ON THE CESÀRO SUMMABILITY OF FOURIER SERIES (III)

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1. Let $\varphi(t)$ be an even integrable function with period 2π and let

$$\varphi(t) \sim \sum_{n=1}^{\infty} a_n \cos nt, \qquad a_0 = 0, \tag{1.1}$$

$$\varphi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \varphi(u)(u-t)^{\alpha-1}\, du \quad (\alpha > 0), \tag{1.2}$$

and S_n^β be the β-th Cesàro sum of the Fourier series of $\varphi(t)$ at $t = 0$, that is,

$$S_n^\beta = \sum_{\nu=0}^{n} A_{n-\nu}^\beta a_\nu \qquad (\beta > -1). \tag{1.3}$$

C. T. Loo [7] proved the following theorem.

THEOREM A. *If $\alpha > 0$ and*

$$S_n^\alpha = o(n^\alpha/\log n) \qquad \text{as } n \to \infty, \tag{1.4}$$

then

$$\varphi_{1+\alpha}(t) = o(t^{1+\alpha}).$$

This theorem is the converse type of Izumi-Sunouchi's theorem [5]. Recently, we proved the following theorem [6]:

THEOREM B. *If*

$$\varphi_\beta(t) = o\left\{t^\beta \Big/ \left(\log \frac{1}{t}\right)^{\frac{1}{\gamma}}\right\} \qquad (\beta, \gamma > 0) \qquad \text{as } t \to 0,$$

and

$$\int_0^t \left| d\left\{ \frac{t\varphi(t)}{\left(\log \frac{1}{t}\right)^\Delta} \right\} \right| = O(t) \qquad (\Delta > 0,\ 0 < t \leqq \eta),$$

then

$$S_n^\alpha = o(n^\alpha),$$

where $\qquad \alpha = (\Delta\gamma\beta - 1)/(1 + \Delta\gamma).$

In the present note we prove a theorem which is the converse type of theorem B.

THEOREM. *If*

$$a_n > -K(\log n)^\alpha/n \qquad (\alpha > 0) \qquad \text{as } n \to \infty \tag{1.5}$$

where K is a constant and

(1.6) $$S_n^\beta = o\{n^\beta/(\log n)^\gamma\} \qquad (\beta, \gamma > 0) \quad \text{as} \quad n \to \infty,$$

then

$$\varphi_\mu(t) = o(t^\mu) \qquad \text{as} \quad t \to 0,$$

where

$$\mu = \alpha(1+\beta)/(\gamma+\alpha).$$

This theorem is also related to G. Sunouchi's theorem [8].

2. For the proof of theorem, we use the Bessel summability. Let $J_\mu(t)$ be the Bessel function of order μ and put

(2.1) $$\alpha_\mu(t) = 2^\mu \Gamma(\mu+1) J_\mu(t)/t^\mu,$$

then

(2.2) $$\Delta^\rho \alpha_\mu(nt) = O(t^\rho) \qquad \text{for} \quad 0 < nt \leqq 1$$

and

(2.3) $$\Delta^\rho \alpha_\mu(nt) = O(t^{\rho-\mu-1/2} n^{-\mu-1/2}) \qquad \text{for } nt > 1,$$

where Δ^ρ $(\rho = 0, 1, 2, \ldots.)$ are the repeated differences of ρ-times. This properties are shown in the theorem 2 of K. Chandrasekharan and O. Szász [3].

If the series

(2.4) $$\sum_{n=0}^{\infty} a_n \{\alpha_\mu(\lambda_n t)\}^k = \varphi_\mu^k(t)$$

converges for some interval $0 < t < t_0$ and

(2.5) $$\varphi_\mu^k(t) = o(1) \qquad \text{as } t \to 0,$$

then the series $\sum a_n$ is said to be summable (J_μ, k, λ) to 0.

LEMMA 1. *If the series* $\sum a_n$ *is summable* $(J_\mu, 1, n)$, *that is* J_μ-*summable, to* s, *then* $t^{-(\mu+1/2)} \varphi_{\mu+1/2}(t)$ *tends to* s *as* $t \to 0$, *and vice versa, where* $\mu > -\dfrac{1}{2}$.

