ON THE CESÁRO SUMMABILITY OF FOURIER SERIES (III)

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1. Let \( \varphi(t) \) be an even integrable function with period \( 2\pi \) and let

(1.1) \[ \varphi(t) \sim \sum_{n=1}^{\infty} a_n \cos nt, \quad a_0 = 0, \]

(1.2) \[ \varphi_a(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} \varphi(u)(u - t)^{\alpha-1} du \quad (\alpha > 0), \]

and \( S_n^\beta \) be the \( \beta \)-th Cesàro sum of the Fourier series of \( \varphi(t) \) at \( t = 0 \), that is,

(1.3) \[ S_n^\beta = \sum_{\nu=0}^{n} A_{\nu}^\beta \quad (\beta > -1). \]

C. T. Loo [7] proved the following theorem.

**THEOREM A.** If \( \alpha > 0 \) and

(1.4) \[ \sup_n S_n^\beta = o(n^\alpha / \log n) \quad \text{as } n \to \infty, \]

then

(1.5) \[ \varphi_{1+\alpha}(t) = o(t^{1+\alpha}). \]

This theorem is the converse type of Izumi-Sunouchi's theorem [5].

Recently, we proved the following theorem [6]:

**THEOREM B.** If

(1.6) \[ \varphi_\beta(t) = o\left( t^\beta \left( \frac{\log \frac{1}{t}}{\log \frac{1}{t}} \right)^{\gamma} \right) \quad (\beta, \gamma > 0) \quad \text{as } t \to 0, \]

and

(1.7) \[ \int_{0}^{t} \left| d \left( \frac{t \varphi(t)}{\log \frac{1}{t}} \right) \right| = O(t) \quad (\Delta > 0, \ 0 < t \leq \eta), \]

then

(1.8) \[ S_n^\alpha = o(n^\alpha), \]

where

(1.9) \[ \alpha = (\Delta \gamma \beta - 1)/(1 + \Delta \gamma). \]

In the present note we prove a theorem which is the converse type of theorem B.

**THEOREM.** If

(1.10) \[ a_n > -K(\log n)^\alpha / n \quad (\alpha > 0) \quad \text{as } n \to \infty \]
where $K$ is a constant and

\[(1.6)\quad S_n^\alpha = o(n^\beta/(\log n)^\gamma) \quad (\beta, \gamma > 0) \quad \text{as} \quad n \to \infty,
\]

then

\[
\varphi_\mu(t) = o(t^\mu) \quad \text{as} \quad t \to 0,
\]

where

\[
\mu = \alpha(1 + \beta)/(\gamma + \alpha).
\]

This theorem is also related to G. Sunouchi’s theorem [8].

2. For the proof of theorem, we use the Bessel summability. Let $J_\mu(t)$ be the Bessel function of order $\mu$ and put

\[
(2.1) \quad \alpha_\mu(t) = 2^\mu \Gamma(\mu + 1) J_\mu(t)/t^\mu,
\]

then

\[
(2.2) \quad \Delta^p \alpha_\mu(nt) = O(t^p) \quad \text{for} \quad 0 < nt \leq 1
\]

and

\[
(2.3) \quad \Delta^p \alpha_\mu(nt) = O(t^{\rho+1/2-n-1/2}) \quad \text{for} \quad nt > 1,
\]

where $\Delta^p$ ($p = 0, 1, 2, \ldots$) are the repeated differences of $p$-times. This properties are shown in the theorem 2 of K. Chandrasekharan and O. Szasz [3].

If the series

\[
(2.4) \quad \sum_{n=0}^\infty a_n (\alpha_\mu(\lambda nt))^k = \varphi_\mu^k(t)
\]

converges for some interval $0 < t < t_0$ and

\[
(2.5) \quad \varphi_\mu^k(t) = o(1) \quad \text{as} \quad t \to 0,
\]

then the series $\sum a_n$ is said to be summable $(J_n, k, \lambda)$ to 0.

**Lemma 1.** If the series $\sum a_n$ is summable $(J_n, 1, n)$, that is $J_\mu$-summable, to $s$, then $t^{-(\mu+1/2)} \varphi_\mu^{1/2}(t)$ tends to $s$ as $t \to 0$, and vice versa, where $\mu > -\frac{1}{2}$.

This is given in [3].

