Tôhoku Math. Journ.
22(1970), 517-524.

ON THE INTEGRABILITY OF A D-KILLING EQUATION IN A SASAKIAN SPACE

SEIICHI YAMAGUCHI

(Received Oct. 27, 1969; revised Jan. 12, 1970)

0. Introduction. In a Riemannian space M^n, a skew symmetric tensor $u_{a_1 \cdots a_p}$ is called a Killing tensor of order p, if it satisfies the Killing-Yano's equation:

$$\nabla_{a_0} u_{a_1 \cdots a_p} + \nabla_{a_1} u_{a_0 a_2 \cdots a_p} = 0,$$

where ∇ denotes the operator of the Riemannian covariant derivative.

The following theorem is well known [1, 5]:

THEOREM A. *A necessary and sufficient condition in order that the Killing-Yano's equaton is completely integrable is that the Riemannian space $M^n(n > 2)$ is a space of constant curvature*

Recently S. Tanno [8, 9] has investigated automorphism groups of almost Hermitian spaces and almost contact Riemannian spaces and classified those spaces which admit automorphism groups of maximum dimensions.

We consider the analogy of Theorem A in a Sasakian space. Denote the structure tensors of a Sasakian space by (φ, η, g). Y. Ogawa [2] has introduced the notion of C-Killing 1-form on a Sasakian space. We call a 1-form u C-Killing if it satisfies $\delta u = 0$ and leaves invariant $g_{ab} - \eta_a \eta_b$ Especially if a C-Killing 1-form u satisfies $\eta_r u^r =$ constant, it is called special C-Killing. For a special C-Killing 1-form u we have

$$\nabla_a u_b + \nabla_b u_a = -2u_r(\varphi_a{}^r \eta_b + \varphi_b{}^r \eta_a).$$

In this paper we shall define a D-Killing vector of type α in a Sasakian space and consider the analogy of Theorem A with respect to a D-Killing vector of type α. In §2 we shall define a D-Killing vector of type α and give some examples. We consider a sufficient condition for a Sasakian space to be a space of constant ϕ-holomorphic sectional curvature in §3. §4 will be devoted to the integrability condition of the D-Killing equation.

The author would like to express here his thanks to Prof. S. Tanno for his valuable criticisms and advices.

1. Preliminaries. An n-dimensional Riemannian space M is called a Sasakian space if it admits a unit Killing vector field η^a such that

$$\nabla_a \nabla_b \eta_c = \eta_b g_{ac} - \eta_c g_{ab}, \tag{1.1}$$

where g_{ab} is the metric tensor of M. Then n is necessarily odd $(=2m+1)$ and M is orientable. With respect to a local coordinate system $\{x^a\}$, if we define a 2-form $\varphi = (1/2)\varphi_{ab} dx^a \wedge dx^b$ by $\varphi_{ab} = \nabla_a \eta_b$, then we have $d\eta = 2\varphi$ and it holds

$$\nabla_a \varphi_{bc} = \eta_b g_{ac} - \eta_c g_{ab}. \tag{1.2}$$

On a Sasakian space M, the following identities are well known:

$$\nabla_r \varphi_a{}^r = (n-1)\eta_a, \tag{1.3}$$

$$R_{abc}{}^r \eta_r = \eta_a g_{bc} - \eta_b g_{ac}. \tag{1.4}$$

When the curvature tensor of a Sasakian space M has components of the form

$$\begin{aligned} 4R_{abcd} = {} & (H+3)(g_{ad}g_{bc} - g_{ac}g_{bd}) \\ & + (H-1)(\eta_b\eta_d g_{ac} + \eta_a\eta_c g_{bd} - \eta_a\eta_d g_{bc} - \eta_b\eta_c g_{ad} \\ & + \varphi_{ad}\varphi_{bc} - \varphi_{bd}\varphi_{ac} - 2\varphi_{ab}\varphi_{cd}), \end{aligned} \tag{1.5}$$

then M is called a space of constant ϕ-holomorphic sectional curvature (a locally C-Fubinian space). In such a space $M(n>3)$, H is necessarily constant.

2. *D*-Killing vector of type α. In this section we define a D-Killing vector of type α.

We call a vector field u^a a D-Killing vector of type α if it satisfies the following equation

$$\nabla_a u_b + \nabla_b u_a = -2\alpha u_r(\varphi_a{}^r \eta_b + \varphi_b{}^r \eta_a), \tag{2.1}$$

where α is constant. We call this the D-Killing equation of type α.