This is given in [3].

Let h denote a positive integer, and we write

$$\Delta_{-h} s_n = \Delta_h^1 s_n = s_{n+h} - s_n$$

and $\Delta_h^p = \Delta_h \Delta_h^{p-1}$ for $p = 1, 2, \ldots.$, where $\Delta_h^0 s_n = s_n$. Similarly! we write

$$\Delta_{-h} s_n = \Delta_{-h}^1 s_n = s_n - s_{n-h}$$

and $\Delta_{-h}^p = \Delta_{-h} \Delta_{-h}^{p-1}$, where $\Delta_{-h}^0 s_n = s_n$.

LEMMA 2. *If* h *and* p *are non-negative intgers and* $0 < \delta \leqq 1$, *then*

(2.6) $$\frac{\Gamma(h+\delta)}{\Gamma(h)} h^p s_n$$

$$= \Delta_h^{p+\delta} S_n^{p+\delta} - \delta \sum_{\nu_0=1}^{h} \frac{\Gamma(h-\nu_0+\delta)}{\Gamma(h-\nu_0+1)} \sum_{\nu_1=1}^{h} \cdots \sum_{\nu_p=1}^{h} (s_{n+\nu_0+\ldots+\nu_p} - s_n)$$

and, if $n > (p+1)h$,

$$\frac{\Gamma(h+\delta)}{\Gamma(h)} h^p s_n \tag{2.7}$$

$$= \Delta_h^{p+\delta} S_n^{p+\delta} + \delta \sum_{\nu_0=1}^{h} \frac{\Gamma(h-\nu_0+\delta)}{\Gamma(h-\nu_0+1)} \sum_{\nu_1=1}^{h} \cdots \sum_{\nu_p=1}^{h} (s_n - s_{n+p+1-\nu_0-\ldots-\nu_p}).$$

This is proved in Bosanquet's paper [2].

LEMMA 3. *If* $0 < m < n, 0 < \delta \leqq 1$, *then*

$$\left|\sum_{\nu=0}^{m} A_{n-\nu}^{\delta-1} s_\nu\right| \leqq \max_{0\leqq\mu\leqq n} |S_\mu^\delta|.$$

This is proved by Bosanquet [1].

LEMMA 4. *If* $r > 0, \beta > 0$, $h > 0$ *and* $S_n^r = o(n^\beta W(n))$ *as* $n \to \infty$, *then* $\Delta_h^r S_n^r = o(n^\beta W(n))$ *and* $\Delta_{-h}^r S_n^r = o(n^\beta W(n))$ *as* $n \to \infty$, *where* $W(n)$ *is positive non-decreasing function of* n.

Using Lemma 3, the proof is done analogous by that of Lemma 7 of Bosanquet's paper [2].

LEMMA 5. *If* $V(n)$ *and* $W(n)$ *are positive increasing for* n, *and*

$$s_n = S_n^0 = O(V(n)), \quad S_n^r = o(W(n)) \qquad (r > 0),$$

then

$$S_n^\alpha = o\left((V(n))^{1-\frac{\alpha}{r}} (W(n))^{\frac{\alpha}{r}}\right) \qquad (0 < \alpha < r).$$

This is the Dixon and Ferrer theorem [4].

LEMMA 6. *Let* $V(n)$ *and* $W(n)$ *are positive and satisfy the following conditions*:

(2.8)
(i) *there exists a real number* $d > 0$ *such that* $n^d V(n)$ *is non-decreasing*;
(ii) $W(n)$ *is non-decreasing*;
(iii) $W(n) = O(V(n))$ *as* $n \to \infty$.

And if

$$(1)\ s_n = O(n^b V(n)) \text{ and } (2)\ S_n^a = o(n^c W(n)) \qquad \text{as } n \to \infty,$$

where $a + b \geqq c > -1$, *then*

$$S_n^{a'} = o\left(n^{(a-a')b/a + a'c/a} (V(n))^{1-\frac{a'}{a}} (W(n))^{\frac{a'}{a}}\right), \tag{2.9}$$

PROOF. We can prove this easily by the simple modification of Bosanquet's paper, but for the sake of completeness we prove the lemma.

Suppose that b is any real number and assume the theorem with b, c replaced by $b+1$, $c+1$. Then, by (ii) of (2.8), $c > -1$ and (2),

$$S_n^{a+1} = \sum_{\nu=0}^{n} S_\nu^a = o\left\{\sum_{\nu=0}^{n} \nu^c W(\nu)\right\} = o(n^{c+1}W(n)).$$

If we put $T_n^a = \sum_{\nu=0}^{n} A_{n-\nu}^{a-1} \nu s_\nu$, then we get

$$T_n^a = (a+n) S_n^a - (a+1) S_n^{a+1} = o(n^c W(n)). \tag{2.10}$$

From (2.10) and $ns_n = O(n^{b+1}V(n))$, the hypotheses of the theorem are satisfied, with s_n replaced by ns_n, b by $a+1$, c by $c+1$. It follows from the case assumed that

$$T_n^{a'} = o\left(n^{(b+1)(a-a')/a+a'(c+1)/a}(V(u))^{1-a'/a}(W(n))^{a'/a}\right) \qquad (0 < a' < a).$$

Now suppose that $a' \geqq 0$ and $a-1 \leqq a' < a$, then by (2)

$$S_n^{a'+1} = \sum_{\nu=0}^{n} A_{n-\nu}^{a'-a} S_\nu^a = o\left\{\sum_{\nu=0}^{n}(n-\nu)^{a'-a}\nu^c W(\nu)\right\} = o(n^{c+a'-a+1} W(n))$$

and so by (2.10)

$$S_n^{a'} = o\left(n^{(a-a')b/a+a'c/a}(V(n))^{1-a'/a}(W(n))^{a'/a}(1+n^{-(a-a')(b-c+a)/a}(W(n)/V(n))^{1-a'/a})\right)$$

Since $a+b \geqq c > -1$ and (iii) of (2.8), we obtain

$$S_n^{a'} = o\left(n^{(a-a')b/a+a'c/a}(V(n)^{1-a'/a}(W(n))^{a'/a}\right).$$

Thus, if $0 < a \leqq 1$, the result follows by induction from Lemma 5. If $a > 1$, we suppose $0 \leqq a' < a-1$, and sssume the result with a' replaced by $a'+1$. Then

$$\begin{aligned} S_n^{a'} &= (a'+1+n)^{-1}\{T_n^{a'} + (a'+1) S_n^{a'+1}\} \\ &= o\left(n^{(a-a)b/a+a'c/a}(V(n))^{1-a'/a}(W(n))^{a'/a}\right). \end{aligned}$$

Thus our result may be proved by induction.

LEMMA 7. *If* (2.8),

$$s_{n+m} - s_n > -Kn^b V(n)m \qquad (K > 0), \tag{2.11}$$

and

$$S_n^a = o(n^c W(n)) \qquad \textit{as } n \to \infty, \tag{2.12}$$

where $a > 0$ *and* $-1 < c \leqq a+b+1$, *we have*

$$S_n^{a'} = o\left(n^{\{b(a-a')+c(a'+1)\}/(a+1)}(V(n))^{1-(a'+1)/(a+1)}(W(n))^{(a'+1)/(a+1)}\right) \tag{2.13}$$

for $0 \leqq a' \leqq a$.