Let $h$ denote a positive integer, and we write

\[
\Delta_{-h} S_n = \Delta_{h} S_n = s_{n+h} - s_n
\]

and $\Delta^p = \Delta_{h} \Delta^{p-1} s_n$ for $p = 1, 2, \ldots$, where $\Delta_{h}^0 s_n = s_n$. Similarly, we write

\[
\Delta_{-h} S_n = \Delta_{h}^1 s_n = s_n - s_{n-h}
\]

and $\Delta^p_{-h} = \Delta_{-h} \Delta^{p-1}_{-h}$, where $\Delta^0_{-h} s_n = s_n$.

**Lemma 2.** If $h$ and $p$ are non-negative integers and $0 < \delta \leq 1$, then

\[
(2.6) \quad \frac{\Gamma(h + \delta)}{\Gamma(h)} \varphi_{h^p s_n}
\]
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\[ \Delta_h^{\nu+\delta} S_h^{\nu+\delta} = \delta \sum_{\nu_0=1}^{h} \frac{\Gamma(h - \nu_0 + \delta)}{\Gamma(h - \nu_0 + 1)} \sum_{\nu_1=1}^{h} \ldots \sum_{\nu_p=1}^{h} (s_{\nu_0+\ldots+\nu_p} - s_n) \]

and, if \( n > (p+1)h \),

\[ \frac{\Gamma(h + \delta)}{\Gamma(h)} h^p s_n \]

This is proved in Bosanquet's paper [2].

**Lemma 3.** If \( 0 < m < n, 0 < \delta \leq 1 \), then

\[ \sum_{\nu=0}^{m} A_{\nu}^{\delta-1} s_{\nu} \leq \max_{\nu \in \mathbb{N}} \vert S_{\nu}^{\delta} \vert. \]

This is proved by Bosanquet [1].

**Lemma 4.** If \( r > 0, \beta > 0, h > 0 \) and

\[ S_n = o(n^{\beta} W(n)) \text{ as } n \to \infty \text{ then } \Delta_n^{\beta} S_n = o(n^{\beta} W(n)) \text{ and } \Delta_n^{\alpha} S_n = o(n^{\beta} W(n)) \text{ as } n \to \infty, \]

where \( W(n) \) is a positive non-decreasing function of \( n \).

Using Lemma 3, the proof is done analogous by that of Lemma 7 of Bosanquet's paper [2].

**Lemma 5.** If \( V(n) \) and \( W(n) \) are positive increasing for \( n \), and

\[ s_n = S_n^0 = O(V(n)), S_n^r = o(W(n)) \quad (r > 0), \]

then

\[ S_n^r = o\left((V(n))^{1-\frac{\alpha}{r}} \left(W(n)\right)^{\frac{\alpha}{r}}\right) \quad (0 < \alpha < r). \]

This is the Dixon and Ferrer theorem [4].

**Lemma 6.** Let \( V(n) \) and \( W(n) \) are positive and satisfy the following conditions:

(i) there exists a real number \( d > 0 \) such that \( n^d V(n) \) is non-decreasing;

(ii) \( W(n) \) is non-decreasing;

(iii) \( W(n) = O(V(n)) \) as \( n \to \infty \).

And if

1. \( s_n = O(n^b V(n)) \) and \( S_n^r = o(n^b W(n)) \) as \( n \to \infty \),

where \( a + b \geq c > -1 \), then

\[ S_n^r = o\left(n^{(a-c)/a+b+ca} (V(n))^{1-\frac{a'}{a}} (W(n))^{\frac{a'}{a}}\right), \]

**Proof.** We can prove this easily by the simple modification of Bosanquet's paper, but for the sake of completeness we prove the lemma.