REMARK 1. A D-Killing vector of type 1 is special C-Killing in a compact Sasakian space [2].

REMARK 2. A D-Killing vector of type α is a special form of an infinitesimal $(m-1)$-conformal transformation [6].

REMARK 3. Let u^a be a D-Killing vector of type α satisfying $\eta_r u^r=$constant. Then we have

$$\theta(u)(g_{ab}-\alpha\eta_a\eta_b)=0 \quad \text{and} \quad \delta u=0,$$

where $\theta(u)$ means the operator of Lie derivative with respect to u^a.

In a Sasakian space M, let us consider a set of differential equations

$$\frac{\delta}{ds}\left(\frac{dx^a}{ds}\right)=\beta\eta_r\varphi_s{}^a\frac{dx^r}{ds}\frac{dx^s}{ds}, \tag{2.2}$$

where s indicates arc-length and δ covariant differentiation along the curve and β is constant. These equations show us that its integral curve is a C-loxodrome, that is, the curve is a loxodrome cutting trajectories of η^a with constant angle [10]. We shall call (2.2) a C-loxodrome of type β.

Now take a C-loxodrome $x^a(s)$ in a Sasakian space and consider the inner product of a D-Killing vector of type $\beta/2$ and a unit tangent vector dx^a/ds to the C-loxodrome. Then along the C-loxodrome we have

$$\frac{\delta}{ds}\left(u_a\frac{dx^a}{ds}\right)=\frac{1}{2}[\nabla_a u_b+\nabla_b u_a+\beta u_r(\varphi_a{}^r\eta_b+\varphi_b{}^r\eta_a)]\frac{dx^a}{ds}\frac{dx^b}{ds}=0, \tag{2.3}$$

which shows that the inner product is constant along the C-loxodrome.

Conversely, if the inner product is constant along any C-loxodrome, then we have (2.3) for any dx^a/ds and we can see that the vector u^a is a D-Killing vector of type $\beta/2$. Thus we have

THEOREM 2.1. *In order that a vector field u^a define a D-Killing vector of type $\beta/2$ in a Sasakian space, it is necessary and sufficient that the inner product of u^a and a unit tangent to an arbitrary C-loxodrome of type β be constant along this C-loxodrome.*

Next, let us consider a D-homothetic deformation [7] $g\to *g$ defined by

$$*g_{ab}=\beta g_{ab}+\beta(\beta-1)\eta_a\eta_b$$

for a positive constant β. Denoting by $W_b{}^a{}_c$ the difference $*\left\{{a \atop bc}\right\}-\left\{{a \atop bc}\right\}$ of Christoffel's symbols, we have in a Sasakian space

(2. 4) $$W_b{}^a{}_c = (\beta - 1)(\varphi_b{}^a\eta_c + \varphi_c{}^a\eta_b).$$

If $\varphi_a{}^b$, η_a, g_{ab} are the structure tensors of a Sasakian space, then

(2. 5) $$*\varphi_b{}^a = \varphi_b{}^a, \quad *\eta_b = \beta\eta_b, \quad *g_{ab} = \beta g_{ab} + \beta(\beta - 1)\eta_a\eta_b$$

are also the structure tensors of a Sasakian space [7].

Let u^a be a Killing vector with respect to g_{ab}. Then we have

$$\nabla_a u_b + \nabla_b u_a = 0.$$

Taking account of (2. 4) and (2. 5), the above equation reduces to

$$*\nabla_a u_b + *\nabla_b u_a = -2(1 - \beta^{-1})u_r(*\varphi_a{}^r *\eta_b + *\varphi_b{}^r *\eta_a),$$

where $*\nabla$ denotes the operator of the covariant derivative with respect to $*g$. Thus we have

THEOREM 2. 2. *For a Killing* 1*-form u_a with respect to g_{ab} on a Sasakian space, $*g^{ab}u_b$ is a D-Killing vector of type* $1-\beta^{-1}$ *with respect to* $*g_{ab}$.

