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A COMPLEMENT TO "ON THE HOMOMORPHISM OF VON NEUMANN ALGEBRA"

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In the proof of Theorem I in [1], the continuum hypothesis has been used. Dr. J.Vesterstrφm has asked the author whether Theorem I in [1] can be proved or not without using the continuum hypothesis. In this paper, we shall show that the fact holds without the continuum hypothesis. Before going into the discussions, the author wishes to his thanks to Dr. J. Vesterstrφm for his indebted communication in the presentation of this paper.

THEOREM. *Let M be a properly infinite von Neumann algebra with the separable predual M_*; then any homomorphism π from M onto a von Neumann algebra N is σ-weakly continuous.*

PROOF. In [1], we have shown the following fact, that is, if any *-homomorphism from a von Neumann algebra M onto a von Neumann algebra N is σ-weakly continuous, then any homomorphism from M onto N is σ-weakly continuous. Therefore, to prove Theorem, we may suppose that π is a *-homomorphism.

If the kernel $\pi^{-1}(0)=I$ of π is σ-weakly closed, then there exists a central projection z of M such that $I=M_{(1-z)}$. Then, the restriction of π to Mz is a *-isomorphism from Mz onto N. Therefore, π is σ-weakly continuous. Next, we assume that I is not σ-weakly closed and derive a contradiction. Let $\tilde{I}$ be the σ-weak closure of I in M. Then there exists a central projection z of M such that $\tilde{I}=Mz$. Since π is a *-homomorphism from M onto N, $\pi(z)$ is a central projection of N. Furthermore, π induces a *-homomorphism from Mz onto $N_{\pi(z)}$. Therefore, we can assume that π is a *-homomorphism from M onto N and the kernel $\pi^{-1}(0)=I$ is σ-weakly dense in M. Since the predual M_* is separable, M is σ-finite and the cardinal number of M_p is not larger than $\mathfrak{c}$.

Now, since the σ-weak closure of I is M and M is σ-finite, there exists a sequence $\{e_n\}_{n=1}^{\infty}$ of mutually orthogonal projections in I with $\sum_{n=1}^{\infty} e_n=1$. That is, there exists a family $\{e_n\}_{n=1}^{\infty}$ of countable orthogonal projections such that $\sum_{n=1}^{\infty} e_n$

$=1$ and $\pi(e_n)=0$ for all n. Then, by the same argument as in the proof of Lemma 1 in [1], there exists a continuum family $\{q_s\}_{s\in R}$ of projections in M such that $\pi(q_s q_{s'})=0$ if $s\neq s'$ and $\pi(q_s)\neq 0$. Indeed, since M is properly infinite, there exists in M a family $\{p_n\}_{n=1}^{\infty}$ of orthogonal projections such as $p_n\sim 1$. Take a partial isometry v_n such that $v_n{}^*v_n=p_n$ and $v_nv_n{}^*=1$ for each n. Define the family $\{q_n\}_{n=1}^{\infty}$ of orthogonal projections by $q_n=v_n{}^*\left(\sum_{k=1}^{n}e_k\right)v_n$.

Let $\{n_i\}_{i=1}^{\infty}$ be an increasing sequence of positive integers, then we have

$$q_{n_{i+1}}=v_{n_{i+1}}^*\left(\sum_{k=1}^{n_{i+1}}e_k\right)v_{n_{i+1}}\geqq v_{n_{i+1}}^*\left(\sum_{k=n_i+1}^{n_{i+1}}e_k\right)v_{n_{i+1}}\sim\sum_{k=n_i+1}^{n_{i+1}}e_k.$$

Hence $\sum_{i=1}^{\infty}q_{ni}\succsim\sum_{k=n_1+1}^{\infty}e_k$, and $\pi\left(\sum_{i=1}^{\infty}q_{n_i}\right)\succsim\pi\left(\sum_{k=n_1+1}^{\infty}e_k\right)=\pi\left(1-\sum_{k=1}^{n_1}e_k\right)=\pi(1)\neq 0$. Therefore, $\pi\left(\sum_{i=1}^{\infty}q_{n_i}\right)\neq 0$. On the other hand $\pi(q_n)=\pi(v_n)^*\left(\sum_{k=1}^{n}\pi(e_k)\right)\pi(v_n)=0$.

Next, let $\{r_n\}_{n=1}^{\infty}$ be a countable set of all rational numbers. For each real number s, we can choose an infinite subsequence $\{r_{n_i}\}_{i=1}^{\infty}$ of $\{r_n\}_{n=1}^{\infty}$ such that, for each positive integer i, we have $0<|r_{n_i}-s|<1/i$ and $n_j<n_i$ for every $j<i$. Now, let us correspond to s the index set $\{n_i\}_{i=1}^{\infty}$ of the above sequence $\{r_{n_i}\}_{i=1}^{\infty}$. Then if $s\neq s'$ for real numbers s and s', $\{n_i\}\cap\{n_i'\}$ is at most a finite set where $\{n_i\}_{i=1}^{\infty}$ and $\{n'_i\}_{i=1}^{\infty}$ are corresponding index sets of s and s' respectively, because $\{r_{n_i}\}_{i=1}^{\infty}$ and $\{r_{n'_i}\}_{i=1}^{\infty}$ converge to s and s' respectively. Therefore, if we set $q_s=\sum_{i=1}^{\infty}q_{n_i}$ for a real number s where $\{n_i\}_{i=1}^{\infty}$ is an index set corresponding to s, we get $\pi(q_s)\neq 0$ and $\pi(q_sq_{s'})=0$ if $s\neq s'$.

Now, there exists a continuum family of orthogonal projections in N and so the cardinal number of the set of all projections in N is not smaller than $2^{\mathfrak{c}}$. But, since the cardinal number of M_p is not larger than $\mathfrak{c}$ and $\pi(M_p)=N_p, N_p\leqq M_p\leqq\mathfrak{c}$. This is a contradiction. Therefore, the kernel $\pi^{-1}(0)=I$ of π is σ-weakly closed in M. By the first argument in this paper, π is σ-weakly continuous. This completes the proof of Theorem.

References

[1] H. Takemoto, On the homomorphism of von Neumann algebra, Tôhoku Math. J., 21(1969), 152–157.

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