Suppose that \( b \) is any real number and assume the theorem with \( b, c \) replaced by \( b + 1, c + 1 \). Then, by (ii) of (2.8), \( c > -1 \) and (2),
If we put \( T_n = \sum_{v=0}^{n} A_{n-v}^{-1} \nu s_n \), then we get

\[
T_n^* = (a + n) S_n^* - (a + 1) S_{n+1}^* = o(n^2 W(n)).
\]

From (2.10) and \( n s_n = O(n^{a+1} V(n)) \), the hypotheses of the theorem are satisfied, with \( s_n \) replaced by \( n s_n \), \( b \) by \( a + 1 \), \( c \) by \( c + 1 \). It follows from the case assumed that

\[
T_n^* = o\left(n^{(a+1)(a-\alpha')/a}(V(n))^{-\alpha'/a}(W(n))^{\alpha'/a}\right) \quad (0 < a' < a).
\]

Now suppose that \( a' \geq 0 \) and \( a - 1 \leq a' < a \), then by (2)

\[
S_n^* = \sum_{v=0}^{n} A_{n-v}^{-1} S_v = o\left(n^{a+1} W(n)\right)
\]

and so by (2.10)

\[
S_n^* = o\left(n^{(a-a')/a+\alpha'\epsilon/a}(V(n))^{-\alpha'/a}(W(n))^{\alpha'/a}\right).
\]

Since \( a + b > c > -1 \) and (iii) of (2.8), we obtain

\[
S_n^* = o\left(n^{(a-a')\epsilon/a+\alpha'\epsilon/a}(V(n))^{-\alpha'/a}(W(n))^{\alpha'/a}\right).
\]

Thus, if \( 0 < a \leq 1 \), the result follows by induction from Lemma 5. If \( a > 1 \), we suppose \( 0 \leq a' < a - 1 \), and assume the result with \( a' \) replaced by \( a' + 1 \). Then

\[
S_n^* = (a' + 1 + n)^{-1}\{T_{n-1}^* + (a' + 1) S_{n-1}^*\}
\]

\[
= o\left(n^{(a-a')\epsilon/a+\alpha'\epsilon/a}(V(n))^{-\alpha'/a}(W(n))^{\alpha'/a}\right).
\]

Thus our result may be proved by induction.

**Lemma 7.** If (2.8),

\[
s_{n+m} - s_n = -Kn^2 V(n) m \quad (K > 0),
\]

and

\[
S_n^* = o(n^a W(n)) \quad \text{as } n \to \infty,
\]

where \( a > 0 \) and \(-1 < c \leq a + b + 1\), we have

\[
S_n^* = o\left(n^{(a-a')\epsilon/a+\alpha'\epsilon/a}(V(n))^{-\alpha'/a}(W(n))^{\alpha'/a}\right)
\]

for \( 0 \leq a' \leq a \).

**Proof.** First suppose that \( b \geq 0 \), \( c > 0 \). Let a small positive \( \epsilon \) be given, and let \( a = p + \delta \), where \( 0 < \delta \leq 1 \) and \( p \) is a non-negative integer. Then, for all sufficiently large \( n \), by (2.6)

\[
\frac{1}{\Gamma(h + \delta)} h^n s_n = \Delta_{s}^{p+\delta} S_n^{p+\delta} - \delta \sum_{v=1}^{n} \frac{1}{\Gamma(h - v_0 + 1)} \sum_{v_1=1}^{h} \ldots \sum_{v_p=1}^{h} (s_{n+v_0+\ldots+v_p} - s_n)
\]
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\[ < \varepsilon^{p+q+1} n^q W(n) + K \sum_{v=1}^{h} \frac{\Gamma(h - v + 1)}{\Gamma(h - v + \delta)} \sum_{v=1}^{h} \sum_{v=1}^{h} n^p V(n) (v_0 + \ldots + v_p) \]

\[ < \varepsilon^{p+q+1} n^q W(n) + (p + 1) K h^{p+q+1} n^q V(n), \]

provided that \((p + 1) h < H n\) and \(K > K'\). Also, by (2.7), for all sufficiently large \(n\),

\[ \frac{\Gamma(h + \delta)}{\Gamma(h)} h^p V_n > -\varepsilon^{p+q+1} n^q W(n) - (p + 1) K h^{p+q+1} n^q V(n), \]

for \((p + 1) h < (1 + H)^{-1} H n\).

Taking \(h = \varepsilon(n^q W(n)/n^q V(n))^{1/(p+q+1)}\) we get, for sufficiently large \(n\),

\[ |S_n| < (1 + (p + 1) K') \varepsilon \left( n^{(p+q)(p+q+1)/(p+q+1)} W(n) \right)^{1 - \frac{p+q+1}{p+q+1}} \]

that is,

\[ (2.14) \quad S_n = o \left( \frac{n^{(p+q+1)} (V(n))^{1 - \frac{1}{p+q+1}} (W(n))^{1/(p+q+1)}} {n^{1/(p+q+1)}} \right) \quad \text{as } n \to \infty. \]

From (i) and (ii) of (2.8), there exists a number \(a' > 0\) such that

\[ n''(V(n))^{1/(p+q+1)} = n'(V(n)/W(n))^{1/(p+q+1)} \]

is non-decreasing for sufficiently large \(n\). Hence, using Lemma 7, we have

\[ S_n' = o(n^{(a-a')/(p+q+1)/(p+q+1)} \left( V(n) \right)^{1 - \frac{a'}{p+q+1}} (W(n))^{1 - \frac{a'}{p+q+1}}), \]

for \(0 < a' < a\), by (2.12) and (2.14).

The rest of the proof is analogous to that of Lemma 7.

3. Proof of the Theorem. By lemma 1, it is sufficient to prove that

the series \(\sum a_n\) is \(J_{-\frac{1}{2}}\)-summable to 0, where \(\mu = \alpha(\beta + 1)/(\gamma + \alpha)\).

First we shall prove that

\[ (3.1) \quad \sum_{n=1}^\infty \frac{|a_n|}{n} = O((\log n)^{\mu}/n). \]

Under the assumption (1.5)

\[ |a_n| - a_n < 2K (\log n)^{\mu}, \]

thus we have

\[ \sum_{n=1}^\infty (|a_n| - a_n) = O((\log n)^{\mu}). \]

On the other hand, since we may put in lemma 2 \(V(n) = n'(\log n)^{\mu}\) and \(W(n) = n'/(\log n)^{\mu}\), where \(\varepsilon\) is any positive number, by (1.5) and (1.6),

\[ S_n = o \left( n^{\mu} (\log n)^{(\beta-h)(\beta+1)/(\beta+1)} \right), \quad \text{for } h = 0, 1, 2, \ldots, k - 1. \]

Especially,
\[ S_n^0 = s_n = O((\log n)^{(\beta \alpha + \gamma)/(\beta + 1)}) = O((\log n)^\alpha), \]

for
\[ \alpha - (\beta \alpha + \gamma)/(\beta + 1) = (\alpha + \gamma)/(\beta + 1) > 0. \]

Thus, we have
\[ \sum_{y=n}^{2n} |a_r| = \sum_{y=n}^{2n} (|a_r| - a_r) + s_{2n} - s_{n-1} = O((\log n)^\alpha). \]

Hence
\[ \sum_{y=n}^{2n} v^{-1}|a_r| \leq \sum_{y=n}^{2n} |a_r| = O((\log n)^\alpha/n), \]

\[ \sum_{y=n+1}^{2n} v^{-1}|a_r| = O((\log 2^y)^\alpha/2^y), \]

and
\[ \sum_{y=1}^{2^k} v^{-1}|a_r| = O\left\{ \sum_{k=0}^{2^k} \frac{(\log 2^k)^\alpha}{2^k} \right\} = O(1). \]

Consequently
\[ \sum_{y=n}^{\infty} v^{-1}|a_r| = \sum_{y=n}^{\infty} \sum_{k=0}^{2^k-1} v^{-1}|a_r| = O\left\{ \sum_{k=0}^{\infty} (\log n 2^{k\alpha}/n 2^k) \right\} \]
\[ = O\left\{ \frac{(\log n)^\alpha}{n} \sum_{k=0}^{\infty} (\log 2^k)^\alpha/2^k \right\} = O((\log n)^\alpha/n) \]

which is the desired inequality (3.1).

Let
\[ (3.3) \quad \psi_1(t) = \sum_{n=0}^{\infty} a_n \alpha_{\mu-\frac{1}{2}}(nt) = \left( \sum_{n=0}^{\infty} + \sum_{n=v+1}^{\infty} \right) a_n \alpha_{\mu-\frac{1}{2}}(nt) = \psi_1(t) + \psi_2(t), \]
say, where \( v \) is to be chosen presently.

Using the inequality (3.1)
\[ (3.4) \quad \psi_2(t) = O\left\{ \sum_{n=v+1}^{\infty} n^{-1} |a_n|(nt)^{-\mu} n \right\} = O\left( t^{-\mu l-\mu} \nu^{-1}(\log \nu)^\alpha \right). \]

This shows that the series \( \sum_{n=0}^{\infty} a_n \alpha_{\mu-\frac{1}{2}}(nt) \) converges for fixed \( t > 0 \).

