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ON THE GENERATION OF SEMI-GROUPS OF LINEAR OPERATORS

Dedicated to Professor Gen-ichirô Sunouchi on his 60th birthday

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1. Introduction. This paper is concerned with the generation of semi-groups of classes $(0, A)$ and $(1, A)$.

Let X be a Banach space and let $B(X)$ be the set of all bounded linear operators from X into itself. A one-parameter family $\{T(t); t \geqq 0\}$ is called a *semi-group* (of operators), if it satisfies the following conditions:

$$T(t) \in B(X) \text{ for } t \geqq 0 . \tag{1.1}$$

$$T(0) = I \text{ (the identity)}, \; T(t+s) = T(t)T(s) \text{ for } t, s \geqq 0 . \tag{1.2}$$

$$\lim_{h \to 0} T(t+h)x = T(t)x \text{ for } t > 0 \text{ and } x \in X . \tag{1.3}$$

Let $\{T(t); t \geqq 0\}$ be a semi-group. By the *infinitesimal generator* A_0 of $\{T(t); t \geqq 0\}$ we mean

$$A_0 x = \lim_{h \to 0+} (T(h)x - x)/h \tag{1.4}$$

whenever the limit exists. If A_0 is closable, then $A = \bar{A}_0$ (the closure of A_0) is called the *complete infinitesimal generator* of $\{T(t); t \geqq 0\}$.

The following basic classes of semi-groups are well known (see [2]). If a semi-group $\{T(t); t \geqq 0\}$ satisfies the condition (C_0) $\lim_{t \to 0+} T(t)x = x$ for $x \in X$, then $\{T(t); t \geqq 0\}$ is said to be of *class* (C_0). In this case A_0 is closed and hence the complete infinitesimal generator coincides with the infinitesimal generator. If a semi-group $\{T(t); t \geqq 0\}$ satisfies the condition

$$\int_0^1 \| T(t) \| dt < \infty \text{ and } \lim_{\lambda \to \infty} \lambda \int_0^\infty e^{-\lambda t} T(t)x \, dt = x \text{ for } x \in X , \tag{1, A}$$

then $\{T(t); t \geqq 0\}$ is said to be of *class* $(1, A)$. If, instead of the condition $(1, A)$, $T(t)$ satisfies the weaker condition

$$\int_0^1 \| T(t)x \| dt < \infty \text{ and } \lim_{\lambda \to \infty} \lambda \int_0^\infty e^{-\lambda t} T(t)x \, dt = x \text{ for } x \in X , \tag{0, A}$$

then a semi-group $\{T(t); t \geqq 0\}$ is said to be of *class* $(0, A)$. Clearly $(C_0) \subset (1, A) \subset (0, A)$ in the set theoretical sense. It is known that in general the infinitesimal generator of a semi-group of class $(1, A)$ need not be closed, and that every semi-group of class $(0, A)$ has the complete infinitesimal generator (see [2, 5]).

Our main results are as follows.

THEOREM 1. *An operator A is the complete infinitesimal generator of a semi-group $\{T(t); t \geqq 0\}$ of class $(0, A)$ if and only if*

(i) *A is densely defined, closed linear operator with domain and range in X,*

(ii) *there is a real ω such that $\{\lambda; \lambda > \omega\} \subset \rho(A)$ (the resolvent set of A),*

(iii) *$\|R(\lambda; A)\| = O(1/\lambda)$ as $\lambda \to \infty$, where $R(\lambda; A)$ is the resolvent of A,*

and $R(\lambda; A)$ satisfies either of the following conditions (iv_1), (iv_2);

(iv_1) *for each $x \in X$ there exists a non-negative measurable function $f(t, x)$ on $(0, \infty)$ satisfying*

(a) *for each $x \in X$, $f(t, x)$ is bounded on every compact subset of the open interval $(0, \infty)$,*

(b) $\int_0^\infty e^{-\omega t} f(t, x)dt < \infty$ *for $x \in X$,*

(c) $\|R(\lambda; A)^n x\| \leqq 1/(n-1)!\int_0^\infty e^{-\lambda t} t^{n-1} f(t, x)dt$ *for $x \in X$, $\lambda > \omega$ and $n \geqq 1$,*

(iv_2) (a') *for every $\varepsilon > 0$ there exist $M_\varepsilon > 0$ and $\lambda_0 = \lambda_0(\varepsilon)$ such that $\|\lambda^n R(\lambda; A)^n\| \leqq M_\varepsilon$ for $\lambda > \lambda_0$ and n with $n/\lambda \in [\varepsilon, 1/\varepsilon]$,*

(b') *there exists an $M > 0$ such that $\|R(\lambda; A)^n x\| \leqq M(\lambda - \omega)^{-n}\|x\|_1$ for $x \in D(A)$, $\lambda > \omega$ and $n \geqq 1$, where $\|x\|_1 = \|x\| + \|Ax\|$,*

(c') $\int_0^\infty e^{-\omega t} \liminf_{n\to\infty} \|T(t; n)x\|dt < \infty$ *for $x \in X$, where*

$$T(t; n) = \left(I - \frac{t}{n}A\right)^{-n} = \left[\frac{n}{t}R\left(\frac{n}{t}; A\right)\right]^n \text{ for } t > 0 \text{ and } n > \omega t \tag{1.5}$$
$$= I \text{ for } t = 0 \text{ and } n \geqq 1 .$$

THEOREM 2. *An operator A is the complete infinitesimal generator of a semi-group $\{T(t); t \geqq 0\}$ of class $(1, A)$ if and only if* (i)—(iii) *in Theorem 1 are satisfied, and $R(\lambda; A)$ satisfies either of the following conditions* (v_1), (v_2);

(v_1) *there exists a non-negative measurable function $f(t)$ on $(0, \infty)$ with the properties*

(a) $\int_0^\infty e^{-\omega t} f(t)dt < \infty$,

(b) $\|R(\lambda; A)^n\| \leqq 1/(n-1)!\int_0^\infty e^{-\lambda t} t^{n-1} f(t)dt$ *for $\lambda > \omega$ and $n \geqq 1$,*

(v_2) (a') *for every $\varepsilon > 0$ there exist $M_\varepsilon > 0$ and $\lambda_0 = \lambda_0(\varepsilon)$ such that $\|\lambda^n R(\lambda; A)^n\| \leqq M_\varepsilon$ for $\lambda > \lambda_0$ and n with $n/\lambda \in [\varepsilon, 1/\varepsilon]$,* (b′) *there exists an* $M>0$ *such that* $\|R(\lambda; A)^n x\| \leqq M(\lambda-\omega)^{-n}\|x\|_1$ *for* $x\in D(A)$, $\lambda>\omega$ *and* $n\geqq 1$,

(c′) $\int_0^\infty e^{-\omega t}\liminf_{n\to\infty}\|T(t; n)\|dt<\infty$.

Theorem 1 is new. To generate semi-groups of class $(0, A)$ the author assumed in [3] that, instead of (iv_1)-(a), for each $x\in X$, $f(t, x)$ is continuous in $t>0$. The condition (v_1) in Theorem 2 was first given by Phillips [2, 5], and the conditions (iv_2) and (v_2) in the above theorems are quite new.

Our proof of Theorem 1 is based on the generation theorem for semi-groups of class $(C_{(k)})$ due to Oharu [4], and Theorem 2 is proved by using Theorem 1. In §2 we shall deal with semi-groups of class $(C_{(k)})$. Proofs of Theorems 1 and 2 are given in §3.

2. **Semi-groups of class $(C_{(k)})$.** In this section we present the classes $(C_{(k)})$, $k=0, 1, 2, \cdots$, of semi-groups introduced by Oharu [4].

Let $\{T(t); t\geqq 0\}$ be a semi-group. It is well known that $\omega_0\equiv\lim_{t\to\infty} t^{-1}\log\|T(t)\|$ is finite or $-\infty$. And ω_0 is called the *type* of $\{T(t); t\geqq 0\}$. According to Feller [1] we define the *continuity set* $\sum$ of $\{T(t); t\geqq 0\}$ by

$$\Sigma=\left\{x\in X; \lim_{t\to 0+} T(t)x=x\right\}.$$

We see that $X_0\equiv\bigcup_{t>0}T(t)[X]\subset\Sigma$ and if $\lambda>\omega_0$ then the Laplace integral $\int_0^\infty e^{-\lambda t}T(t)xdt$ exists for each $x\in\Sigma$.

LEMMA 2.1. *If X_0 is dense in X and if there exists an $\omega>\omega_0$ such that for each $\lambda>\omega$ there is an operator $R(\lambda)\in B(X)$ with the properties* (a) $R(\lambda)x=\int_0^\infty e^{-\lambda t}T(t)xdt$ *for* $x\in X_0$ *and* (b) $R(\lambda)$ *is invertible, then* $A\equiv\bar{A}_0$ *exists and* $R(\lambda)=R(\lambda; A)$ *for* $\lambda>\omega$.