PROOF. First suppose that $b \geqq 0$, $c > 0$. Let a small positive ε be given, and let $a = p + \delta$, where $0 < \delta \leqq 1$ and p is a non-negative integer. Then, for all sufficiently large n, by (2.6)

$$\frac{\Gamma(h+\delta)}{\Gamma(h)} h^p s_n = \Delta_h^{p+\delta} S_n^{p+\delta} - \delta \sum_{\nu_0=1}^{h} \frac{\Gamma(h-\nu_0+\delta)}{\Gamma(h-\nu_0+1)} \sum_{\nu_1=1}^{h} \cdots \sum_{\nu_p=1}^{h} (s_{n+\nu_0+\dots+\nu_p} - s_n)$$

$$< \varepsilon^{p+\delta+1} n^c W(n) + K\delta \sum_{\nu_0=1}^{h} \frac{\Gamma(h-\nu_0+\delta)}{\Gamma(h-\nu_0+1)} \sum_{\nu_1=1}^{h} \cdots \sum_{\nu_p=1}^{h} n^b V(n)(\nu_0+\ldots+\nu_p)$$

$$< \varepsilon^{p+\delta+1} n^c W(n) + (p+1) K' h^{p+\delta+1} n^b V(n),$$

provided that $(p+1)h < Hn$ and $K' > K$. Also, by (2.7), for all sufficiently large n,

$$\frac{\Gamma(h+\delta)}{\Gamma(h)} h^p s_n > -\varepsilon^{p+\delta+1} n^c W(n) - (p+1) K' h^{p+\delta+1} n^b V(n),$$

for $(p+1)h < (1+H)^{-1} Hn$.

Taking $h = \varepsilon(n^c W(n)/n^b V(n))^{1/(p+\delta+1)}$ we get, for sufficiently large n,

$$|S_n| < \left(1+(p+1)K'\right)\varepsilon\left(n^{c-(p+\delta)(c-b)/(p+\delta+1)}(W(n))^{1-\frac{a'}{a+1}}(V(n))^{\frac{a}{a+1}}\right),$$

that is,

$$s_n = o\left(n^{\frac{c+ab}{a+1}}(V(n))^{1-\frac{1}{a+1}}(W(n))^{\frac{1}{a+1}}\right) \quad \text{as } n \to \infty. \tag{2.14}$$

From (i) and (ii) of (2.8), there exists a number $d' > 0$ such that $n^{d'}(V(n))^{1-\frac{1}{a+1}}(W(n))^{\frac{1}{a+1}} = n^{d'} V(n)(W(n)/V(n))^{\frac{1}{a+1}}$ is non-decreasing for sufficiently large n. Hence, using Lemma 7, we have

$$S_n^{a'} = o\left(n^{(a-a')(c+ab)/a(a+1)+a'c/a}\left((V(n))^{1-\frac{1}{a+1}}(W(n))^{1-\frac{a'}{a}}\right)^{1-\frac{a'}{a}}(W(n))^{\frac{a'}{a}}\right)$$

$$= o\left(n^{\{(a-a')b+(a'+1)c\}/(a+1)}(V(n))^{1-\frac{a'+1}{a+1}}(W(n))^{\frac{a'+1}{a+1}}\right),$$

for $0 < a' < a$, by (2.12) and (2.14).

The rest of the proof is analogous to that of Lemma 7.

3. **Proof of the Theorem.** By lemma 1, it is sufficient to prove that the series $\sum a_n$ is $J_{\mu-\frac{1}{2}}$ summable to 0, where $\mu = \alpha(\beta+1)/(\gamma+\alpha)$.

First we shall prove that

$$\sum_{\nu=n}^{\infty} \frac{|a_\nu|}{\nu} = O((\log n)^\alpha/n). \tag{3.1}$$

Under the assumption (1.5)

$$|a_n| - a_n < 2K\frac{(\log n)^\alpha}{n},$$

thus we have

$$\sum_{\nu=n}^{2n}(|a_\nu| - a_\nu) = O\{(\log n)^\alpha\}.$$

On the other hand, since we may put in lemma 2 $V(n) = n^\varepsilon(\log n)^\alpha$ and $W(n) = n^\varepsilon/(\log n)^\gamma$, where ε is any positive number, by (1.5) and (1.6),

$$S_n^h = o\left(n^h(\log n)^{\{(\beta-h)\alpha-(h+1)\gamma\}/(\beta+1)}\right). \quad \text{for } h = 0,1,2,\ldots\ldots k-1. \tag{3.2}$$

