# ON FRACTIONAL INTEGRATION

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**1. Introduction.** The present paper is devoted to give certain results for fractional integration which are related to the work of I. I. Hirschman, Jr.

Let $u(\theta)$ be a function in the class $L^p(0, 2\pi), p \geqq 1$ with mean value zero and its Fourier series be

$$u(\theta) \sim \sum' a_n e^{in\theta}$$

where $-\infty < n < \infty$ and $n \neq 0$.

The fractional integral $u_\alpha(\theta)$ of $u(\theta)$ of order $\alpha$ is defined by

$$u_\alpha(\theta) \sim \sum' a_n (in)^{-\alpha} e^{in\theta}$$

and let the Abel mean of $u(\theta)$ and $u_\alpha(\theta)$ be

$$u(r, \theta) = \sum' a_n r^{|n|} e^{in\theta}$$

$$u_\alpha(r, \theta) = \sum' a_n (in)^{-\alpha} r^{|n|} e^{in\theta}$$

We consider the following functions, the first due to I. I. Hirschman, Jr. [1] and the remains to G. Sunouchi [3],

$$g(\alpha ; \theta) = \left\{ \int_0^1 (1-r)^{1-2\alpha} |u_{\alpha-1}(r, \theta)|^2 dr \right\}^{1/2}$$

$$g^*(\alpha, \beta ; \theta) = \left\{ \int_0^1 (1-r)^{2(\beta-\alpha)} dr \int_0^{2\pi} \frac{|u_{\alpha-1}(r, \theta+t)|^2}{|1-re^{it}|^{2\beta}} dt \right\}^{1/2}$$

$$\delta(\alpha, k ; \theta) = \left\{ \int_0^{2\pi} |\Delta_{t(k)}^{k+1} u_\alpha(\theta)|^2 t^{-2\alpha-1} dt \right\}^{1/2},$$

where

$$\Delta_t^1 u_\alpha(\theta) = u_\alpha(\theta + t) - u_\alpha(\theta - t)$$

$$\Delta_t^{k+1} u_\alpha(\theta) = \Delta_t^1(\Delta_t^k u_\alpha(\theta))$$

$$t(k) = t/2(k+1).$$

The main purpose of this paper is to prove the following:

THEOREM 1. *Let $u(\theta) \in L^p(0, 2\pi)$, $p > 1$, then we have*

$$\|\delta(\alpha, k;\ \theta)\|_p \leqq A_p \|u\|_p$$

*where $0 < \alpha < k+1\ (2 < p < \infty)$, $2/p - 1 < \alpha < k+1\ (1 < p < 2)$ and $k$ is a positive integer or zero.*

*The constant $A_p$ depends only on $p$, and not on the function $u(\theta)$.*

We shall use constants, not necessarily the same at each occurrence, which depend only on indicated indices. The case $k=0$ is due to I. I. Hirschman, Jr., but his result is not quite right, as G. Sunouchi [3] indicates. The author thanks to Professor G. Sunouchi who gave him valuable suggestions and advices and also to Mr. C. Watari.

2. For the proof of theorem 1 we need the following two lemmas.

LEMMA 1. *Let $u(\theta) \in L(0, 2\pi)$, and its mean value be zero, then we have for $\alpha > 0$*

$$\delta(\alpha, 0; \theta) \leqq A_\alpha g^*(\alpha, 1; \theta) + B_\alpha g^*(\alpha, (\alpha+1)/2; \theta) \qquad a.e.\ \theta.$$

LEMMA 2. *Let $u(\theta) \in L(0, 2\pi)$, and its mean value be zero, then we have*

$$\delta(\alpha, k; \theta) \leqq A_{\alpha,k} g^*(\alpha-k, 1; \theta) + B_{\alpha,k} g^*(\alpha-k, (\alpha-k+2)/2; \theta) \qquad a.e.\ \theta,$$

*for $\alpha > k-1$, and*

$$\delta(\alpha, k; \theta) \leqq A_{\alpha,k} g^*(\alpha-j, 1; \theta) + B_{\alpha,k} g^*(\alpha-j, (\alpha-j+2)/2; \theta) \qquad a.e.\ \theta,$$

