HOMOGENEOUS RIEMANNIAN MANIFOLDS OF NEGATIVE CURVATURE

SHOSHICHI KOBAYASHI

(Received August 1, 1962)

In my recent note [1], I announced the following theorem:

THEOREM. *Let M be a homogeneous Riemannian manifold with non-positive sectional curvature and negative definite Ricci tensor. Then M is simply connected.*

B.O'Neill kindly pointed out to the author that the proof of the lemma used in [1] contains an error. The purpose of this paper is to give a complete proof of the above theorem together with the correct proof of the lemma.

Recently, Wolf [3] proved that if M is a homogeneous Riemannian manifold with non-positive sectional curvature, then M is isometric with the product of a flat torus and a simply connected homogeneous Riemannian manifold, thus giving an affirmative answer to qeustion (a) raised in my note [1].

Following Wolf we call an isometry of a Riemannian manifold a *Clifford translation* if the distance between a point and its image is the same for every point. The following lemma is due to Wolf [2]:

LEMMA 1. *Let M and N be Riemannian manifolds and p: $N \rightarrow M$ a locally isometric covering projection. If M is homogeneous, then any homeomorphism φ of N onto itself such that $p \circ \varphi = p$ is a Clifford translation of N.*

PROOF. Let G be a connected Lie group of isometries acting transitively on M and $\mathfrak{g}$ the Lie algebra of G. Considering every $X \in \mathfrak{g}$ as an infinitesimal isometry of M, let X^* be the lift of X to N. Then the set of all X^* thus obtained forms a Lie algebra of infinitesimal isometries of N, which will be denoted by $\mathfrak{g}^*$. Let G^* be the transitive Lie group of isometries of N generated by $\mathfrak{g}^*$. Since φ induces the identity transformation of M, it leaves every $X^* \in \mathfrak{g}^*$ invariant. Hence φ commutes with every element of G^*. For any two points y and y' of N, let ψ be an element of G^* such that $y' = \psi(y)$. Then

$$\begin{aligned} \text{distance } (y',\varphi(y')) &= \text{distance } (\psi(y),\ \varphi\circ\psi(y)) \\ &= \text{distance } (\psi(y),\ \psi\circ\varphi(y)) \\ &= \text{distance } (y,\varphi(y)). \end{aligned}$$

This completes the proof of Lemma 1.

For the proof of the following lemma, I am indebted to Wolf.

LEMMA 2. *Let M,N and φ be as in Lemma 1. Let $y_0 \in N$, $y_1 = \varphi(y_0)$ and $\tau^* = y_t$, $0 \leqq t \leqq 1$, be a minimizing geodesic from y_0 to y_1 where t is an affine parameter. Set $x_t = p(y_t)$. Then $\tau = x_t$, $0 \leqq t \leqq 1$, is a smooth closed geodesic, that is, the outgoing direction of τ at x_0 coincides with the incoming direction of τ at x_1.*

PROOF. Let r be a small positive number such that the r-neighborhoods $V(y_i;r)$ of y_i, $i = 0,1$, are homeomorphic with the r-neighborhood $U(x_0;r)$ of $x_0 = x_1$ by the projection p. Assume τ is not smooth at $x_0 = x_1$. Then there is a small positive number a, such that the point x_{1-a} and x_a can be joined by a curve σ in $U(x_0;r)$ whose length is less than the length of τ from x_{1-a} through $x_1 = x_0$ to x_a. Let σ^* be the curve in $V(y_1;r)$ such that $p(\sigma^*) = \sigma$. Let y^* be the end point of σ^*. Then $y^* = \varphi(y_a)$. The distance between y_a and y^* is at most the sum of the length of τ^* from y_a to y_{1-a} and the length of σ^*. Hence, we have

$$\text{distance } (y_a,\varphi(y_a)) = \text{distance } (y_a,y^*) < \text{distance } (y_0,y_1).$$

This contradicts Lemma 1.

In order to make the paper self-contained, I repeat the argument in my note [1]. Assuming that M is not simply connected, let N be the universal covering manifold of M and let $\tau = x_t$, $0 \leqq t \leqq 1$, be a smooth closed geodesic of M as in Lemma 2.

Let V be any infinitesimal isometry of M. We define a non-negative function $f(t)$, $-\infty < t < \infty$, as follows:

$$f(t) = \text{the square of the length of } V \text{ at } x_t \text{ for } 0 \leqq t \leqq 1,$$

and then extend it to a periodic function of period 1. By Lemma 2, $f(t)$ is differentiable for all values of t.

Let X be the vector field along τ tangent to τ. Let V' and V'' be the first and the second covariant derivatives of V in the direction of X. If we denote by g and R the metric tensor and the curvature tensor of M, then we have, for $0 \leqq t \leqq 1$,

$$\begin{aligned} f(t) &= g(V,V)_{x_t}, \\ f'(t) &= 2.g(V',V)_{x_t}, \\ f''(t) &= 2.g(V',V')_{x_t} - 2.g(R(V,X)X,V)_{x_t} \geqq 0, \end{aligned}$$

as the sectional curvature is non-positive. Since $f(t)$ is a periodic differentiable function and since $f''(t) \geqq 0$, $f(t)$ is a constant function. Hence, $f''(t) = 0$. In particular, $g(V',V')=0$ and $g(R(V,X)X,V)=0$.

On the other hand, if M is a homogeneous Riemannian manifold with negative definite Ricci tensor, there exists an infinitesimal isometry V of M such that $g(R(V,X)X,V)_{x_0} < 0$. This contradiction comes from the assumption that

M is not simply connected.

BIBLIOGRAPHY

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UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA.