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ON THE CHARACTER RING OF REPRESENTATIONS OF A COMPACT GROUP

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In Pontrjagin's theory of duality for compact abelian groups, the following theorem is well known:

Let G be a compact abelian group, G^* the dual group. Then the topological dimension of G, in the sense of Lebesgue, is equal to the rank of discrete abelian group G^*.

In his paper [3], S. Takahashi has formulated the corresponding theorem in non-commutative case as follows.

THEOREM A. *Let G be an arbitrary compact group, $G^\wedge$ the aggregate of continuous finite dimensional representations of G, $C[G^\wedge]$ the algebra over the complex numbers C generated by the coefficients of representations in $G^\wedge$, i.e., the representative ring of G in the sense of C. Chevalley in* [1]. *Then the topological dimension of G, in the sense of Lebesgue, is equal to the transcendental degree of $C[G^\wedge]$ over C.*

Another form of corresponding theorem is the following:

THEOREM B. *Let $\overline{G}$ be the space consisting of conjugate classes of a compact group G, G^* the characters of representations in $G^\wedge$, $C[G^*]$ the algebra over C generated by G^*. Then the topological dimension of $\overline{G}$ is equal to the transcendental degree of $C[G^*]$ over C.*

Theorem A was affirmatively solved, but Theorem B was merely justified for connected compact Lie groups, reducing it to the following fact; the transcendental degree of $C[G^*]$ over C is equal to the rank of G, i.e., the topological dimension of a maximal abelian subgroup of G. Prof. T. Tannaka has called my attention to justify Theorem B for arbitrary compact groups. Unfortunately we have not succeeded to prove it by now, accordingly we wish to content ourselves with proving the following fact:

THEOREM C. *Let G be a compact group with the finite topological* *dimension, r the maximum number of topological dimension of abelian subgroups in G. Then the transcendental degree of $C[G^*]$ over C is equal to r.*

The purpose of the present paper is to give the proof for Theorem C.

1. Notations. We shall use following notations for arbitrary compact group.

$\dim(G)$: the topological dimension of G in the sense of Lebesgue.

$r(G)$: the maximum number of topological dimension of abelian subgroups of G. If G is a connected Lie group, $r(G)$ is equal to the topological dimension of space $\overline{G}$ consisting of conjugate classes of G.

$C[G^*|S]$ ($C[G^\wedge|S]$) for any subset S of G: the algebra over C consisting of restriction of elements of character ring $C[G^*]$ ($C[G^\wedge]$) to S.

$r(G^*) = \langle C[G^*] : C \rangle$: the transcendental degree of $C[G^*]$ over C.

$r(G^*|S)$ for any subset S of G: the transcendental degree of $C[G^*|S]$ over C.

2. Reduction of the theorem to a connected case.

LEMMA 1. *Let G be a compact topological group, G_0 the connected component of unit element of G. Then it holds $r(G) = r(G_0)$.*

PROOF. Let T be any abelian subgroup of G, T_0 the connected component of unit element of T. Since T, T_0 are compact subgroups, they are projective limit of compact Lie groups, i.e., $T = \varprojlim T_\alpha$, $T_0 = \varprojlim T_{\alpha,0}$, $\alpha \in \Lambda$ where T_α are compact Lie groups, and for each α, $T_{\alpha,0}$ is a connected component of unit element of T_α. Since $\dim T$ ($\dim T_0$) is the maximum number of $\dim T_\alpha$ ($\dim T_{\alpha,0}$), and $\dim T_\alpha = \dim T_{\alpha,0}$, it holds $\dim T = \dim T_0$. Another one, T_0 is contained in G_0, therefore, $r(G) = r(G_0)$. q.e.d.

In order to reduce our theorem to a connected case we must prove $r(G^*) = r(G_0^*)$ and we shall begin with proving this equality for compact Lie groups.

LEMMA 2. *Let G be a compact Lie group, G_0 connected component of unit element of G. Then it holds $r(G_0^*) = r(G^*|G_0)$*

