frac{1}{NR_N} < \frac{\log \mu(R_N)}{N}.$$

Hence

$$\frac{\log \mu(R_{N+1})}{\nu(R_{N+1})} < \frac{\log \mu(R_N)}{N} < \frac{1}{K} + o(1),$$

and so $N+1 \subset E$. Similarly $N+2$, $N+3$, $\ldots\ldots \subset E$. Let $N > \mathfrak{N}$. Then

$$R_{N+p} \leqq R_N + \sum_{n=N}^{N+p-1} n^{-2} < \text{a constant},$$

which leads to a contradiction since R_{N+p} tends to infinity with p. Hence alternative (B) is not possible and (2. 3) holds and the theorem is proved.

REMARKS. (i) It is obvious from our proof that if S and s be any functions of ν such that $s(\nu)$ is an integer and $S(\nu) = O\left(\frac{\nu}{\log \nu}\right)$, $s(\nu) = o\left(\frac{\nu}{\log \nu}\right)$ as $r \to \infty$, then

$$\varlimsup_{r\to\infty} \frac{\log\{r^S M(r, f^{(s)})\}}{\nu(r,f)} = 0.$$

(ii) It is known [2] that for any function of finite or infinite order

$$\nu(r,f) \leqq \nu(r,f') \leqq \ldots\ldots .$$

Hence it follows that for an entire function of infinite order

$$\varlimsup_{r\to\infty} \frac{\log\{r^S M(r, f^{(s)})\}}{\nu(r, f^{(i)})} = 0$$

where S and s are as defined already and i may have any integer value.

3. If $f(z)$ is of infinite order, we have for any (arbitrarily large) positive constant H, $\varlimsup_{x\to\infty} \frac{\nu(x)}{x^H} = \infty$. Hence we can choose a sequence of points $\bar{r}_n$

$(n = 1, 2, 3, \ldots\ldots)$ such that

$$\frac{\nu(x)}{x^H} \leqq \frac{\nu(\bar{r}_n)}{\bar{r}_n^H} \qquad \text{for } a \ (= \text{constant}) \leqq x \leqq \bar{r}_n.$$

Then

$$\frac{\bar{r}_n^m}{\nu(\bar{r}_n)} \int_a^{\bar{r}_n} \frac{\nu(x)}{x^{m+1}}\, dx = \frac{\bar{r}_n^m}{\nu(\bar{r}_n)} \int_a^{\bar{r}_n} \frac{x^{H-m-1}\nu(x)}{x^H}\, dx$$

$$\leqq \frac{1}{\bar{r}^{H-m}} \int_a^{\bar{r}_n} x^{H-m-1}\, dx$$

$$< \frac{1}{H-m}.$$

Hence

$$\lim_{r \to \infty} \frac{r^m}{\nu(r)} \int_a^r \frac{\nu(x)}{x^{m+1}}\, dx = 0 \tag{3. 1}$$

for every real finite m. For a finite positive m (3. 1) is sharper than (1. 1), since

$$r^m \int_a^r \frac{\nu(x)}{x^{m+1}}\, dr \geqq \frac{r^m}{r^m} \int_a^r \frac{\nu(x)}{x}\, dx$$

$$= \log \mu(r) - O(1).$$

References

[1] J. Clunie, Note on integral functions of infinite order, Quart. J. of Math. (Oxford Series), 6(1955), 88-90.

[2] Q. I. Rahman, On the derivatives of integral functions, Math. Student, (to appear).

[3] S. M. Shah, The maximum term of an entire series, Math. Student, 10 (1942), 80-82.

[4] S. M. Shah, The maximum term of an entire series (IV), Quart. J. of Math. (Oxford Series), 1(1950), 112-116.

[5] S. M. Shah and Girja Khanna, On entire functions of infinite order, Math. Student, 21(1953), 47-48.

[6] T. Vijayaraghavan, On derivatives of integral functions, J. London Math. Soc., 10(1935), 116-117.

[7] G. Valiron, Lectures on the general theory of integral functions, Toulouse (1923).

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