*for $j-1 < \alpha < j+1$ $(j = 1, 2, \ldots, k-1)$ and $k$ is any positive integer.*

PROOF OF LEMMA 1. The proof runs on the line of A. Zygmund [4]. Let

$$\begin{aligned}\Delta^1_{t/2} u_\alpha(\theta) &= \{\Delta^1_{t/2} u_\alpha(\theta) - \Delta^1_{t/2} u_\alpha(r_t, \theta)\} + \Delta^1_{t/2} u_\alpha(r_t, \theta) \\ &= V + W \qquad \text{say,}\end{aligned}$$

where $1 - r_t = 1 - t/4\pi$ and then $1/2 \leqq r \leqq 1$ are mapped on $0 \leqq t \leqq 2\pi$, We shall first estimate the $W$. We have

$$W = \int_{-t/2}^{t/2} u_{\alpha-1}(r_t, \theta+v)\, dv$$

$$W^2 \leqq At \int_{-t/2}^{t/2} |u_{\alpha-1}(r_t, \theta+v)|^2\, dv$$

and so

$$\begin{aligned}\int_0^{2\pi} W^2 t^{-2\alpha-1}\, dt &\leqq A \int_0^{2\pi} t^{-2\alpha}\, dt \int_{-t/2}^{t/2} |u_{\alpha-1}(r_t, \theta+v)|^2\, dv \\ &\leqq A_\alpha \int_0^{1/2} \delta^{-2\alpha}\, d\delta \int_{-2\pi\delta}^{2\pi\delta} |u_{\alpha-1}(r, \theta+v)|^2\, dv,\end{aligned}$$

where $\delta = 1 - r$. Since in the region: $0 < \delta = 1 - r \leqq 1/2$, $|t| \leqq k\delta \leqq \pi$, it holds that $|1 - re^{it}|^{-1} \sim 1/\delta$, and hence

$$\begin{aligned}\int_0^{2\pi} W^2 t^{-2\alpha-1}\, dt &\leqq A_\alpha \int_0^{1/2} \delta^{-2\alpha+1} d\delta \int_{-2\pi\delta}^{2\pi\delta} |u_{\alpha-1}(r, \theta+v)|^2 P(r, v)\, dv \\ &\leqq A_\alpha (g^*(\alpha, 0; \theta))^2.\end{aligned}$$

We have next

$$V = \int_{r_t}^{1} \Delta^1_{t/2} u_{\alpha-1}(r, \theta)\, dr$$

$$= \int_0^{\delta t} \delta^{(\alpha-1)/2}\, \delta^{(-\alpha+1)/2} \Delta^1_{t/2} u_{\alpha-1}(r, \theta)\, d\delta$$

and so, for $\alpha > 0$ by Schwarz' inequality, it follows that

$$V^2 \leqq A_\alpha\, \delta_t^2 \int_0^{\delta_t} \delta^{-\alpha+1} |\Delta^1_{t/2} u_{\alpha-1}(r, \theta)|^2\, d\delta,$$

$$\int_0^{2\pi} V^2 t^{-2\alpha-1}\, dt$$

$$\leqq A_\alpha \int_0^{2\pi} t^{-\alpha-1}\, dt \int_0^{\delta_t} \delta^{-\alpha+1} (|u_{\alpha-1}(r, \theta + t/2)|^2 + |u_{\alpha-1}(r, \theta - t/2)|^2) d\delta$$

$$= A_\alpha \int_0^{1/2} \delta^{-\alpha+1}\, d\delta \int_{2\pi\delta}^{\pi} (|u_{\alpha-1}(r, \theta + t)|^2 + |u_{\alpha-1}(r, \theta - t)|^2)\, t^{-\alpha-1}\, dt$$

Since, in the region: $0 < \delta \leqq 1/2$, $k\delta \leqq |t| \leqq \pi$, it holds that $|1 - re^{it}|^{-1} \sim 1/t$, we have

$$\int_0^{2\pi} V^2 t^{-2\alpha-1}\, dt$$

$$\leqq A_\alpha \int_0^{1/2} \delta^{-\alpha+1}\, d\delta \int_{2\pi\delta}^{\pi} \frac{|u_{\alpha-1}(r, \theta + t)|^2 + |u_{\alpha-1}(r, \theta - t)|^2}{|1 - r\, e^{it}|^{2(\alpha+1)/2}}\, dt$$

$$\leqq A_\alpha (g^*(\alpha,\ (\alpha+1)/2\,;\, \theta))^2.$$

We have thus proved Lemma 1 completely.