Clearly it holds $C[G_0^*]) \supseteq C[G^*|G_0]$. Let χ be an irreducible character of a continuous finite dimensional representation of G_0, D a matrix representation corresponding to χ. Let

$$G = a_1 G_0 + a_2 G_0 + \cdots + a_n G_0$$

be the coset decomposition of G with respect to G_0. We now extend the domain of definition of D as follows:

$$\dot{D}(y) = \begin{cases} D(y) & y \in G \\ 0 & y \notin G. \end{cases}$$

Then D^G affords a matrix representation D^G given by

$$D^G(x) = (\dot{D}(a_j^{-1} x a_i))_{1 \leqq i, j \leqq n}, \quad x \in G.$$

Then the induced character χ^G obtained from this satisfies

$$\chi^G(x) = \sum_{i=1}^{n} \dot{\chi}(a_i^{-1} x a_i), \quad x \in G.$$

Let $I(G)$ be the set of inner automorphisms of G, and for $\sigma \in I(G)$, $f \in C[G_0^*]$, we define f^σ by

$$f^\sigma(x) = f(a^{-1} x a) \quad \text{for all} \quad x \in G_0$$

where σ means a mapping $x \to a^{-1}xa$. Then $I(G_0)$ is a normal subgroup of $I(G)$, and $I(G)/I(G_0)$ is a finite group. Since $f^\sigma = f$ for any $\sigma \in I(G_0)$, it is natural to think that $\bar{\sigma}$ is an element of $I(G)/I(G_0)$ operating on $C[G_0^*]$. If a character χ an element of $C[G_0^*]$ is invariant under the operation of any $\bar{\sigma} \in I(G)/I(G_0)$, it holds

$$\chi^G(x) = \sum_{i=1}^{n} \chi(a_i^{-1} x a_i) = n\chi(x) \quad \text{for all } x \in G_0,$$

that is, if we set $\chi^G | G_0$ the restriction of χ^G to G_0, then it holds the following relation:

$$\frac{1}{n} \chi^G | G_0 = \chi, \text{ i.e., } \chi \in C[G^* | G_0].$$

Let χ be any character element of $C[G_0^*]$, and we consider the following equation:

$$(X - \chi^{\bar{\sigma}})(X - \chi^{\bar{\tau}}) \cdots (X - \chi^{\bar{\rho}}) = 0$$

where $I(G)/I(G_0)=\{\bar{\sigma}, \bar{\tau}, \cdots, \bar{\rho}\}$, and X unknown element. Then χ is a root of this equation, and each coefficients of X^i, $i=1,2,\cdots,n$ are characters belonging to $C[G_0^*]$ and are invariant under the operation of $\bar{\sigma}\in I(G)/I(G_0)$, therefore their each coefficient belongs to $C[G^*|G_0]$, that is, χ is algebraic over $C[G^*|G_0]$. In S. Takahashi [3] Lemma 2, it is proved that $C[G_0^*]$ has no zero-divisor, then $C[G_0^*]$ is algebraic over $C[G^*|G_0]$, i.e., $r(G_0^*) = r(G^*|G_0)$. q. e. d.

LEMMA 3. *Let G be a compact Lie group, G_0 connected component of unit element of G, and let the coset decomposition of G with respect to G_0 be*

$$G = G_1 + G_2 + \cdots + G_n .$$

If it holds $r(G^*|G_i)\leqq m$, $i = 1,2,\cdots,n$, then we have $r(G^*)\leqq m$.

PROOF. Let $f_1,f_2,\cdots,f_{m+1}$ be arbitrary elements of $C[G^*]$. Since $f_1(x),f_2(x),\cdots,f_{m+1}(x)$, $x\in G_i$ are algebraically dependent, there exists a non-trivial polynomial $F_i(X_1, X_2,\cdots, X_{m+1})$ in the polynomial ring $C[X_1, X_2, \cdots, X_{m+1}]$ over C generated by indeterminates $X_1, X_2,\cdots, X_{m+1}$ such that:

$$F_i(f_1(x),f_2(x),\cdots,f_{m+1}(x)) = 0, \quad x\in G_i .$$

Therefore

$$\prod_{i=1}^{n} F_i(f_1(x),f_2(x),\cdots,f_{m+1}(x)) = 0, \quad x\in G.$$

This means that $f_1,f_2,\cdots,f_{m+1}$ are algebraically dependent, i.e., $r(G^*)\leqq m$. q. e. d.

LEMMA 4. *Let G, G_i $(i = 1,2,\cdots,n)$ be as in Lemma 3. If U is any open subset of G_i, then $r(G^*|U) = r(G^*|G_i)$.*

PROOF. Clearly it holds $r(G^*|U)\leqq r(G^*|G_i)$. Let $r = r(G^*|U)$, and f_1, $f_2,\cdots,f_{r+1}$ any elements in $C[G^*|G_i]$. Then there exists a non-trivial polynomial $F(X_1, X_2,\cdots, X_{r+1})$ in $C[X_1, X_2,\cdots, X_{r+1}]$ such that:

$$F(f_1(x),f_2(x),\cdots,f_{r+1}(x)) = 0 \quad x\in U .$$

This polynomial F is an analytic function on G_i. Since G_i is connected, this equality holds everywhere on G_i, i.e., $r(G^*|U)\geqq r(G^*|G_i)$. q. e. d.

LEMMA 5. *Let* G, G_i $i = 1, 2, \cdots, n$ *be as in Lemma* 3, $C[G^\wedge]$ *the representative ring,* $C[G^\wedge | G_i]$ *the restriction of* $C[G^\wedge]$ *to* G_i. *Then* $C[G^\wedge | G_i]$ *has no zero divisor.*

PROOF. Assume $f_1(x) f_2(x) = 0$ everywhere on G_i, where f_1, f_2 are two functions belonging to $C[G^\wedge | G_i]$. We must then show that at least one of f_1, f_2 is zero everywhere on G_i. Now as f_1, f_2 are analytic functions on G_i, by property of analytic functions at least one of f_1, f_2 is zero in a sufficiently small open set of G_i. Since G_i is connected, this holds everywhere on G_i.

LEMMA 6. *Let* G, G_0 *be as in Lemma* 2, T *a maximal abelian subgroup of* G_0. *Then it holds* $r(G_0^*) = r(T^*)$.

PROOF. See S. Takahashi [3] Theorem B.

LEMMA 7. *Let* G, G_0 *be as in Lemma* 2. *Then it holds* $r(G^*) = r(G_0^*)$.

PROOF. We have clearly $r(G^*) \geqq r(G^* | G_0)$, accordingly it holds by Lemma 2,

$$r(G^*) \geqq r(G_0^*) .$$

Let the coset decomposition of G with respect to G_0 be

$$G = G_1 + G_2 + \cdots + G_n ,$$

and T be a maximal abelian subgroup of G_0. In order to deduce $r(G^*) \leqq r(G_0^*)$, it is sufficient to prove $r(G^* | G_i) \leqq r(T^*)$ for $i = 1, 2, \cdots, n$, by Lemmas 3 and 6.

We put now $r(T^*) = r$, and take out arbitrary $r+1$ elements of character $\chi_1, \chi_2, \cdots, \chi_{r+1} \in C[G^* | G_i]$. Let each corresponding matrix representations be $D_1, D_2, \cdots, D_{r+1}$, that is, $D_l(a^l_{\alpha\beta}(x))$ $l = 1, 2, \cdots, r+1$, $a^l_{\alpha\beta}(x) \in C[G^\wedge | G_i]$ for $x \in G_i$, and let the characteristic equation of D_l be

$$\Phi_l(X) = X^{n_l} + F^l_1 X^{n_l-1} + \cdots + F^l_{n_l} = 0 ,$$

where $F^l_\gamma \in C[G^\wedge | G_i]$. If there is a reducible characteristic equation $\Phi_l = 0$ over $C[G^\wedge | G_i]$, we decompose this equation into irreducible equations over $C[G^\wedge | G_i]$, $\Phi^1_l = 0$, $\Phi^2_l = 0, \cdots, \Phi^{s_l}_l = 0$. Since $C[G^\wedge | G_i]$ is the integral domain by Lemma 5, the discriminants $\mathscr{D}^\delta_l$ of $\Phi^\delta_l = 0$ are all non-zero and belong to $C[G^\wedge | G_i]$, that is:

$$\mathcal{D}_1^1 \cdots \mathcal{D}_1^{s_1} \mathcal{D}_2^1 \cdots \mathcal{D}_2^{s_2} \cdots \mathcal{D}_{r+1}^1 \cdots \mathcal{D}_{r+1}^{s_{r+1}} \neq 0 .$$

Then there exists an element g in G_i such that $\mathcal{D}_l^\delta(g) \neq 0$ for $l = 1, 2, \cdots, r+1$, $\delta = 1, 2, \cdots, s_l$. Therefore, since elements of $C[G^\wedge | G_i]$ are analytic on G_i, there exists a neighborhood U of g such that $U \subset G_i$, and roots $\lambda_l^j(x)$ $(j = 1, 2, \cdots, n_l)$ of $\Phi_l(x) = 0$ $(l = 1, 2, \cdots, r+1)$ for $x \in U$ are analytic on U. Let $C[\lambda]$ be the algebra generated by $\lambda_l^j(x)$ $x \in U$ over C. In the proof of Lemma 5, it is clear that $C[\lambda]$ is the integral domain and contain $\chi_1(x)$, $\chi_2(x), \cdots, \chi_{r+1}(x)$, $x \in U$. Now we may assume that the representation of the maximal abelian subgroup T by D_l is diagonal :

$$D_l(t) = \begin{pmatrix} h_l^1(t) & & & 0 \\ & h_l^2(t) & & \\ & & \ddots & \\ 0 & & & h_l^{n_l}(t) \end{pmatrix} \qquad t \in T .$$

