NOTE ON THE CANONICAL DECOMPOSITION OF CONTRACTION

TEISHIRÔ SAITÔ AND TAKASHI YOSHINO

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1. The purpose of this note is to prove some results concerning spectra of contractions as applications of the canonical decomposition.

Let T be a contraction (i.e. $\|T\| \leqq 1$) on a Hilbert space $\boldsymbol{H}$. Then there exists a unitary operator U on a larger Hilbert space $\boldsymbol{K}(\supset H)$ such that

$$T^n = PU^n \qquad (n = 0, \pm 1, \pm 2, \cdots\cdots),$$

where P denotes the orthogonal projection of K onto $\boldsymbol{H}$, and $\boldsymbol{K}$ is spanned by all $U^n x\,(x \in \boldsymbol{H}\,;\ n = 0, \pm 1, \pm 2, \cdots)$. In this case, U is called a minimal unitary dilation of T. An operator T on a Hilbert space $\boldsymbol{H}$ is called completely non-unitary if for each non-zero $x \in \boldsymbol{H}$ there is an integer $n > 0$ such that either $\|T^n x\| \neq \|x\|$ or $\|T^{*n} x\| \neq \|x\|$ is true. A unitary operator $U = \int_0^{2\pi} e^{i\theta}\, dE_\theta$ on a Hilbert space $\boldsymbol{H}$ is called absolutely continuous if $(E_\theta x, y)$ is absolutely continuous with respect to Lebesgue measure on the unit circle. The main result in [6] is stated as follows.

THEOREM 1. (i) *For any contraction T on a Hilbert space $\boldsymbol{H}$, there corresponds a unique direct sum decomposition $\boldsymbol{H} = \boldsymbol{H}^{(u)} \oplus \boldsymbol{H}^{(o)}$ with the following properties*:

(a) *$\boldsymbol{H}^{(u)}$ and $\boldsymbol{H}^{(o)}$ reduce T.*

(b) *the restriction $T^{(u)} = T|\boldsymbol{H}^{(u)}$ is unitary part and the restriction $T^{(o)} = T|\boldsymbol{H}^{(o)}$ is completely non-unitary part.*

(ii) *A minimal unitary dilation of a completely non-unitary contraction is absolutely continuous.*

This decomposition is called the canonical decomposition of T.

By a unilateral shift operator with shifted space $\boldsymbol{H}$, we mean the operator S_H on $\widetilde{\boldsymbol{H}} = \left\{ (x_n)_{n=0}^{\infty}\,;\ x_n \in H,\ \sum_{n=0}^{\infty} \|x_n\|^2 < \infty \right\}$ defined by the following way:

$$S_H(x_0, x_1, x_2, \cdots\cdots) = (0, x_0, x_1, x_2, \cdots\cdots) \quad \text{for} \quad (x_n)_{n=0}^{\infty} \in \widetilde{\boldsymbol{H}}.$$

As a special case of Theorem 1 (i), the result of [2; Lemma 2. 1] is stated as follows.

THEOREM 2. *Let T be an isometric operator on a Hilbert space $\boldsymbol{H}$ and $T=T^{(u)}\oplus T^{(o)}$ the canonical decomposition of T. Then the completely non-unitary part $T^{(o)}$ is unitarily equivalent to a unilateral shift operator with shifted space $\boldsymbol{H}_0=\boldsymbol{H}\ominus T(\boldsymbol{H})$.*

2. Throughout this section, $\sigma(T), P_\sigma(T), A_\sigma(T), C_\sigma(T)$ and $R_\sigma(T)$ mean the spectrum, the point spectrum, the approximate point spectrum (see [3]), the continuous spectrum and the residual spectrum of an operator T respectively.

It is well-known that for a unilateral shift operator T $\sigma(T)=\{\lambda: |\lambda|\leqq 1\}$, but we obtain the following result.

LEMMA 1. *If T is a unilateral shift operator with shifted space $\boldsymbol{H}$, $1\in A_\sigma(T)$, that is, there exists a sequence $\{\widetilde{x}_m\}_{m=1}^\infty$, $\widetilde{x}_m=(x_n^{(m)})_{n=0}^\infty\in\widetilde{\boldsymbol{H}}$ such that $\|\widetilde{x}_m\|=1$ for all m and $\|T\widetilde{x}_m-\widetilde{x}_m\|\to 0$ as $m\to\infty$.*

PROOF. For each m, we define $\widetilde{x}_m=(x_n^{(m)})_{n=0}^\infty\in\boldsymbol{H}$ as follows:

$$x_0^{(m)}=\cdots=x_{m-1}^{(m)}=\frac{x}{\sqrt{m}} \text{ for a unit vector } x\in H, \text{ and } x_n^{(m)}=0$$

for $n\geqq m$.

Then we have $\|\widetilde{x}_m\|=1$ for each m and $\|T\widetilde{x}_m-\widetilde{x}_m\|=\sqrt{\dfrac{2}{m}}\to 0$ as $m\to\infty$, which completes the proof.

By Theorem 2 and Lemma 1, the following slight generalization of [1: Lemma 2] is easily proved.