PROOF OF LEMMA 2. We prove the case $k = 1$, and for the remaining case we only sketch the proof.

(a) the case $k = 1$. As in Lemma 1, let us put

$$\Delta^2_{t/4} u_\alpha(\theta) = \{\Delta^2_{t/4} u_\alpha(r_t, \theta) - \Delta^2_{t/4} u_\alpha(r_t, \theta)\} + \Delta^2_{t/4} u_\alpha(r_t, \theta)$$
$$= V + W \qquad \text{say.}$$

Concerning $W$, we have

$$W = \int_{-t/4}^{t/4} \Delta^1_{t/4} u_{\alpha-1}(\theta + v)\, dv$$

$$= \int_{-t/4}^{t/4} dv \int_{-t/4}^{t/4} u_{\alpha-2}(\theta + v_1)\, dv_1$$

and then

$$W^2 \leqq At^2 \int_{-t/4}^{t/4} dv \int_{-t/4}^{t/4} |u_{\alpha-2}(\theta + v_1)|^2 \, dv_1.$$

Changing the order of integration, we have

$$W^2 \leqq At^3 \int_{-t/2}^{t/2} |u_{\alpha-2}(\theta + v)|^2 \, dv,$$

$$\int_0^{2\pi} W^2 t^{-2\alpha-1} \, dt \leqq A \int_0^{2\pi} t^{-2\alpha+2} \, dt \int_{-t/2}^{t/2} |u_{\alpha-2}(r_t, \theta + v)|^2 \, dv$$

$$\leqq A \int_0^{1/2} \delta^{-2\alpha+2} \, d\delta \int_{-2\pi\delta}^{2\pi\delta} |u_{\alpha-2}(r, \theta + v)|^2 \, dv$$

$$\leqq A(g^*(\alpha - 1, 1; \theta))^2.$$

By integration by parts, we have

$$V = (1 - r_t) \frac{\partial}{\partial r} \Delta^2_{t/4} u_\alpha(r_t, \theta) + \int_{r_t}^1 (1 - r) \frac{\partial^2}{\partial r^2} \Delta^2_{t/4} u_\alpha(r, \theta) \, dr$$

$$= V_1 + V_2 \qquad \text{say}.$$

Since $0 \leqq \delta_t = 1 - r_t \leqq 1/2$ for $0 \leqq t \leqq 2\pi$, we get

$$V_1^2 = \delta_2^t r_t^{-2} (\Delta^2_{t/4} u_{\alpha-1}(r_t, \theta))^2$$

$$= At^2 \Big( \int_{-t/4}^{t/4} \Delta^1_{t/4} u_{\alpha-2}(r_t, \theta + v) \, dv \Big)^2$$

$$\leqq At^3 \int_{-t/2}^{t/2} |u_{\alpha-2}(r_t, \theta + v)|^2 \, dv.$$

Similarly as for $W$, we obtain

$$\int_0^{2\pi} V_1^2 t^{-2\alpha-1} \, dt \leqq A(g^*(\alpha - 1, 1; \theta))^2.$$