Since for each $x \in U$, $\{\lambda_l^j(x)\}^n$ is an eigen-value of a matrix $D_l(x^n)$, we set $\lambda_l^j(x^n) = \{\lambda_l^j(x)\}^n$. Since the order of G/G_0 is n, we have $x^n \in G_0$ for $x \in U$, and there exists an element g in G_0 with $g^{-1}x^n g = t \in T$. Therefore, for any λ_l^j, $x \in U$, there exist $t \in T$ and h_l^k with

$$\lambda_l^j(x^n) = h_l^k(t) .$$

Now we shall show that the transcendental degree of $C[\lambda]$ over C is at most r. By taking arbitrary $r+1$ elements of eigen-values λ_λ^j from $C[\lambda]$, we can choose without loss of generality

$$\lambda_1^1, \lambda_1^2, \cdots, \lambda_1^{s_1}, \lambda_2^1, \cdots, \lambda_2^{s_2}, \cdots, \lambda_m^1, \cdots, \lambda_m^{s_m} ,$$

where $s_1 + s_2 + \cdots + s_m = r+1$. Let Ψ_l, $l = 1, 2, \cdots, m$ be the aggregates of combinations $(i_1, i_2, \cdots, i_{s_l})$ to choose s_l elements from $1, 2, \cdots, n_l$. Since the transcendental degree of $C[T^*]$ over C is r, there exist non-trivial polynomials $F_{\alpha_1, \alpha_2, \cdots, \alpha_m}(X_1, X_2, \cdots, X_{r+1})$ in the polynomial ring $C[X_1, X_2, \cdots, X_{r+1}]$ generated by indeterminates $X_1, X_2, \cdots, X_{r+1}$ over C such that :

$$F_{\alpha_1, \alpha_2, \cdots, \alpha_m}(h_1^{i_1}(t), h_1^{i_2}(t), \cdots, h_1^{i_{s_1}}(t), h_2^{k_1}(t), \cdots, h_2^{k_{s_2}}(t), \cdots, h_m^{t_1}(t), \cdots, h_m^{t_{s_m}}(t)) = 0$$

there by $t \in T$, $\alpha_l \in \Psi_l$, $\alpha_1 = (i_1, \cdots, i_{s_1})$, $\alpha_2 = (k_1, \cdots, k_{s_2}), \cdots$, $\alpha_m = (t_1, \cdots, t_{s_m})$. Let Θ_l be the aggregates of permutations $\begin{pmatrix} 1, 2, \cdots, s_l \\ j_1, j_2, \cdots, j_{s_l} \end{pmatrix}$ of s_l elements and we set

$$F^{\beta_1,\beta_2,\cdots,\beta_m}_{\alpha_1,\alpha_2,\cdots,\alpha_m}(\lambda_1^1,\cdots,\lambda_1^{s_1},\ \lambda_2^1,\cdots,\lambda_2^{s_2},\cdots,\lambda_m^1,\cdots,\lambda_m^{s_m})$$
$$= F_{\alpha_1,\alpha_2,\cdots,\alpha_m}(\lambda_1^{j_1},\cdots,\lambda_1^{j_{s_1}},\ \lambda_2^{k_1},\cdots,\lambda_2^{k_{s_2}},\cdots,\lambda_m^{t_1},\cdots,\lambda_m^{t_{s_m}})$$

where

$$\alpha_l\in\Theta_l,\quad \beta_1=\begin{pmatrix}1,2,\cdots,s_1\\ j_1,j_2,\cdots,j_{s_1}\end{pmatrix},\ \beta_2=\begin{pmatrix}1,2,\cdots,s_2\\ k_1,k_2,\cdots,k_{s_2}\end{pmatrix},\ \cdots,\ \beta_m=\begin{pmatrix}1,2,\cdots,s_m\\ t_1,t_2,\cdots,t_{s_m}\end{pmatrix}.$$

As $g^{-1}x^ng=t\in T$, we have

$$\prod_{\alpha_l\in\Psi_l,\beta_l'\in\Theta_l'} F^{\beta_1,\beta_2,\cdots,\beta_m}_{\alpha_1,\alpha_2,\cdots,\alpha_m}(\lambda_1^1(x^n),\cdots,\lambda_1^{s_1}(x^n),\lambda_2^1(x^n),\cdots,\lambda_2^{s_2}(x^n),$$
$$\cdots,\lambda_m^1(x^n),\cdots,\lambda_m^{s_m}(x^n))=0\qquad x\in U.$$