3. A sufficient condition for M to be a space of constant ϕ-holomorphic sectional curvature. In this section and next section, we shall deal with a *D*-Killing vector of type α. Then we have

(3. 1) $$\theta(u)g_{cd} = u_{cd} + u_{dc} = -2\alpha u_r(\varphi_c{}^r\eta_d + \varphi_d{}^r\eta_c),$$

where we put $u_{cd} = \nabla_c u_d$. Into the following formula

$$2\theta(u)\left\{ {r \atop bc} \right\} = g^{rs}(\nabla_b\theta(u)g_{cs} + \nabla_c\theta(u)g_{bs} - \nabla_s\theta(u)g_{bc}),$$

we substitute (3. 1), then we have

(3. 2) $$\left(\theta(u)\left\{ {r \atop bc} \right\}\right)g_{rd} = u_{bcd} + u^r R_{rbcd}$$
$$= \alpha[u_b{}^r(\varphi_{rc}\eta_d + \varphi_{rd}\eta_c) + u_c{}^r(\varphi_{rb}\eta_d + \varphi_{rd}\eta_b) - u_d{}^r(\varphi_{rb}\eta_c + \varphi_{rc}\eta_b)$$
$$+ 2u^r\{\eta_r\eta_d g_{bc} - \eta_c\eta_d g_{rb} - \eta_b\eta_d g_{rc} + \eta_b\eta_c g_{rd} + \varphi_{rc}\varphi_{bd} + \varphi_{rb}\varphi_{cd}\}],$$

where we put $u_{bcd} = \nabla_b\nabla_c u_d$. By virtue of (3. 2), we can write the following

$$u_{bcd} = \alpha[u_b{}^r(\varphi_{rc}\eta_d + \varphi_{rd}\eta_c) + u_c{}^r(\varphi_{rb}\eta_d + \varphi_{rd}\eta_b) - u_d{}^r(\varphi_{rb}\eta_c + \varphi_{rc}\eta_b)] + [*],$$

where [*] means the term which contains components u^a.

Let us prove the following:

THEOREM 3.1. *Let M be a Sasakian space. For any point Q of M and for any constants C_a and C_{ab} such that C_{ab} satisfies*

$$C_{ab} + C_{ba} = -2\alpha C_r(\varphi_a{}^r\eta_b + \varphi_b{}^r\eta_a), \tag{3.4}$$

if there exists (locally) a D-Killing vector u^a of type α satisfying $u_a(Q)=C_a$ and $\nabla_a u_b(Q) = C_{ab}$, then M is a space of constant ϕ-holomorphic sectional curvature with $H=1-4\alpha$.

PROOF. If we substitute (3.2) into the following formula

$$g_{dr}\theta(u)R_{abc}{}^r = g_{dr}\left(\nabla_a\theta(u)\left\{ {r \atop bc} \right\} - \nabla_b\theta(u)\left\{ {r \atop ac} \right\}\right),$$

we can get

$$\begin{aligned} u_a{}^r R_{rbcd} + u_b{}^r R_{arcd} + u_c{}^r R_{abrd} - u_{rd}R_{abc}{}^r \\ = u_a{}^r A_{dcbr} + u_b{}^r A_{cdar} + u_c{}^r A_{badr} + u_d{}^r A_{abcr} + [*], \end{aligned} \tag{3.5}$$

where we have put

$$A_{dcbr} = \alpha[(\alpha-2)(\eta_d\eta_b g_{rc} - \eta_c\eta_b g_{rd}) + \eta_r(\eta_d g_{bc} - \eta_c g_{bd}) + \varphi_{rc}\varphi_{bd} - \varphi_{rd}\varphi_{bc} + 2\varphi_{rb}\varphi_{cd}].$$

As an initial condition at Q we consider $C_a = 0$ and non-zero C_{ab} such that $C_{ab}+C_{ba} = 0$. Then the equation (3.5) can be rewritten

$$u_a{}^r T_{dcbr} + u_b{}^r T_{cdar} + u_c{}^r T_{badr} + u_d{}^r T_{abcr} = 0, \tag{3.6}$$

where we have put

$$T_{dcbr} = R_{dcbr} - A_{dcbr}. \tag{3.7}$$

The tensor T_{abcd} satisfies the following relations:

$$T_{abcd} + T_{bcad} + T_{cabd} = 0, \quad T_{abcd} + T_{bacd} = 0, \quad T_{abr}{}^r = 0, \tag{3.8}$$

$$T_{rcd}{}^{r} = R_{cd} + 2\alpha g_{cd} + \alpha[(n-1)\alpha - 2n]\eta_c\eta_d ,$$

$$T_{dcbr}g^{cb} = R_{dr} + \alpha(\alpha+1)g_{dr} - \alpha(\alpha+n)\eta_d\eta_r .$$