For a given positive number \( C \), we put
\[ (3.5) \quad v = \rho(t) = \left[ C\left( \frac{1}{t} \right)^{\sigma/\mu} t^{-1} \right]. \]

Then from (3.4), we obtain
\[ \psi_2(t) = O\left\{ C^{-\mu} \left( \log \frac{1}{t} \right)^{-\sigma/\mu} \left( \log \frac{C\left( \frac{1}{t} \right)^{\sigma/\mu}}{t} \right)^{\alpha} \right\} = O(C^{-\mu}). \]
Thus if we take \( C \) sufficiently large, we get
\[
\psi(t) = o(1) \quad \text{as } t \to 0.
\]

Now there is an integer \( k \geq 1 \) such that \( k - 1 < \beta < k \). We suppose that \( k - 1 < \beta < k \), for the case \( \beta = k \) can be easily deduced by the following argument.

Now
\[
S_k^n = \sum_{n=0}^{n} A_{n,v}^{-\beta-1} S_0^v = o\left\{ \sum_{n=2}^{n}(n-v)^{-\beta-1} v^\gamma \right\} = o(v^\gamma \log v)^{-\gamma}. \tag{3.7}
\]

Concerning \( \psi(t) \), by Abel’5 lemma on partial summation \( k \)-times we have
\[
\psi(t) = \sum_{n=0}^{n} \sum_{n=0}^{n} A_{n,v}^{-\beta-1} S_0^v + \sum_{n=0}^{n} A_{n,v}^{-\beta-1} S_0^v = o(v^\gamma \log v)^{-\gamma} \quad \text{say.}
\]

Using (2.3), (3.5) and (3.7), we have
\[
\psi(t) = S_k^n \Delta^k \alpha_{\beta-1}^\gamma \psi(t) = o(v^\gamma \log v)^{-\gamma} \quad \text{as } t \to 0.
\]

Also, by (2.3), (3.5) and (3.2)
\[
S_k^n \Delta^k \alpha_{\beta-1}^\gamma \psi(t) = o(v^\gamma \log v)^{-\gamma} \quad \text{say.}
\]

Since the exponent of \( \log \frac{1}{t} \) is
\[
(h - \mu) \mu + \beta \alpha - \gamma - h(\alpha + \gamma) = \frac{h \alpha}{\mu} - \alpha + \beta \alpha - \gamma - \frac{\mu}{\beta + 1} < 0,
\]

we have
\[
\psi(t) = o(1) \quad \text{as } t \to 0.
\]

Concerning \( \psi(t) \), we split up four parts
\[
\psi(t) = \sum_{n=0}^{n} \psi(t) + \psi(t) + \psi(t) + \psi(t) = \psi(t) + \psi(t) + \psi(t) + \psi(t), \quad \text{say.}
\]
\[
= O\left\{ \sum_{m=0}^{[1/\ell]} \sum_{n=m+1}^{m+1} (n - m)^{k-\beta-1} t^{k+1} \right\} = o\left\{ t^{k+1} \sum_{m=2}^{[1/\ell]} m^\beta (\log m)^{-\gamma} t^{-(k-\beta)} \right\}
\]
(3.11) \[= o\left( \left( \frac{\log \frac{1}{t}}{t} \right)^{-\gamma} \right) = o(1) \quad \text{as } t \to 0.\]

From (1.6) and (2.3), we also get
\[
\psi(t) = \sum_{m=1/\ell+1} \sum_{n=m}^{m+1} A^{k-\beta-1} m^{\alpha-1/2} (nt)
\]
\[
= o\left\{ \sum_{m=1/\ell+1} \sum_{n=m}^{m+1} m^\beta (\log m)^{-\gamma} (n - m)^{k-\beta-1} t^{k+1} n^{-\gamma} \right\}
\]
\[
= o\left\{ \sum_{m=1/\ell+1} \sum_{n=m}^{m+1} m^\beta (\log m)^{-\gamma} m^{-\gamma} t^{k+1} n^{-\gamma} \right\}
\]
\[
= o\left\{ t^{k+1} \sum_{m=1/\ell+1} m^{\beta+1/2} (\log m)^{-\gamma} t^{-(k-\beta)} \right\} = o\left( t^{\beta+1} (\log m)^{-\gamma} \right) \quad \text{as } t \to 0,
\]
for \( \beta + 1 - \mu = \frac{\gamma (\beta + 1)}{\gamma + \delta} > 0. \) Hence
(3.12) \[\psi(t) = o\left( t^{\beta+1} t^{-(3+1-\mu)} \left( \frac{\log \frac{1}{t}}{t} \right)^\mu (\beta+1-\mu)^{-\gamma} \right) = o(1) \quad \text{as } t \to 0,
\]
by (3.5), for \( \frac{\alpha}{\mu} - (\beta + 1 - \mu) = \gamma = 0. \)