PROOF. It is easy to see that $R(\lambda)x=\int_0^\infty e^{-\lambda t}T(t)xdt$ for $x\in\Sigma$. Hence $A_0R(\lambda)x=\lim_{h\to 0+}A_hR(\lambda)x=\lim_{h\to 0+}R(\lambda)A_hx=\lambda R(\lambda)x-x$ for $x\in\Sigma$, where $A_h=(T(h)-I)/h$. Since $D(A_0)\subset\Sigma$, we have $R(\lambda)A_0x=\lambda R(\lambda)x-x$ for $x\in D(A_0)$. To show the closability of A_0 let $x_n\in D(A_0)$, $x_n\to 0$ and $A_0x_n\to y$ as $n\to\infty$. Since $R(\lambda)A_0x_n=\lambda R(\lambda)x_n-x_n$, we obtain $R(\lambda)y=0$ and hence $y=0$ by (b). Therefore $A\equiv\bar{A}_0$ exists and $R(\lambda)Ax=\lambda R(\lambda)x-x$, i.e., $R(\lambda)(\lambda-A)x=x$ for $x\in D(A)$. Let $x\in X$. Since X_0 is dense in X, there is a sequence $\{x_n\}$ in X_0 such that $x_n\to x$ as $n\to\infty$. Hence $R(\lambda)x_n\to R(\lambda)x$ and $A_0R(\lambda)x_n=\lambda R(\lambda)x_n-x_n\to\lambda R(\lambda)x-x$ as $n\to\infty$. This means that $R(\lambda)x\in D(A)$ and $AR(\lambda)x=\lambda R(\lambda)x-x$, i.e., $(\lambda-A)R(\lambda)x=x$ for $x\in X$.

Thus $\{\lambda; \lambda > \omega\} \subset \rho(A)$ and $R(\lambda) = R(\lambda; A)$ for $\lambda > \omega$. Q.E.D.

DEFINITION 2.1. A semi-group $\{T(t); t \geqq 0\}$ is said to be of class $(C_{(k)})$, where k is a nonnegative integer, if it satisfies the following conditions:

(a_1) X_0 is dense in X.

(a_2) There exists an $\omega > \omega_0$ such that for each $\lambda > \omega$ there is an operator $R(\lambda) \in B(X)$ with the properties

(a) $R(\lambda)x = \int_0^\infty e^{-\lambda t} T(t)x dt$ for $x \in X_0$,

(b) $R(\lambda)$ is invertible.

(a_3) $D(A^k) \subset \Sigma$, where A is the complete infinitesimal generator of $\{T(t); t \geqq 0\}$ and $A^0 = I$.

It follows from the definition that $(C_{(k)}) \subset (C_{(k+1)})$ and $(C_{(0)})$ is nothing else but the class (C_0). If $\{T(t); t \geqq 0\}$ is a semi-group of class $(0, A)$, then (a_1) and (a_2) are satisfied, and moreover $\lim_{t\to 0+} T(t)x = x$ for $x \in D(A)$, namely, $D(A) \subset \Sigma$ (see [2]). This means $(0, A) \subset (C_{(1)})$. And an example in [2] shows that $(0, A) \neq (C_{(1)})$ (see [2; p. 371, example 1]).

We now mention the generation theorem for semi-groups of class $(C_{(k)})$ due to Oharu [4].

THEOREM A. *An operator A is the complete infinitesimal generator of a semi-group $\{T(t); t \geqq 0\}$ of class $(C_{(k)})$ if and only if*

(α_1) *A is densely defined, closed linear operator with domain and range in X,*

(α_2) *there is a real ω such that $\{\lambda; \lambda > \omega\} \subset \rho(A)$,*

(α_3) *there exists an $M > 0$ such that*

$$\|R(\lambda; A)^n x\| \leqq M(\lambda - \omega)^{-n}\|x\|_k \text{ for } x \in D(A^k),\ \lambda > \omega \text{ and } n \geqq 1\,,$$

where $\|x\|_k = \|x\| + \|Ax\| + \cdots + \|A^k x\|$,

(α_4) *for every $\varepsilon > 0$ and $x \in D(A^k)$ there are $M_\varepsilon > 0$ and $\lambda_0 = \lambda_0(\varepsilon, x)$ such that $\|\lambda^n R(\lambda; A)^n x\| \leqq M_\varepsilon \|x\|$ for $\lambda > \lambda_0$ and n with $n/\lambda \in [\varepsilon, 1/\varepsilon]$.*

Then the semi-group $\{T(t); t \geqq 0\}$ generated by A has the following property; for each $x \in D(A^k)$

$$T(t)x = \lim_{n\to\infty} T(t; n)x = \lim_{n\to\infty} \left(I - \frac{t}{n}A\right)^{-n} x$$

uniformly on every compact interval of $[0, \infty)$.

