Tôhoku Math. Journ.
21(1969), 56-64.

ON CONFORMAL KILLING TENSOR IN A RIEMANNIAN SPACE

SHUN-ICHI TACHIBANA

(Received July 5, 1968)

0. Let M^n be an n dimensional Riemannian space. A vector field v_a[1] is called a Killing vector if it satisfies the Killing's equation:

$$\nabla_a v_b + \nabla_b v_a = 0,$$

where ∇_a means the operator of the covariant derivation with respect to the Riemannian connection. A Killing tensor v_{bc} is, by definition, a skew symmetric tensor satisfying the Killing-Yano's equation:

$$\nabla_a v_{bc} + \nabla_b v_{ac} = 0. \tag{0.1}$$

In recent papers[2] we discussed the integrability condition of the equation (0.1) and determined such tensors completely in the Euclidean space and the sphere.

A conformal Killing vector u_b is a vector field satisfying

$$\nabla_a u_b + \nabla_b u_a = 2\rho g_{ab}, \tag{0.2}$$

where ρ is a scalar function and g_{ab} the Riemannian metric. As for a generalization of such a vector it is not suitable to define a conformal Killing tensor as a skew symmetric tensor field u_{bc} satisfying

$$\nabla_a u_{bc} + \nabla_b u_{ac} = 2\rho_c g_{ab},$$

where ρ_c is a certain vector field. Because we can easily show that a conformal Killing tensor in this sense is a Killing tensor, i.e., we have $\rho_c = 0$. Thus this definition of conformal Killing tensor is meaningless.

In this paper we shall define a conformal Killing tensor in another way and generalize some results about a conformal Killing vector to the conformal Killing tensor. The definition which we shall adopt is suggested by the following fact. A parallel vector field in the Euclidean space E^{n+1} induces a

1) We adopt the identification of a vector field with a 1-form by virtue of the Riemannian metric.

2) S. Tachibana [1], S. Tachibana and T. Kashiwada [2].

conformal Killing vector on the sphere S^n of constant curvature. Thus a tensor field on S^n induced from a parallel tensor field in E^{n+1} is to be a model of conformal Killing tensor.

We shall concern only with tensor of degree 2 and the general case will be discussed in Kashiwada's forthcoming paper [6].

1. Preliminaries. Consider an n dimensional Riemannian space M^n whose Riemannian metric is given by g_{bc} with respect to local coordinates $\{x^a\}$[3].

Let $R_{abc}{}^d$ be the Riemannian curvature tensor. Then Ricci's identity for any tensor $u_{ab}{}^e$ is given by

$$\nabla_a\nabla_b u_{cd}{}^e - \nabla_b\nabla_a u_{cd}{}^e = -R_{abc}{}^f u_{fd}{}^e - R_{abd}{}^f u_{cf}{}^e + R_{abf}{}^e u_{cd}{}^f .$$

Especially we obtain the following formula for any skew symmetric tensor u_{bc},

$$\begin{aligned}(1.1)\qquad 2\nabla_b\nabla_c u^{bc} &= \nabla_b\nabla_c u^{bc} - \nabla_c\nabla_b u^{bc} = R_{bce}{}^b u^{ec} + R_{bce}{}^c u^{be} \\ &= R_{ce} u^{ec} - R_{be} u^{be} = 0,\end{aligned}$$

where $R_{ce} = R_{bce}{}^b$ is the Ricci tensor.

The conformal curvature tensor $C_{abc}{}^d$ is defined by

$$\begin{aligned}C_{abc}{}^d = R_{abc}{}^d &+ \frac{1}{n-2}(R_{ac}\delta_b{}^d - R_{bc}\delta_a{}^d + g_{ac}R_b{}^d - g_{bc}R_a{}^d) \\ &- \frac{R}{(n-1)(n-2)}(g_{ac}\delta_b{}^d - g_{bc}\delta_a{}^d),\end{aligned}$$

where R denotes the scalar curvature.

If the tensor $C_{abc}{}^d$ vanishes identically, then $M^n(n>3)$ is called to be conformally flat.

A space of constant curvature $(n>2)$ is a Riemannian space satisfying

$$R_{abc}{}^d = k(g_{bc}\delta_a{}^d - g_{ac}\delta_b{}^d)$$

and then k is a constant given by $k=R/n(n-1)$.

A space of constant curvature is necessarily conformally flat.