Especially,

$$S_n^0 = s_n = o\{(\log n)^{(\beta\alpha+\gamma)/(\beta+1)}\} = o((\log n)^\alpha),$$

for

$$\alpha - (\beta\alpha + \gamma)/(\beta + 1) = (\alpha + \gamma)/(\beta + 1) > 0.$$

Thus, we have

$$\sum_{\nu=n}^{2n} |a_\nu| = \sum_{\nu=n}^{2n} (|a_\nu| - a_\nu) + s_{2n} - s_{n-1} = O((\log n)^\alpha).$$

Hence

$$\sum_{\nu=n}^{2n} \nu^{-1}|a_\nu| \leqq n^{-1} \sum_{\nu=n}^{2n} |a_\nu| = O\{(\log n)^\alpha/n\},$$

$$\sum_{\nu=2k}^{2^{k+1}-1} \nu^{-1}|a_\nu| = O\{(\log 2^k)^\alpha/2^k\},$$

and

$$\sum_{\nu=1}^{2^l} \nu^{-1}|a_\nu| = O\left\{\sum_{k=0}^{l} \frac{(\log 2^k)^\alpha}{2^k}\right\} = O(1).$$

Consequently

$$\sum_{\nu=n}^{\infty} \nu^{-1}|a_\nu| = \sum_{k=0}^{\infty} \sum_{n\cdot 2k}^{n\cdot 2^{k+1}-1} \nu^{-1}|a_\nu| = O\left\{\sum_{k=0}^{\infty} \frac{(\log n2^k)^\alpha}{n\cdot 2^k}\right\}$$

$$= O\left\{\frac{(\log n)^\alpha}{n} \sum_{k=0}^{\infty} \frac{(\log 2^k)^\alpha}{2^k}\right\} = O\{(\log n)^\alpha/n\}$$

which is the desired inequality (3.1).

Let

$$\psi_\mu(t) = \sum_{n=0}^{\infty} a_n \,\alpha_{\mu-\frac{1}{2}}(nt) = \left(\sum_{n=0}^{\nu} + \sum_{n=\nu+1}^{\infty}\right) a_n \,\alpha_{\mu-\frac{1}{2}}(nt) = \psi_1(t) + \psi_2(t), \tag{3.3}$$

say, where ν is to be chosen presently.

Using the inequality (3.1)

$$\psi_2(t) = O\left\{\sum_{n=\nu+1}^{\infty} n^{-1}|a_n|(nt)^{-\mu}\, n\right\} = O\left(t^{-\mu}\nu^{1-\mu}\nu^{-1}(\log \nu)^\alpha\right). \tag{3.4}$$

This shows that the series $\sum_{n=0}^{\infty} a_n \,\alpha_{\mu-\frac{1}{2}}(nt)$ converges for fixed $t > 0$.

For a given positive number C, we put

$$\nu = \rho(t) = \left[C\left(\log \frac{1}{t}\right)^{\alpha/\mu} t^{-1}\right]. \tag{3.5}$$

Then from (3.4), we obtain

$$\psi_2(t) = O\left\{C^{-\mu}\left(\log \frac{1}{t}\right)^{-\alpha}\left(\log \frac{C\left(\log \frac{1}{t}\right)^{\alpha/\mu}}{t}\right)^\alpha\right\} = O(C^{-\mu}).$$

Thus if we take C sufficiently large, we get

$$\psi_2(t) = o(1) \qquad \text{as } t \to 0. \tag{3.6}$$

Now there is an integer $k \geqq 1$ such that $k-1 < \beta \leqq k$. We suppose that $k-1<\beta<k$, for the case $\beta = k$ can be easily deduced by the following argument.