We have for $V_2$,

$$= V_2 \int_{r_t}^1 (1 - r) r^{-2} \Delta^2_{t/4} u_{\alpha-2}(r, t) \, dr$$

$$A \leqq \int_{r_t}^1 (1 - r)(|u_{\alpha-2}(r, \theta + t/2)| + 2|u_{\alpha-2}(r, \theta)| + |u_{\alpha-2}(r, \theta - t/2)|) \, dr$$

$$= A(V_{21} + V_{22} + V_{23}) \qquad \text{say}.$$

For $V_{22}$, if we write $\delta = \delta^{(\alpha-1)/2} \delta^{(3-\alpha)/2}$ and apply the Schwarz inequality, then we have for $\alpha > 0$

$$V_{22} \leqq t^{\alpha} \int_0^{\delta_t} \delta^{3-\alpha} |u_{\alpha-2}(r,\theta)|^2 \, d\delta,$$

and so,

$$\int_0^{2\pi} V_{22}^2 t^{-2\alpha-1} \, dt \leqq \int_0^{2\pi} t^{-\alpha-1} \, dt \int_0^{\delta_t} \delta^{3-\alpha} |u_{\alpha-2}(r,\theta)|^2 \, d\delta$$

$$\leqq \int_0^{1/2} \delta^{3-\alpha} |u_{\alpha-2}(r,\theta)|^2 \, d\delta \int_{4\pi\delta}^{\infty} t^{-\alpha-1} \, dt$$

$$\leqq A_\alpha \int_0^{1/2} \delta^{3-\alpha} |u_{\alpha-2}(r,\theta)|^2 \, d\delta$$

$$\leqq A(g(\alpha-1\,;\theta))^2 \leqq A(g^*(\alpha-1, 1\,;\theta))^2.$$

For $V_{21}$, we have similarly as for $V_{22}$,

$$V_{21}^2 \leqq t^{\alpha} \int_0^{\delta_t} \delta^{3-\alpha} |u_{\alpha-2}(r,\theta+t/2)|^2 \, d\delta,$$

and so,

$$\int_0^{2\pi} V_{21}^2 \, t^{-2\alpha-1} \, dt \leqq \int_0^{2\pi} t^{-1-\alpha} \, dt \int_0^{\delta_t} \delta^{3-\alpha} |u_{\alpha-2}(r,\theta+t/2)|^2 \, dt$$

$$\leqq A_\alpha \int_0^{1/2} \delta^{3-\alpha} \, d\delta \int_{2\pi\delta}^{\pi} |u_{\alpha-2}(r,\theta+t)|^2 t^{-1-\alpha} \, dt$$

$$\leqq A_\alpha \int_0^{1/2} \delta^{3-\alpha} \, d\delta \int_{2\pi\delta}^{\pi} \frac{|u_{\alpha-2}(r,\theta+t)|^2}{|1-re^{it}|^{2(1+\alpha)/2}} \, dt$$

$$\leqq A(g^*(\alpha-1,\ (\alpha+1)/2\,;\theta))^2.$$

Similarly, we have

$$\int_0^{2\pi} V_{22}^2 t^{-2\alpha-1} \, dt \leqq A_\alpha(g^*(\alpha-1,\ (\alpha+1)/2\,;\theta))^2.$$

Thus, we have established completely the Lemma of typical case.

(b) general case $k \geqq 2$. First we prove for $\alpha > k-1$.

Let

$$\begin{aligned} \Delta_{t(k)}^{k+1} u_\alpha(\theta) &= \{\Delta_{t(k)}^{k+1} u_\alpha(\theta) - \Delta_{t(k)}^{k+1} u_\alpha(r_t,\theta)\} + \Delta_{t(k)}^{k+1} u_\alpha(r_t,\theta) \\ &= V + W \qquad\qquad \text{say.} \end{aligned}$$

For $W$, we have

$$W = \int_{-t(k)}^{t(k)} dv_1 \int_{v_1-t(k)}^{v_1+t(k)} dv_2 \ldots . \int_{v_k-t(k)}^{v_k+t(k)} u_{\alpha-k-1}(r_t, \theta+v_{k+1}) \, dv_{k+1}.$$