2. Conformal Killing tensor. We shall call a skew symmetric tensor u_{cd} a *conformal Killing tensor* if there exists a vector field ρ_c such that

3) Indices $a, b, \cdots$ run over $1, \cdots, n$. Throughout this paper we assume that $n>3$.

$$(2.1)\qquad \nabla_b u_{cd} + \nabla_c u_{bd} = 2\rho_d g_{bc} - \rho_b g_{cd} - \rho_c g_{bd}.$$

We call ρ_c the associated vector of u_{cd}. And if ρ_c vanishes identically, then u_{cd} is called a Killing tensor.[4]

First we shall seek for differential equations of second order satisfied by u_{cd}.

Transvecting (2.1) with g^{bc}, we have

$$(2.2)\qquad \nabla^b u_{bd} = (n-1)\rho_d,$$

where $\nabla^b = g^{bc}\nabla_c$. Taking account of (1.1) it follows that

$$(2.3)\qquad \nabla^c\nabla^b u_{bc} = 0, \qquad \nabla^c \rho_c = 0.$$

In the following we shall write ρ_{ab} instead of $\nabla_a \rho_b$ for brevity.

Operating ∇_a to (2.1) we get

$$(2.4)\qquad \nabla_a\nabla_b u_{cd} + \nabla_a\nabla_c u_{bd} = 2\rho_{ad} g_{bc} - \rho_{ab} g_{cd} - \rho_{ac} g_{bd}.$$

By interchanging indices a, b, c as $a \to b \to c \to a$ in this equation we obtain the following two equations:

$$(2.5)\qquad \nabla_b\nabla_c u_{ad} + \nabla_b\nabla_a u_{cd} = 2\rho_{bd} g_{ca} - \rho_{bc} g_{ad} - \rho_{ba} g_{cd},$$

$$(2.6)\qquad \nabla_c\nabla_a u_{bd} + \nabla_c\nabla_b u_{ad} = 2\rho_{cd} g_{ab} - \rho_{ca} g_{bd} - \rho_{cb} g_{ad}.$$

If we form (2.4)+(2.5)−(2.6), then it follows that

$$(2.7)\qquad 2\nabla_a\nabla_b u_{cd} - 2R_{cba}{}^e u_{de} - R_{bad}{}^e u_{ce} - R_{acd}{}^e u_{be} - R_{bcd}{}^e u_{ae}$$
$$= 2(\rho_{ad} g_{bc} + \rho_{bd} g_{ca} - \rho_{cd} g_{ab}) + (\rho_{cb} - \rho_{bc}) g_{ad} + (\rho_{ca} - \rho_{ac}) g_{bd}$$
$$- (\rho_{ab} + \rho_{ba}) g_{cd}.$$

We shall deform (2.7) into another form. By $b \to c \to d \to b$ in (2.7) we have

$$(2.8)\qquad 2\nabla_a\nabla_c u_{db} - 2R_{dca}{}^e u_{be} - R_{cab}{}^e u_{de} - R_{adb}{}^e u_{ce} - R_{cdb}{}^e u_{ae}$$
$$= 2(\rho_{ab} g_{cd} + \rho_{cb} g_{da} - \rho_{db} g_{ac}) + (\rho_{dc} - \rho_{cd}) g_{ab} + (\rho_{da} - \rho_{ad}) g_{cb}$$
$$- (\rho_{ac} + \rho_{ca}) g_{db},$$

4) S. Tachibana, [1].

$$\text{(2.9)} \qquad 2\nabla_a\nabla_d u_{bc} - 2R_{bda}{}^e u_{ce} - R_{dac}{}^e u_{be} - R_{abc}{}^e u_{de} - R_{dbc}{}^e u_{ae}$$
$$= 2(\rho_{ac}g_{db} + \rho_{dc}g_{ba} - \rho_{bc}g_{ad}) + (\rho_{bd} - \rho_{db})g_{ac} + (\rho_{ba} - \rho_{ab})g_{dc}$$
$$- (\rho_{ad} + \rho_{da})g_{bc}.$$

Adding (2.8) and (2.9) to (2.7) side by side we can get

$$\text{(2.10)} \qquad 2\nabla_a\nabla_b u_{cd} + R_{bca}{}^e u_{de} + R_{dba}{}^e u_{ce} + R_{cda}{}^e u_{be}$$
$$= (\rho_{bd} - \rho_{db})g_{ac} + (\rho_{dc} - \rho_{cd})g_{ab} + (\rho_{cb} - \rho_{bc})g_{ad} + 2\rho_{ad}g_{bc}$$
$$- 2\rho_{ac}g_{bd},$$

where we have used the following equations which follows from (2.1):

$$\nabla_a\nabla_b u_{cd} + \nabla_a\nabla_c u_{db} + \nabla_a\nabla_d u_{bc} = 3(\nabla_a\nabla_b u_{cd} + \rho_{ac}g_{bd} - \rho_{ad}g_{bc}).$$

Next we shall obtain algebraic relations between components of u_{cd} and the curvature tensor, ((2.14) below).