Now

$$S_n^k = \sum_{\nu=0}^{n} A_{n-\nu}^{k-\beta-1} S_\nu^\beta = o\left\{ \sum_{\nu=2}^{n} (n-\nu)^{k-\beta-1} \nu^\beta (\log \nu)^{-\gamma} \right\} = o\{n^k (\log n)^{-\gamma}\}. \tag{3.7}$$

Concerning $\psi_1(t)$, by Abel's lemma on partial summation k-times we have

$$\psi_1(t) = \sum_{n=0}^{\nu} a_n \alpha_{\mu-\frac{1}{2}}(nt) = \sum_{n=0}^{\nu-(k+1)} S_n^k \Delta^{k+1} \alpha_{\mu-\frac{1}{2}}(nt) + \sum_{h=0}^{k-1} S_{\nu-h}^h \Delta^h \alpha_{\mu-\frac{1}{2}}(\overline{\nu-h}\,t)$$

$$+ S_{\nu-k}^k \Delta^k \alpha_{\mu-\frac{1}{2}}(\overline{\nu-k}\,t) = \psi_3(t) + \psi_4(t) + \psi_5(t), \qquad \text{say.}$$

Using (2.3), (3.5) and (3.7), we have

$$\psi_5(t) = S_{\nu-k}^k \Delta^k \alpha_{\mu-\frac{1}{2}}(\overline{\nu-k}\,t) = o\{\nu^k (\log \nu)^{-\gamma} t^{k-\mu} \nu^{-\mu}\}$$

$$= o\left\{ t^{-(k-\mu)} t^{k-\mu} \left(\log \frac{1}{t}\right)^{\alpha(k-\mu)/\mu} \left(\log \frac{1}{t}\right)^{-\gamma} \right\}$$

$$= o\left\{ \left(\log \frac{1}{t}\right)^{\alpha(k-\beta-1)/\mu} \right\} = o(1) \qquad \text{as } t \to 0, \tag{3.8}$$

Also, by (2.3), (3.5) and (3.2)

$$S_{\nu-h}^h \Delta^h \alpha_{\mu-\frac{1}{2}}(\overline{\nu-h}\,t) = o\{\nu^{h-\mu} (\log \nu)^{\{\beta\alpha-\gamma-h(\alpha+\gamma)\}/(\beta+1)} t^{h-\mu}\}$$

$$= o\left\{ t^{h-\mu} t^{-(h-\mu)} \left(\log \frac{1}{t}\right)^{\{(h-\mu)\alpha/\mu + (\beta\alpha-\gamma-h(\alpha+\gamma))/(\beta+1)\}} \right\}.$$

Since the exponent of $\log \frac{1}{t}$ is

$$(h-\mu)\frac{\alpha}{\mu} + \frac{\beta\alpha-\gamma-h(\alpha+\gamma)}{\beta+1} = \frac{h\alpha}{\mu} - \alpha + \frac{\beta\alpha-\gamma}{\beta+1} - \frac{\alpha}{\mu} h = -\frac{\alpha+\gamma}{\beta+1} < 0,$$

we have

$$\psi_4(t) = o(1) \qquad \text{as } t \to 0. \tag{3.9}$$

Concerning $\psi_3(t)$, we split up four parts

$$\psi_3(t) = \sum_{n=0}^{\nu-(k+1)} \Delta^{k+1} \alpha_{\mu-\frac{1}{2}}(nt) \sum_{m=0}^{n} A_{n-m}^{k-\beta-1} S_m^\beta = \sum_{m=0}^{\nu-(k+1)} S_m^\beta \sum_{n=m}^{\nu-(k+1)} A_{n-m}^{k-\beta-1} \Delta^{k+1} \alpha_{\mu-\frac{1}{2}}(nt)$$

$$= \sum_{m=0}^{[1/t]} \sum_{n=m}^{m+[1/t]} + \sum_{m=[1/t]+1}^{\nu-(k+1)} \sum_{n=m}^{m+[1/t]} + \sum_{m=0}^{\nu-(k+1)-[1/t]-1} \sum_{n=m+[1/t]+1}^{\nu-(k+1)} - \sum_{m=\nu-(k+1)-[1/t]}^{\nu-(k+1)} \sum_{n=\nu-(k+1)+1}^{m+[1/t]}$$

$$= \psi_6(t) + \psi_7(t) + \psi_8(t) + \psi_9(t), \qquad \text{say.} \tag{3.10}$$