Here if we apply Schwarz' inequality and then change the order of integration repeatedly, we have

$$W^2 \leqq A_k t^{k+1} \int_{-t(k)}^{t(k)} dv_1 \int_{v_1-t(k)}^{v_1+t(k)} dv_2 \dots \int_{v_k-t(k)}^{v_k+t(k)} |u_{\alpha-k-1}(r_t, \theta + v_{k+1})|^2 \, dv_{k+1}$$

$$\leqq A_k t^{k+2} \int_{-2t(k)}^{2t(k)} dv_2 \dots \int_{v_k-t(k)}^{v_k+t(k)} |u_{\alpha-k-1}(r_t, \theta + v_{k+1})|^2 \, dv_{k+1}$$

$$\dots\dots$$

$$\leqq A_k t^{2k+1} \int_{-t/2}^{t/2} |u_{\alpha-k-1}(r_t, \theta + v)|^2 \, dv,$$

and we obtain

$$\int_0^{2\pi} W^2 t^{2\alpha-1} \, dt \leqq A_{k,\alpha} (g^*(\alpha - k, 1; \theta))^2.$$

For $V$, we have

$$V = \int_{-t(k)}^{t(k)} dv_1 \int_{v_1-t(k)}^{v_1+t(k)} dv_2 \dots \int_{v_{k-2}-t(k)}^{v_{k-2}+t(k)} \Delta^2_{t(k)} \{u_{\alpha-k+1}(\theta + v_{k-1}) - u_{\alpha-k+1}(r_t, \theta + v_{k-1}))\} \, dv_{k-1}.$$

Integrating by parts the integrand as in the case (a), we have

$$V = \int_{-t(k)}^{t(k)} dv_1 \int_{v_1-t(k)}^{v_1+t(k)} dv_2 \dots \int_{v_{k-2}-t(k)}^{v_{k-2}+t(k)} dv_{k-1}$$

$$\cdot \left\{ (1 - r_t) \frac{\partial}{\partial r} \Delta^2_{t(k)} u_{\alpha-k+1}(r_t, \theta + v_{k-1}) + \int_{r_t}^{1} (1 - r) \frac{\partial^2}{\partial r^2} \Delta^2_{t(k)} u_{\alpha-k+1}(r, \theta + v_{k-1}) \, dr \right\}$$

$$= V_1 + V_2, \quad \text{say}.$$

We have

$$V_1 = A \cdot \frac{t}{4\pi} \int_{-t(k)}^{t(k)} dv_1 \int_{v_1-t(k)}^{v_1+t(k)} dv_2 \dots \int_{v_{k-2}-t(u)}^{v_{k-2}+t(u)} dv_{k-1} \int_{v_{k-1}}^{v_{k-1}+2t(k)} \Delta^1_{t(k)} u_{\alpha-k-1}(r_t, \theta + v_k) \, dv_k,$$

$$V_1^2 \leqq A_\alpha t^{2k+1} \int_{-t/2}^{t/2} |u_{\alpha-k-1}(r_t, \theta + v)|^2 \, dv,$$

and

$$\int_0^{2\pi} V_1^2 t^{-2\alpha-1} \, dt \leqq A_\alpha (g^*(\alpha - k, 1; \theta))^2.$$

For $V_2$, we have

$$V_2 = \int_{-t(k)}^{t(k)} dv_1 \int_{v_1-t(k)}^{v_1+t(k)} dv_2 \dots \int_{v_{k-2}-t(k)}^{v_{k-2}+t(k)} dv_{k-1} \int_{r_t}^{1} (1 - r) r^{-2} \Delta^2_{t(k)} u_{\alpha-k-1}(r, \theta) \, dr$$

$$= V_{21} + V_{22} + V_{23}, \qquad \text{say.}$$

It follows that

$$V_{21} = \int_{-t(k)}^{t(k)} dv_1 \int_{v_1-t(k)}^{v_1+t(k)} dv_2 \dots \int_{v_{k-2}+t(k)}^{v_{k-2}+3t(k)} dv_{2-1} \int_{rt}^{1} (1-r)\, u^{\alpha-k-1}(r, \theta + v_{k-1}) dr$$

and

$$V_{21}^2 \leqq A_k\, t^{2k-3} \int_{-t/2}^{t/2} dv \left( \int_{rt}^{1} (1-r) u_{\alpha-k-1}(r, \theta + v)\, dr \right)^2.$$