Since it holds $\lambda_l^j(x^n)=(\lambda_l^j(x))^n$ for $x\in U$, $\lambda_1^1,\lambda_1^2,\cdots,\lambda_1^{s_1},\ \lambda_2^1,\cdots,\lambda_2^{s_2},\cdots,\lambda_m^1,\cdots,\lambda_m^{s_m}$ are algebraically dependent, i.e., the transcendental degree of $C[\lambda]$ over C is at most r. Hence $\chi_1(x),\chi_2(x),\cdots,\chi_{r+1}(x)$, $x\in U$ are algebraically dependent, Since $C[G^*|U]$ is the integral domain, we have $r(G^*|U)\leqq r$. By Lemmas 3 and 4 we then have $r(G^*)\leqq r$. q.e.d.

Now we shall refer to the reduction theory under compact topological groups.

LEMMA 8. *Let G be a compact topological group, G_0 the connected component of unit element of G. Then we have $r(G^*)=r(G_0^*)$.*

PROOF. We set $n=r(G^*)$ and $n_0=r(G_0^*)$, and let $f_1,f_2,\cdots,f_{n_0+1}$ be any n_0+1 elements in $C[G^*]$. Then there is a compact Lie group G' such that G is homomorphic to G' and $f_1,f_2,\cdots,f_{n_0+1}$ are defined on G', i.e., they belong to $C[G'^*]$. Since G_0 is mapped onto a connected component G_0' of unit element of G', $f_1,f_2,\cdots,f_{n_0+1}$ are algebraically dependent, by Lemma 7, hence $n\leqq n_0$. On the other hand, let $f_1,f_2,\cdots,f_{n+1}$ be $n+1$ elements in $C[G_0^*]$. Since by Van Kampen's theorem, any irreducible representation of G_0 is contained in the restriction of a representation of G, there is a compact Lie group G' such that, if G_0' denotes the connected component of unit of G', then $f_1,f_2,\cdots,f_{n+1}$ are defined on G_0', that is to say $f_i\in C[G_0'^*]$, $i=1,2,\cdots,n+1$. Since $C[G'^*]\subset C[G^*]$, $f_1,f_2,\cdots,f_{n+1}$ are algebraically dependent by Lemma 7, so that, we have $n\geqq n_0$.

3. On compact connected topological groups. Let G be a compact connected finite dimensional topological group. Then in Pontrjagin [4] example 107, it is shown that G is isomorphic to $(L\times H)/D$, where L is a compact simply connected semi-simple Lie group, and H a compact connected abelian group, $\dim H < \infty$, and D is a finite normal subgroup contained in the center of the direct product $L \times H$, $H \cap D = \{e\}$.

LEMMA 9. *Let G, L, H be as above. Then* $r(G^*) = r(L^*) + r(H^*)$.

PROOF. See S. Takahashi [3] Lemma 5.

LEMMA 10. *Let G, L, H, D be as in Lemma 9, T_G a maximal abelian subgroup of G, T_L a maximal abelian subgroup of L. Then it holds* $\dim T_G = \dim T_L + \dim H$.

PROOF. By applying Theorem A, we have easily that $\dim T_L + \dim H \leqq \dim T_G$. Let f be the canonical mapping of $L \times H$ onto G, and we set $f(L) = L'$, $f(H) = H'$. Then it holds $G = L' \cdot H'$ where $L' \cap H'$ is a finite normal subgroup of G, and H' is contained in the center of G. Then it holds

$$G/H' = L' \cdot H'/H' \cong L'/L' \cap H'$$

where $L'/L' \cap H'$ is a connected semi-simple Lie group. Let A be a maximal abelian subgroup of $L'/L' \cap H'$, then it holds clearly $\dim T_L = \dim A$. T_G/H' is isomorphic to a subgroup of A, so that, T_G/H' is a compact Lie group. Then by Pontrjagin [4] Theorem 69, we have $\dim (T_G/H') = \dim T_G - \dim H'$, that is to say $\dim T_G \leqq \dim H' + \dim T_L$. Since $\dim H = \dim H'$, we have $\dim T_G = \dim H + \dim T_L$.

In S. Takahashi [3], it is shown that $\dim T_L = r(L^*)$, $\dim H = r(H^*)$. Therefore by Lemmas 9 and 10, the following theorem is established:

THEOREM. *Let G be a compact connected finite dimensional topological group, T a maximal abelian subgroup. Then it holds* $r(G^*) = \dim T$.

Theorem C is now completely proved by the above theorem and Lemmas 1 and 8.

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