First by subtraction (2.7) from (2.10) we can get

$$\text{(2.11)} \qquad R_{bca}{}^e u_{de} + R_{adb}{}^e u_{ce} + R_{dac}{}^e u_{be} + R_{cbd}{}^e u_{ae}$$
$$= \sigma_{bd}g_{ca} + \sigma_{ca}g_{bd} - \sigma_{cd}g_{ab} - \sigma_{ab}g_{cd},$$

where we have put

$$\sigma_{bd} = \rho_{bd} + \rho_{db}.$$

Transvecting (2.11) with g^{ab} and making use of

$$R_{dbce}u^{be} + R_{cbde}u^{be} = 0,$$

we obtain

$$\text{(2.12)} \qquad \sigma_{bd} = \frac{1}{(n-2)}(R_c{}^e u_{de} + R_d{}^e u_{ce}).$$

We substitute (2.12) into (2.11) and put

$$\text{(2.13)} \qquad T_{bca}{}^e = (n-2)R_{bca}{}^e - R_b{}^e g_{ca} + R_c{}^e g_{ba},$$

so it follows that

$$(2.14)\qquad (T_{bca}{}^{e}\delta_{d}{}^{f}+T_{adb}{}^{e}\delta_{c}{}^{f}+T_{dac}{}^{e}\delta_{b}{}^{f}+T_{cbd}{}^{e}\delta_{a}{}^{f})u_{fe}=0.$$

Now we shall show the following

THEOREM 1.[5] *If there exists* (*locally*) *a conformal Killing tensor which takes any preassigned* (*skew symmetric*) *value at any point of an* n (>3) *dimensional Riemannian space, then the space is conformally flat.*

PROOF. Under the assumption as the skew symmetric parts of coefficients of u_{fe} in (2.14) vanish, we have

$$\begin{aligned}T_{bca}{}^{e}\delta_{d}{}^{f}+T_{adb}{}^{e}\delta_{c}{}^{f}+T_{dac}{}^{e}\delta_{b}{}^{f}+T_{cbd}{}^{e}\delta_{a}{}^{f}&\\ =T_{bca}{}^{f}\delta_{d}{}^{e}+T_{adb}{}^{f}\delta_{c}{}^{e}+T_{dac}{}^{f}\delta_{b}{}^{e}+T_{cbd}{}^{f}\delta_{a}{}^{e}.&\end{aligned}$$

Contracting d and f in this equation we get

$$T_{bca}{}^{e}=\left(-R_{ab}+\frac{R}{n-1}g_{ab}\right)\delta_{c}{}^{e}+\left(R_{ac}-\frac{R}{n-1}\,g_{ac}\right)\delta_{b}{}^{e}.$$

Substituting this into (2.13) it follows that $C_{bca}{}^{e}=0$. Q.E.D.

3. A sufficient condition to be a conformal Killing tensor. Let u_{cd} be a conformal Killing tensor. Then we can get

$$(3.1)\qquad \nabla^{a}\nabla_{a}u_{cd}-R_{c}{}^{e}u_{de}-R^{b}{}_{cd}{}^{e}u_{be}=-(n-3)\rho_{cd}-\rho_{dc},$$

by transvection (2.7) with g^{ab}. Taking the skew symmetric part of (3.1), we have the following equations:

$$(3.2)\qquad 2\nabla^{a}\nabla_{a}u_{cd}-R_{c}{}^{e}u_{de}+R_{d}{}^{e}u_{ce}-R_{dc}{}^{be}u_{be}=(n-4)(\rho_{dc}-\rho_{cd}).$$

In this section we shall show that a skew symmetric tensor u_{cd} satisfying (3.1) or (3.2) is a conformal Killing tensor provided that M^{n} is compact. To this purpose we prepare an integral formula about a tensor field.

Define a tensor A_{bcd} by

$$(3.3)\qquad A_{bcd}=\nabla_{b}u_{cd}+\nabla_{c}u_{bd}-2\rho_{d}g_{bc}+\rho_{b}g_{cd}+\rho_{c}g_{bd}$$

for a skew symmetric tensor u_{cd}, where ρ_{c} is given by

5) Analogous theorem is well known for a conformal Killing vector. As to a Killing tensor, see S. Tachibana [1].

$$(n-1)\rho_c = \nabla^b u_{bc}.$$

Simple computations give us the following equations:

$$u^{cd}\nabla^b A_{bcd} = u^{cd}(\nabla^b\nabla_b u_{cd} - R_c{}^e u_{de} - R^b{}_{cd}{}^e u_{be} + (n-3)\rho_{cd} + \rho_{dc}), \tag{3.4}$$

$$A_{bcd}A^{bcd} = 2A_{bcd}\nabla^b u^{cd}, \tag{3.5}$$

where we have used (1.1) and the relation:

$$\begin{aligned}\nabla_b\nabla_c u^b{}_d &= \nabla_c\nabla_b u^b{}_d + R_{bce}{}^b u^e{}_d - R_{bcd}{}^e u^b{}_e \\ &= (n-1)\rho_{cd} - R_c{}^e u_{de} - R^b{}_{cd}{}^e u_{be}.\end{aligned}$$

Substituting (3.4) and (3.5) into

$$\nabla^b(A_{bcd}u^{cd}) = u^{cd}\nabla^b A_{bcd} + A_{bcd}\nabla^b u^{cd},$$

we obtain the following

THEOREM 2. *In a compact orientable Riemannian space M, the following integral formula is valid for any skew symmetric tensor field* u_{cd}:

$$\int_M [u^{cd}(\nabla^a\nabla_a u_{cd} - R_c{}^e u_{de} - R^b{}_{cd}{}^e u_{be} + (n-3)\rho_{cd} + \rho_{dc}) + (1/2)A_{bcd}A^{bcd}]d\sigma = 0,$$

where $d\sigma$ *means the volume element of* M *and* $(n-1)\rho_{cd} = \nabla_c\nabla^b u_{bd}$.

Thus we have

THEOREM 3.[6)] *In a compact Riemannian space a necessary and sufficient condition for a skew symmetric tensor field* u_{cd} *to be a conformal Killing tensor is* (3.1) (*or* (3.2)).

4. Conformal Killing tensor in a space of constant curvature. For a conformal Killing tensor u_{cd} we have

$$\nabla_b u_{cd} + \nabla_c u_{bd} = 2\rho_d g_{bc} - \rho_b g_{cd} - \rho_c g_{bd}, \tag{4.1}$$

$$\nabla^b u_{bc} = (n-1)\rho_c, \tag{4.2}$$

6) I. Sato [3] for a conformal Killing vector and K. Yano and S. Bochner [4] for a Killing tensor.

$$\nabla_c \rho_d + \nabla_d \rho_c = \frac{1}{n-2}(R_c{}^e u_{de} + R_d{}^e u_{ce}). \tag{4.3}$$

The following theorem is a trivial consequence of (4.3).

THEOREM 4. *In an Einstein space, the associated vector of a conformal Killing tensor is a Killing vector.*

In the following we shall assume the space under consideration is a space of constant curvature.

Let v_c be a Killing vector. Then as is well known we have

$$\nabla_a \nabla_b v_c + R_{eabc} v^e = 0.$$

Then by virture of

$$R_{eabc} = k(g_{ec} g_{ab} - g_{ac} g_{eb}), \qquad k = R/n(n-1),$$

the last equation turns to

$$\nabla_a \nabla_b v_c = k(v_b g_{ac} - v_c g_{ab})$$

and hence we obtain

$$\nabla_a \nabla_b v_c + \nabla_b \nabla_a v_c = k(-2v_c g_{ab} + v_b g_{ac} + v_a g_{bc}). \tag{4.4}$$

This equation shows that $\nabla_b v_c$ is a conformal Killing tensor.

Now if u_{cd} is a conformal Killing tensor, then its associated vector ρ_c is a Killing vector and hence $\nabla_b \rho_c$ is a conformal Killing tensor whose associated vector is given by $-k\rho_c$. Thus we have

$$\nabla_b \nabla_c \rho_d + \nabla_c \nabla_b \rho_d = -k(2\rho_d g_{bc} - \rho_b g_{cd} - \rho_c g_{bd}). \tag{4.4}$$

Let us assume that $k \neq 0$ (i.e., $R \neq 0$). If we put

$$p_{cd} = u_{cd} + (1/k) \nabla_c \rho_d,$$

then by virtue of (4.1) and (4.4), it follows that

$$\nabla_b p_{cd} + \nabla_c p_{bd} = 0,$$

which means p_{cd} is a Killing tensor. Consequently a conformal Killing tensor

u_{cd} is decomposed in the form:

$$u_{cd} = p_{cd} + q_{cd},$$

where p_{cd} is a Killing tensor and $q_{cd}=(-1/k)\nabla_c\rho_d$ is a conformal Killing tensor. Hence we have

THEOREM 5.[7] *In a space $M^n(n>2)$ of constant curvature with $k=R/n(n-1)\neq 0$, a conformal killing tensor u_{cd} is uniquely decomposed in the form*:

$$u_{cd} = p_{cd} + q_{cd}, \tag{4.5}$$

where p_{cd} is a Killing tensor and q_{cd} is a closed conformal Killing tensor. In this case q_{cd} is the form

$$q_{cd} = (-1/k)\nabla_c\rho_d$$

where ρ_d is the associated vector of u_{cd}.