By virtue of the assumption of the theorem, from (3. 6) we can get

$$T_{dcb}{}^{r}\delta_a^e + T_{cda}{}^{r}\delta_b^e + T_{bad}{}^{r}\delta_c^e + T_{abc}{}^{r}\delta_d^e - T_{dcb}{}^{e}\delta_a^r - T_{cda}{}^{e}\delta_b^r - T_{bad}{}^{e}\delta_c^r - T_{abc}{}^{e}\delta_d^r = 0 .$$

Contracting the last equation on a and e and making use of (3. 8), it follows that

$$(n-1)T_{dcbr} = g_{dr}T_{ebc}{}^{e} - g_{cr}T_{ebd}{}^{e} . \tag{3. 9}$$

Again, transvecting (3. 9) with g^{cd}, we have

$$R_{dr} = \frac{R-(n+1)\alpha}{n} g_{dr} + (n+1)\alpha\eta_d\eta_r , \tag{3. 10}$$

where we have used (3. 8) and R means the scalar curvature. By contraction (3. 10) with $\eta^r\eta^d$, it follows that

$$R = (n-1)(n-(n+1)\alpha) .$$

If we substitute the last equation, (3. 7), (3. 8) and (3. 10) into (3. 9), then we have (1. 5) with $H = 1-4\alpha$. Hence M is a space of constant ϕ-holomorphic sectional curvature with $H = 1-4\alpha$.

4. Integrability condition of a *D*-Killing equation*). In a Sasakian space M we consider the D-Killing equation of type α as a system of partial differential equations of unknown function u_a. This system is equivalent to the following system of partial differential equations with unknown functions u_a and u_{ab}:

$$u_{cd} + u_{dc} = -2\alpha u_r(\varphi_c{}^r\eta_d + \varphi_d{}^r\eta_c) , \tag{4. 1}$$

$$\nabla_c u_d = u_{cd} , \tag{4. 2}$$

$$\begin{aligned} \nabla_b u_{cd} = & -u^r R_{rbcd} + \alpha[u_b{}^r(\varphi_{rc}\eta_d + \varphi_{rd}\eta_c) + u_c{}^r(\varphi_{rb}\eta_d + \varphi_{rd}\eta_b) \\ & - u_d{}^r(\varphi_{rb}\eta_c + \varphi_{rc}\eta_b) + 2u^r\{\eta_r\eta_d g_{bc} - \eta_c\eta_d g_{rb} \\ & + \eta_b\eta_c g_{rd} - \eta_b\eta_d g_{rc} + \varphi_{rc}\varphi_{bd} + \varphi_{rb}\varphi_{cd}\}] . \end{aligned} \tag{4. 3}$$

*) In this section we assume that M and all quantities are real analytic.

We shall show that the system is completely integrable if M is a space of constant ϕ-holomorphic sectional curvature with $H=1-4\alpha$, that is, if a curvature tensor of M has the form (1.5) with $H=1-4\alpha$.

From our assumption, we can replace (4.3) by the following equation:

$$\begin{aligned}\nabla_b u_{cd} = {} & \alpha[u_b{}^r(\varphi_{rc}\eta_d+\varphi_{rd}\eta_c)+u_c{}^r(\varphi_{rb}\eta_d+\varphi_{rd}\eta_b) \\ & -u_d{}^r(\varphi_{rb}\eta_c+\varphi_{rc}\eta_b)]-(1-\alpha)(g_{bc}u_d-g_{bd}u_c) \\ & +\alpha u^r(\eta_r\eta_d g_{bc}+\eta_r\eta_c g_{bd}-\eta_b\eta_d g_{rc}-2\eta_c\eta_d g_{rb}+\eta_b\eta_c g_{rd} \\ & +\varphi_{rc}\varphi_{bd}+\varphi_{bc}\varphi_{rd}).\end{aligned} \tag{4.3)'}$$

The equation obtained from (4.1) by differentiation:

$$\partial_b u_{cd}+\partial_b u_{dc} = -2\alpha\partial_b[u_r(\varphi_c{}^r\eta_d+\varphi_d{}^r\eta_c)]$$

are satisfied identically by (4.1), (4.2) and (4.3)$'$.