If $\alpha > k-1$, we write $\delta = \delta^{(-k+\alpha)/2}\, \delta^{(k-\alpha+2)/2}$, and applying the Schwarz inequality, we have

$$V_{21}^2 \leqq A_{\alpha,k}\, t^{k+\alpha-2} \int_{-t/2}^{t/2} dv \int_{rt}^{1} (1-r)^{k-\alpha+2} |u_{\alpha-k-1}(r, \theta + v)|^2\, dr.$$

Hence, we have

$$\int_0^{2\pi} V_{21}^2\, t^{-2\alpha-1}\, dt$$

$$\leqq \int_0^{2\pi} t^{k-\alpha-3}\, dt \int_{-t/2}^{t/2} dv \int_0^{\delta t} \delta^{k-\alpha+2} |u_{\alpha-k-1}(r, \theta + v)|^2\, d\delta$$

$$\leqq \int_0^{1/2} \delta^{k-\alpha+2}\, d\delta \int_{4\pi\delta}^{2\pi} t^{k-\alpha-3}\, dt \int_{-t/2}^{t/2} |u_{\alpha-k-1}(r, \theta + v)|^2\, dv.$$

Since $\alpha > k-1$, integrating by parts the second integral, we have

$$\int_0^{2\pi} V_{21}^2\, t^{-2\alpha-1}\, dt$$

$$\leqq A_{k,\alpha} \int_0^{1/2} \delta^{-2(\alpha-k)}\, d\delta \int_{-2\pi\delta}^{2\pi\delta} |u_{\alpha-k-1}(r, \theta + v)|^2\, dv$$

$$+ B_{k,\alpha} \int_0^{1/2} \delta^{k-\alpha+2}\, d\delta \int_{2\pi\delta}^{\pi} (|u_{\alpha-k-1}(r, \theta + t)|^2 + |u_{\alpha-k-1}(r, \theta - t)|^2)\, t^{k-\alpha-2}\, dt$$

$$\leqq A_{k,\alpha}(g^*(\alpha - k, 1\,;\theta))^2 + B_{k,\alpha}(g^*(\alpha - k, (\alpha - k + 2)/2\,;\theta))^2.$$

The same argument may be used for the estimation of the $V_{22}$ and $V_{23}$. Combining these estimations we obtain the lemma in general case for $\alpha > k-1$.

Now the remaining case $0 < \alpha \leqq k-1$ is estimated easily by the following inequality.

Let $j-1 < \alpha < j+1$ $(j = 1, 2, \dots, k-1)$, then

$$|\Delta_{t(k)}^{k+1} u_\alpha|^2 = |\Delta_{t(k)}^{k-j} \Delta_{t(k)}^{j+1} u_\alpha|^2$$

$$\leqq A_k \sum_{l=-(k-j)}^{k-j} |\Delta_{t(k)}^{l+1} u_\alpha(\theta + l\, t(k))|^2.$$

We now need the following lemma due to G. Sunouchi [3].

LEMMA 3. *Let $u(\theta) \in L(0, 2\pi)$, and its mean value be zero, then we have for $\beta > \alpha > -\infty$*

$$g^*(\alpha, \beta\,; \theta) \leqq A_{\alpha,\beta}\, g^*(0,\ \beta\,; \theta)$$

Combining Lemmas 1, 2 and 3, we get the following lemmas.

LEMMA 4. *Let $u(\theta) \in L(0, 2\pi)$, and have mean value zero, then we have for $0 < \alpha < 1$*

$$\delta(\alpha, 0\,; \theta) \leqq A_\alpha\, g^*(\alpha,\ 1\,; \theta) + B_\alpha\, g^*(0,\ (\alpha+1)/2\,; \theta) \qquad a.e.\ \theta.$$

LEMMA 5. *Under the same assumptions, we have*

$$\delta(\alpha, k\,; \theta) \leqq A_{\alpha,k}\, g^*(\alpha - j, 1\,; \theta) + B_{\alpha,k}\, g^*(0,\ (\alpha - j + 2)/2\,; \theta)$$

*where $j-1 < \alpha < j+2$ $(j = 1, 2, \ldots, k)$, $k$ is a positive integer.*

In order to complete the proof of the Theorem, we quote the following results due to G. Sunouchi [2], [3].

THEOREM A. *Let $u(\theta) \in L^p(0, 2\pi)$, $p > 1$, and its mean value be zero, then we have*

$$\|g^*(0, \beta\,; \theta)\|_p \leqq A_p \|u\|_p$$

*where $1/2 < \beta$ $(2 < p < \infty)$, $1/p < \beta$ $(1 < p < 2)$. We have also*

$$\|g^*(\alpha,\ 1\,; \theta)\|_p \leqq A_p \|u\|_p$$

*where $-\infty < \alpha < 1$.*

Now we can now complete the proof of the Theorem 1 combining Theorem A, Lemmas 4 and 5.

REMARK. The difference $\Delta_{t(k)}^{k+1} u_\alpha(\theta)$ in our theorem, may be replaced by $\Delta_t^{k+1} u_\alpha(\theta)$, since the contribution for the integral is influenced only by the behavior of $u(\theta)$ in the neighbourhood of the point $t = 0$.

Finally we prove a converse theorem of Theorem 1.

THEOREM 2. *Let $u(\theta) \in L^p(0, 2\pi)$, $p > 1$ and its mean value be zero, then we have*

$$B_{p,\alpha} \|u\|_p \leqq \|\delta(\alpha, 1\,; \theta)\|_p$$

*where $0 < \alpha < 2$.*

We begin to prove the following lemma.

LEMMA 6. *Under the assumption of Theorem 2, we have*

$$B_\alpha\, g(\alpha - 1\,; \theta) \leqq \delta(\alpha, 1\,; \theta)$$

PROOF OF LEMMA 6. Let

$$u_{\alpha-2}(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} u_\alpha(t)\, P_{\theta\theta}(r, \theta - t)\, dt$$

then, since $P_{tt}(r,t)$ is even function and $|P_{tt}(r,t)| < A|1-re^{it}|^{-3}$, we have

$$|u_{\alpha-2}(r,\theta)|^2 = \left| \frac{1}{2\pi}\int_0^{2\pi} \Delta^2_{t/2}\, u_\alpha(\theta)\, P_{tt}(r,t)\, dt \right|^2$$

$$\leqq A\int_0^{2\pi} |\Delta^2_{t/2}\, u_\alpha(\theta)|^2 |1-re^{it}|^{-3-\alpha}\, dt \int_0^{2\pi} |1-re^{it}|^{\alpha-3}\, dt$$

$$\leqq A(1-r)^{\alpha-2}\int_0^{2\pi} |\Delta^2_{t/2}\, u_\alpha(\theta)|^2 |1-re^{it}|^{-3-\alpha}\, dt,$$

provided that $\alpha < 2$. Hence it follows that

$$|g(\alpha-1;\theta)|^2 \leqq \int_0^1 (1-r)^{-\alpha+1}\, dr \int_0^{2\pi} |\Delta^2_{t/2}\, u_\alpha(\theta)|^2 |1-re^{it}|^{-3-\alpha}\, dt$$

$$\leqq A\int_0^{2\pi} |\Delta^2_{t/2}\, u_\alpha(\theta)|^2\, dt \int_0^1 (1-r)^{-\alpha+1}\, |1-re^{it}|^{-3-\alpha}\, dr.$$

Since

$$\int_0^1 (1-r)^{-\alpha+1}|1-re^{it}|^{-3-\alpha}\, dr \leqq At^{-2\alpha-1} \qquad (\alpha < 2),$$

we have

$$(g(\alpha-1;\theta))^2 \leqq A\int_0^{2\pi} |\Delta^2_{t/2}\, u_\alpha(\theta)|^2 t^{-2\alpha-1}\, dt$$

$$\leqq A(\delta(\alpha, 1;\ \theta))^2.$$

This is the required. Theorem 2 follows now immediately from Lemma 6 and the following theorem [1]:

THEOREM B. *Under the assumption of Theorem* 2, *we have*

$$B_{p,\alpha}\|u\|_p \leqq \|g(\alpha\,;\theta)\|_p$$

*for* $-1 < \alpha < \infty$.

## LITERATURE

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