LEMMA 2. *For every isometric operator T, $A_\sigma(T)\neq\emptyset$ and $A_\sigma(T)\subseteqq\{\lambda: |\lambda|=1\}$.*

PROOF. Since the assertion is trivial for a unitary operator, we may assume that T is not unitary. Then, in the canonical decomposition $T=T^{(u)}\oplus T^{(o)}$, $T^{(o)}\neq 0$. By Theorem 2, $T^{(o)}$ is unitarily equivalent to a unilateral shift operator, and so $A_\sigma(T)\neq\phi$ by Lemma 1. To prove the last part, let $\lambda\in A_\sigma(T)$ and $\{x_n\}$ a sequence of vectors such as $\|Tx_n-\lambda x_n\|\to 0$ $(n\to\infty)$. Then

$$|1-|\lambda||=|\|Tx_n\|-\|\lambda x_n\||\leqq\|Tx_n-\lambda x_n\|\to 0 \ (n\to\infty);$$

thus $|\lambda|=1$.

As an application of Theorem 1, we have the following theorem.

THEOREM 3. *Let T be a contraction on a Hilbert space $\boldsymbol{H}$ and U its minimal unitary dilation on a larger space $\boldsymbol{K} \supseteqq \boldsymbol{H}$. Then a complex number λ of modulus* 1 *belongs to $P_\sigma(T)$ if and only if $\lambda \in P_\sigma(U)$, and the corresponding proper vectors coincide.*

This theorem was proved in [5] in a different way.

PROOF OF THEOREM 3. To prove necessity part, let $Tx = \lambda x$ for $x \in H$. Then we have

$$\begin{aligned} \| Ux - \lambda x \| &= \| Ux \|^2 - (Ux, \lambda x) - (\lambda x, Ux) + \| \lambda x \|^2 \\ &= 2 \| x \|^2 - (PUx, \lambda x) - (\lambda x, PUx) \\ &= 2 \| x \|^2 - (Tx, \lambda x) - (\lambda x, Tx) \\ &= 2 \| x \|^2 - 2 \| x \|^2 = 0 \end{aligned}$$

and so $Ux = \lambda x$.

Conversely, suppose that $\lambda \in P_\sigma(U)$. We may assume that T is a non-unitary contraction. Now let $T = T^{(u)} \oplus T^{(o)}$ be the canonical decomposition and $U^{(o)}$ the minimal unitary dilation of $T^{(o)}$, then $U = T^{(u)} \oplus U^{(o)}$. Since $U^{(o)}$ is absolutely continuous, $\lambda \notin P_\sigma(U^{(o)})$ by [8], and so $\lambda \in P_\sigma(T^{(u)})$. Hence $\lambda \in P_\sigma(T)$. The last assertion is obvious from our above discussion.

As a consequence of Theorem 3 we have

THEOREM 4. *Let T be a contraction on a Hilbert space $\boldsymbol{H}$, U its minimal unitary dilation and C_0 the unit circle, then the following results hold.*

(i) $R_\sigma(T) \cap C_0 = \emptyset$

(ii) $A_\sigma(T^*) \cap C_0 = \overline{A_\sigma(T)} \cap C_0$, *where* $\overline{A_\sigma(T)} = \{\bar{\lambda} : \lambda \in A_\sigma(T)\}$.

(iii) *If T is completely non-unitary,* $\sigma(T) \cap C_0 \subseteqq C_\sigma(T)$.

PROOF. Ad. (i). If $\lambda \in R_\sigma(T) \cap C_0, \bar{\lambda} \in P_\sigma(T^*) \cap C_0 = P_\sigma(U^*)$ by Theorem 3, and so $\lambda \in P_\sigma(U) \cup R_\sigma(U)$. Since U is unitary, $R_\sigma(U) = \emptyset$, and thus $\lambda \in P_\sigma(U) = P_\sigma(T) \cap C_0$ by Theorem 3 which is a contradiction. Hence $R_\sigma(T) \cap C_0 = \emptyset$.

Ad. (ii). By (i), $\sigma(T) \cap C_0 = A_\sigma(T) \cap C_0$ and $\sigma(T^*) \cap C_0 = A_\sigma(T^*) \cap C_0$, and since $\sigma(T^*) = \overline{\sigma(T)}$, the result is obvious.

Ad. (iii). By (i) it is sufficient to show that $P_\sigma(T) \cap C_0 = \emptyset$. But, by

Theorem 3, $P_\sigma(T)\cap C_0=P_\sigma(U)$. Since T is completely non-unitary, $P_\sigma(U)=\emptyset$ by [8], and the proof is completed.

ADDED IN PROOF. The fact that $P_\sigma(T)\cap C_0=\emptyset$ for a completely non-unitary contraction is also proved by a simple calculation as follows. If $P_\sigma(T)\cap C_0\neq\emptyset$, there is a λ of modulus 1 and a non-zero vector x such as $Tx=\lambda x$. Then we have

$$T^n x=\lambda^n x \qquad \text{for all } n=0,1,2,\cdots.$$

On the other hand, for each integer $n>0$ $T^n x=\lambda^n x$ if and only if $T^{*n}=\bar{\lambda}^n x$ (see [7]). Hence we have

$$\| T^n x\| = \|\lambda^n x\| = \|x\| = \|\bar{\lambda}^n x\| = \|T^{*n}x\| \quad \text{for all} \quad n=0,1,2,\cdots.$$

which is a contradiction.

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TÔHOKU UNIVERSITY
AND
HACHINOHE TECHNICAL COLLEGE.