Next, discuss the integrability condition (4.2):

$$\nabla_b\nabla_c u_d-\nabla_c\nabla_b u_d = -R_{bcdr}u^r. \tag{4.4}$$

Taking account that M is a space of constant ϕ-holomorphic sectional curvature with $H=1-4\alpha$, we have

$$\begin{aligned}-R_{bcdr}u^r = {} & -(1-\alpha)(g_{cd}u_b-g_{bd}u_c)+\alpha(\varphi_{cd}\varphi_b{}^r u_r-\varphi_{bd}\varphi_c{}^r u_r \\ & -2\varphi_{bc}\varphi_d{}^r u_r-g_{cd}\eta_b\eta^r u_r+g_{bd}\eta_c\eta^r u_r-\eta_c\eta_d u_b+\eta_b\eta_d u_c).\end{aligned}$$

On the other hand, by virtue of (4.1), (4.2) and (4.3)$'$, $\nabla_b\nabla_c u_d-\nabla_c\nabla_b u_d$ becomes the right hand side of the above equation. Thus (4.4) holds good.

The integrability condition of (4.3)$'$ is

$$\nabla_a\nabla_b u_{cd}-\nabla_b\nabla_a u_{cd} = -R_{abc}{}^r u_{rd}-R_{abd}{}^r u_{cr}. \tag{4.5}$$

Since M is a space of constant ϕ-holomorphic sectional curvature with $H=1-4\alpha$, we can get

$$\begin{aligned}-R_{abc}{}^r u_{rd}-R_{abd}{}^r u_{cr} = {} & -(1-\alpha)(g_{bc}u_{ad}-g_{ac}u_{bd}+g_{ad}u_{bc}-g_{bd}u_{ac}) \\ & +\alpha[u_c{}^r(g_{bd}\eta_a\eta_r-g_{ad}\eta_b\eta_r+\varphi_{br}\varphi_{ad}-\varphi_{ar}\varphi_{bd}+2\varphi_{ab}\varphi_{dr}) \\ & +u^r{}_d(g_{bc}\eta_a\eta_r-g_{ac}\eta_b\eta_r+\varphi_{br}\varphi_{ac}-\varphi_{ar}\varphi_{bc}+2\varphi_{ab}\varphi_{cr}) \\ & +2(1-\alpha)u^r\{(\varphi_{ar}\eta_c+\varphi_{cr}\eta_a)g_{bd}-(\varphi_{br}\eta_c+\varphi_{cr}\eta_b)g_{ad}\}],\end{aligned}$$

where we have used (4. 1). On the other hand, making use of (4. 1), (4. 2) and (4.3)′, $\nabla_a \nabla_b u_{cd} - \nabla_b \nabla_a u_{cd}$ reduces to the right hand side of the above equation. Therefore (4. 5) holds good.

Summing up the results obtained above, we get

THEOREM 4. 1. *A necessary and sufficient condition in order that the D-Killing equation of type α is completely integrable is that the Sasakian space M $(n > 3)$ is a space of constant ϕ-holomorphic sectional curvature with $H = 1 - 4\alpha$.*

BIBLIOGRAPHY

[1] L. P. EISENHART, Continuous groups of transformations, Princeton Univ. Press, 1933.
[2] Y. OGAWA, On C-Killing forms in a compact Sasakian space, Tôhoku Math. J., 19(1967), 467–484.
[3] S. SASAKI, On differentiable manifolds with certain structures which are closely related to almost contact structure, I, Tôhoku Math. J., 12(1960), 459–476.
[4] S. SASAKI AND Y. HATAKEYAMA, On differentiable manifolds with contact metric structures, J. Math. Soc. Japan, 14(1962), 249–271.
[5] S. TACHIBANA AND T. KASHIWADA, On the integrability of Killing-Yano's equation, J. Math. Soc. Japan, 21(1969), 259–265.
[6] S. TANNO, Partially conformal transformations with respect to $(m-1)$-dimensional distributions of m-dimensional Riemannian manifolds, Tôhoku Math. J., 17(1965), 358–409.
[7] S. TANNO, The topology of contact Riemannian manifolds, Illinois J. Math., 12(1968), 700–717.
[8] S. TANNO, The automorphism groups of almost Hermitian manifolds, Trans. Amer. Math. Soc., 137(1969), 269–275.
[9] S. TANNO, The automorphism groups of almost contact Riemannian manifolds, Tôhoku Math. J., 21(1969), 21–38.
[10] Y. TASHIRO AND S. TACHIBANA, On Fubinian and C-Fubinian manifolds, Kōdai Math. Sem. Rep., 15(1963), 176–183.

DEPARTMENT OF MATHEMATICS
SCIENCE UNIVERSITY OF TOKYO
TOKYO, JAPAN