For the estimation of \( \psi(t) \), if we use partial summation in the inner series, then
\[
\psi(t) = \psi(t) + \psi''(t) + \psi''''(t), \text{ say.}
\]
\[
\psi(t) = O\left\{ \sum_{m=0}^{\ell-1/\ell} \sum_{n=m+1}^{m+1} (n - m)^{k-\beta-2} n^{-\mu} t^{k-\mu} \right\}
\]
\[
= o\left\{ \sum_{m=0}^{\ell-1/\ell} m^{\beta} (\log m)^{-\gamma} m^{-\gamma} t^{-(k-\beta)} \right\} = o(1) \quad \text{as } t \to 0.
\]
(3.14)
\[ \psi_{N}^{(r)}(t) = o \left\{ \sum_{m=2}^{\nu-k-l} m^{\alpha} (\log m)^{-\gamma} t^{-(k-\beta-1)} (m + [t^{-1}] + 1)^{-\mu} t^{k-\mu} \right\} \]
\[ = o \{ t^{\beta+1-\mu} (\log t)^{-\gamma} \rho^{2+1-\mu} \} \]
\[ = o \left( \log \frac{1}{t} \right)^{\alpha} (k-\mu - \gamma) \]
\[ = o(1) \text{ as } t \to 0. \]

(3.15)

\[ \psi_{N}^{(r)}(t) = o \left\{ \sum_{m=2}^{\nu-k-l} m^{\alpha} (\log m)^{-\gamma} (\nu - k - 1 - m)^{k-\beta-1} (\nu - k)^{-\mu} t^{k-\mu} \right\} \]
\[ = o \{ t^{\beta-\mu} \rho^{\mu+k-\beta-1} (\log \rho)^{-\gamma} \rho^{\beta+1} \} \]
\[ = o \left( \log \frac{1}{t} \right)^{\alpha} (k-\mu - \gamma) \]
\[ = o(1) \text{ as } t \to 0. \]

(3.16)

Thus, from (3.14), (3.15), (3.16) and (3.13)

(3.17) \[ \psi_{N}(t) = o(1) \text{ as } t \to 0. \]

Moreover, we get

\[ \psi_{N}(t) = - \sum_{m=\nu-k-l}^{\nu-k-1} S_{m}^{\delta} \sum_{n=\nu-k}^{m+1/2} A_{n}^{\alpha-1} \Delta^{\alpha+1} \alpha_{\mu-1/2} (nt) \]
\[ = O \left\{ \sum_{m=\nu-k-l}^{\nu-k-1} S_{m}^{\delta} \sum_{n=\nu-k}^{m+1/2} (n - m)^{k-\beta-1} n^{-\mu} t^{k+1-\mu} \right\} \]
\[ = o \left\{ t^{k+1-\mu} \sum_{n=\nu-k}^{\nu-k-1} m^{\alpha} (\log m)^{-\gamma} t^{-(k-\beta)} \right\} \]
\[ = o \{ t^{\beta+1-\mu} (\log t)^{-\gamma} \rho^{\beta+1-\mu} \} \]
\[ = o \left( \log \frac{1}{t} \right)^{\alpha} (k-\mu - \gamma) \]
\[ = o(1) \text{ as } t \to 0. \]

(3.18)

Summing up (3.10), (3.11), (3.12), (3.17) and (3.18), we obtain

(3.19) \[ \psi_{N}(t) = o(1) \text{ as } t \to 0. \]

Thus, from (3.1), (3.6), (3.8), (3.9) and (3.19), we have

\[ \psi_{N}(t) = o(1) \text{ as } t \to 0, \]

which is the required.

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