From (1.6) and (2.2), we get

$$\psi_6(t) = \sum_{m=0}^{[1/t]} S_m^\beta \sum_{n=m}^{m+[1/t]} A_{n-m}^{k-\beta-1} \Delta^{k+1} \alpha_{\mu-\frac{1}{2}}(nt)$$

$$= O\left\{\sum_{m=0}^{[1/t]} S_m^\beta \sum_{n=m}^{m+[1/t]} (n-m)^{k-\beta-1} t^{k+1}\right\} = o\left\{t^{k+1} \sum_{m=2}^{[1/t]} m^\beta (\log m)^{-\gamma} t^{-(k-\beta)}\right\}$$

$$(3.11) \qquad = o\left(\left(\log\frac{1}{t}\right)^{-\gamma}\right) = o(1) \qquad \text{as } t \to 0.$$

From (1.6) and (2.3), we also get

$$\psi_7(t) = \sum_{m=[1/t]+1}^{\nu-(k+1)} S_m^\beta \sum_{n=m}^{m+[1/t]} A_{m-n}^{k-\beta-1} \alpha_{\mu-\frac{1}{2}}(nt)$$

$$= o\left\{\sum_{m=[1/t]+1}^{\nu-(k+1)} m^\beta (\log m)^{-\gamma} \sum_{n=m}^{m+[1/t]} (n-m)^{k-\beta-1} t^{k+1-\mu} n^{-\mu}\right\}$$

$$= o\left\{\sum_{m=[1/t]+1}^{\nu-(k+1)} m^\beta (\log m)^{-\gamma} m^{-\mu} t^{k+1-\mu} \sum_{n=m}^{m+[1/t]} (n-m)^{k-\beta-1}\right\}$$

$$= o\left\{t^{k+1-\mu} \sum_{m=[1/t]+1}^{\nu-(k+1)} m^{\beta-\mu} (\log m)^{-\gamma} t^{-(k-\beta)}\right\} = o\{t^{\beta+1-\mu} (\log \nu)^{-\gamma} \nu^{\beta+1-\mu}\}$$

for $\beta + 1 - \mu = \dfrac{\gamma(\beta+1)}{\gamma+\delta} > 0$. Hence

$$(3.12) \qquad \psi_7(t) = o\left\{t^{\beta+1-\mu} t^{-(\beta+1-\mu)} \left(\log\frac{1}{t}\right)^{\frac{\alpha}{\mu}(\beta+1-\mu)-\gamma}\right\} = o(1) \qquad \text{as } t \to 0,$$

by (3.5), for $\dfrac{\alpha}{\mu}(\beta+1-\mu) - \gamma = 0$.

For the estimation of $\psi_8(t)$, if we use partial summation in the inner series, then

$$\psi_8(t) = \sum_{m=0}^{\nu-(k+1)-[1/t]-1} S_m^\beta \sum_{n=m+[1/t]+1}^{\nu-(k+1)} A_{n-m}^{k-\beta-1} \Delta^{k+1} \alpha_{\mu-\frac{1}{2}}(nt)$$

$$= \sum_{m=0}^{\nu-(k+2)-[1/t]} S_m^\beta \sum_{n=m+[1/t]+2}^{\nu-(k+1)} A_{n-m}^{k-\beta-2} \Delta^k \alpha_{\mu-\frac{1}{2}}(nt)$$

$$+ \sum_{m=0}^{\nu-(k+2)-[1/t]} S_m^\beta A_{[1/t]+1}^{k-\beta-1} \Delta^k \alpha_{\mu-\frac{1}{2}}(\overline{m+[t^{-1}]+1}\, t)$$

$$- \sum_{m=0}^{\nu-(k+2)-[1/t]} S_m^\beta A_{\nu-k-1-m}^{k-\beta-1} \Delta^k \alpha_{\mu-\frac{1}{2}}(\overline{\nu-k}\, t)$$