Conversely if p_{cd} is a Killing tensor and ρ_d is a Killing vector, then u_{cd} given by (4.5) *is a conformal Killing tensor.*

The uniqueness of the decomposition follows from the following

LEMMA. *Under the assumption of Theorem* 5, *if a Killing tensor is closed, then it is a zero tensor.*

PROOF OF LEMMA. Let u_{cd} be a closed Killing tensor. Then we have

$$\nabla_b u_{cd} + \nabla_c u_{db} + \nabla_d u_{bc} = 0,$$

$$\nabla_b u_{cd} + \nabla_c u_{db} = 0.$$

Hence we get $\nabla_b u_{cd}=0$. Thus by virtue of Ricci's identity it follows that

$$R_{aeb}{}^f u_{fc} + R_{aec}{}^f u_{bf} = 0.$$

As the space is of constant curvature, we can obtain by a transvection with g^{ab}

$$(n-2)ku_{eb} = 0.$$

EXAMPLE. Let E^{n+1} be the Euclidean space with orthogonal coordinates

7) For a conformal Killing vector, see K. Yano and T. Nagano [5].

$\{y^\lambda\}, \lambda=1, \cdots, n+1$. Consider the unit sphere S^n and let $\{x^a\}$ be its local coordinates. Putting $B_a{}^\lambda=\partial y^\lambda/\partial x^a$, we see that the second fundamental tensor $H_{ba}{}^\lambda$ is given by

$$H_{ba}{}^\lambda \equiv \nabla_b B_a{}^\lambda \equiv \partial_b B_a{}^\lambda - B_c{}^\lambda\left\{{c \atop ba}\right\} + B_a{}^\mu\left\{{\lambda \atop \nu\mu}\right\} B_b{}^\nu, \qquad \partial_b \equiv \partial/\partial x^b.$$

As S^n is totally umbilic we have $H_{ba}{}^\lambda = g_{ba}N^\lambda$, where N^λ means the unit normal vector : $N^\lambda = -y^\lambda$.

Let $v_{\mu\nu}$ be a parallel skew symmetric tensor field and define a tensor field u_{bc} on S^n by $u_{bc}=B_b{}^\mu B_c{}^\nu v_{\mu\nu}$. Operating ∇_a to this equation we have

$$\begin{aligned}\nabla_a u_{bc} &= B_a{}^\lambda \nabla_\lambda v_{\mu\nu} B_b{}^\mu B_c{}^\nu + v_{\mu\nu}(H_{ab}{}^\mu B_c{}^\nu + B_b{}^\mu H_{ac}{}^\nu)\\ &= v_{\mu\nu}(N^\mu B_c{}^\nu g_{ab} + N^\nu B_b{}^\mu g_{ac}).\end{aligned}$$

If we put $\rho_c = v_{\mu\nu}N^\mu B_c{}^\nu$, then it follows that

$$\nabla_a u_{bc} = \rho_c g_{ab} - \rho_b g_{ac}.$$

Thus we get

$$\nabla_a u_{bc} + \nabla_b u_{ac} = 2\rho_c g_{ab} - \rho_a g_{bc} - \rho_b g_{ac},$$

which shows u_{bc} is a conformal Killing tensor on S^n.

BIBLIOGRAPHY

[1] S. TACHIBANA, On Killing tensor in a Riemannian space, Tôhoku Math. J., 20(1968), 255–262.
[2] S. TACHIBANA AND T. KASHIWADA, On the integrability of Killing-Yano's equation, to appear.
[3] I. SATO, On conformal Killlng tensor fields, Bull. Yamagata Univ., 3 (1956), 175–180.
[4] K. YANO AND S. BOCHNER, Curvature and Betti numbers, Ann. of Math., Stud. 32, 1953.
[5] K. YANO AND T. NAGANO, Some theorems on projective and conformal transformations, Ind. Math., 14 (1957), 451-458.
[6] T. KASHIWADA, On conformal Killing tensor, to appear in Nat. Sci. Rep. of the Ochanomizu University.

DEPARTMENT OF MATHEMATICS
OCHANOMIZU UNIVERSITY
TOKYO, JAPAN