$$(3.13) \qquad = \psi_8'(t) + \psi_8''(t) + \psi_8'''(t), \text{ say.}$$

$$\psi_8'(t) = O\left\{\sum_{m=0}^{\nu-(k+2)-[1/t]} S_m^\beta \sum_{n=m+[1/t]+2}^{\nu-(k+1)} (n-m)^{k-\beta-2} n^{-\mu} t^{k-\mu}\right\}$$

$$= o\left\{t^{k-\mu} \sum_{m=2}^{\nu-(k+2)-[1/t]} m^\beta (\log m)^{-\gamma} m^{-\mu} t^{-(k-\beta-1)}\right\}$$

$$(3.14) \qquad = o\left\{\left(\log\frac{1}{t}\right)^{\frac{\alpha}{\mu}(\beta+1-\mu)-\gamma}\right\} = o(1) \qquad \text{as } t \to 0.$$

$$\psi_8''(t) = o\left\{\sum_{m=2}^{\nu-(k+2)-[1/t]} m^{\beta}(\log m)^{-\gamma}\, t^{-(k-\beta-1)}\,(m+[t^{-1}]+1)^{-\mu}\, t^{k-\mu}\right\}$$

$$= o\{t^{\beta+1-\mu}(\log \nu)^{-\gamma}\,\nu^{\beta+1-\mu}\}$$

$$= o\left\{\left(\log\frac{1}{t}\right)^{\frac{\alpha}{\mu}(\beta+1-\mu)-\gamma}\right\} = o(1) \qquad \text{as } t\to 0. \tag{3.15}$$

$$\psi_8'''(t) = o\left\{\sum_{m=2}^{\nu-(k+2)-[1/t]} m^{\beta}(\log m)^{-\gamma}(\nu-k-1-m)^{k-\beta-1}(\nu-k)^{-\mu}t^{k-\mu}\right\}$$

$$= o\{t^{k-\mu}\,\nu^{-\mu+k-\beta-1}\,(\log \nu)^{-\gamma}\,\nu^{\beta+1}\}$$

$$= o\left\{\left(\log\frac{1}{t}\right)^{\frac{\alpha}{\mu}(k-\mu)-\gamma}\right\} = o(1) \qquad \text{as } t\to 0 \tag{3.16}$$

Thus, from (3.14), (3.15), (3.16) and (3.13)

$$\psi_8(t) = o(1) \qquad \text{as } t\to 0. \tag{3.17}$$

Moreover, we get

$$\psi_9(t) = -\sum_{m=\nu-(k+1)-[1/t]}^{\nu-k-1} S_m^{\beta} \sum_{n=\nu-k}^{m+[1/t]} A_{n-m}^{k-\beta-1}\,\Delta^{k+1}\,\alpha_{\mu-\frac{1}{2}}(nt)$$

$$= O\left\{\sum_{m=\nu-k-1-[1/t]}^{\nu-k-1} S_m^{\beta} \sum_{n=\nu-k}^{m+[1/t]} (n-m)^{k-\beta-1}\,n^{-\mu}\,t^{k+1-\mu}\right\}$$

$$= o\left\{t^{k+1-\mu}\sum_{n=\nu-k-[1/t]}^{\nu-k-1} m^{\beta}\,(\log m)^{-\gamma}\,\nu^{-\mu}\,t^{-(k-\beta)}\right\}$$

$$= o\{t^{\beta+1-\mu}(\log \nu)^{-\gamma}\,\nu^{\beta+1-\mu}\}$$

$$= o\left\{\left(\log\frac{1}{t}\right)^{\frac{\alpha}{\mu}(\beta+1-\mu)-\gamma}\right\} = o(1) \qquad \text{as } t\to 0. \tag{3.18}$$

Summing up (3.10), (3.11), (3.12), (3.17) and (3.18), we obtain

$$\psi_3(t) = o(1) \qquad \text{as } t\to 0. \tag{3.19}$$

Thus, from (3.1), (3.6), (3.8), (3.9) and (3.19), we have

$$\psi(t) = o(1) \qquad \text{as } t\to 0,$$

which is the required.

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