3. Proofs of Theorems 1 and 2. We start from the following

LEMMA 3.1. *Let A be a closed linear operator with domain and range in X.*

Suppose that

(i) *there is a real ω such that $\{\lambda; \lambda > \omega\} \subset \rho(A)$,*

(ii) *for each* $x \in X$ *there exists a non-negative measurable fuction* $f(t, x)$ *on* $(0, \infty)$ *satisfying the following properties*

(ii$_1$) $\int_0^\infty e^{-\omega t} f(t, x)dt < \infty$ *for* $x \in X$,

(ii$_2$) $\|R(\lambda; A)^n x\| \leqq 1/(n-1)! \int_0^\infty e^{-\lambda t} t^{n-1} f(t, x)dt$ *for* $x \in X, \lambda > \omega$ *and* $n \geqq 1$.

Then we have

(i′) *there exists a constant* $M > 0$ *such that*

$$\|R(\lambda; A)^n x\| \leqq M(\lambda - \omega)^{-n}\|x\|_1 \text{ for } x \in D(A), \lambda > \omega \text{ and } n \geqq 1 ,$$

(ii′) $\int_0^\infty e^{-\omega t} \liminf_{n\to\infty} \|T(t; n)x\| dt < \infty$ *for* $x \in X$, *where* $T(t; n)$ *are operators defined by* (1.5).

PROOF. (i′) Let $\lambda > \omega$ and $x \in D(A)$. Since $R(\lambda; A)^k(A - \omega)x = (\lambda - \omega)R(\lambda; A)^k x - R(\lambda; A)^{k-1}x$, we obtain from (ii$_2$) that

$$\begin{aligned}
&\|(\lambda - \omega)^k R(\lambda; A)^k x - (\lambda - \omega)^{k-1} R(\lambda; A)^{k-1} x\| \\
&= \|(\lambda - \omega)^{k-1} R(\lambda; A)^k (A - \omega)x\| \leqq \frac{(\lambda - \omega)^{k-1}}{(k-1)!}\int_0^\infty e^{-\lambda t} t^{k-1} f(t, (A - \omega)x)dt
\end{aligned}$$

for $k \geqq 1$. Hence

$$\begin{aligned}
\|(\lambda - \omega)^n R(\lambda; A)^n x - x\| &\leqq \int_0^\infty e^{-\lambda t} \sum_{k=1}^n \frac{(\lambda - \omega)^{k-1} t^{k-1}}{(k-1)!} f(t, (A - \omega)x)dt \\
&\leqq \int_0^\infty e^{-\omega t} f(t, (A - \omega)x)dt \text{ for } n \geqq 1 .
\end{aligned}$$

Since $R(\lambda; A)^n$, $\lambda > \omega$, $n \geqq 1$, are bounded linear operators from the Banach space $D(A)$ with the norm $\|x\|_1 = \|x\| + \|Ax\|$ into X, the above inequality implies that there is an $M > 0$ such that

$$\|(\lambda - \omega)^n R(\lambda; A)^n x\| \leqq M\|x\|_1$$

for $x \in D(A)$, $\lambda > \omega$ and $n \geqq 1$ (the uniform boundedness principle).

(ii′) Let $T > 0$ be arbitrary but fixed, and let $x \in X$. Then for each integer n with $n > T|w|$, $T(t; n)$ is well defined on $[0, T]$ and by (ii$_2$)

$$\|T(t; n)x\| \leqq \frac{(n/t)^n}{(n-1)!}\int_0^\infty e^{-ns/t} s^{n-1} f(s, x)ds \text{ for } 0 < t \leqq T .$$

For each integer $n \geqq 1$ let us define a function E_n by

$$E_n(t) = \begin{cases} (1 - \omega t/n)^n & \text{for } 0 \leqq t \leqq n/|\omega| \\ 0 & \text{for } n/|\omega| < t \end{cases} \quad \text{if } \omega \neq 0 ,$$

and $E_n(t) \equiv 1$ if $\omega = 0$. Then

$$\int_0^T E_n(t)\,\|T(t;n)x\|dt \leqq \int_0^\infty E_n(t)\left[\frac{(n/t)^n}{(n-1)!}\int_0^\infty e^{-ns/t}s^{n-1}f(s,x)ds\right]dt$$
$$= \int_0^\infty s^{n-1}f(s,x)\left[\frac{1}{(n-1)!}\int_0^\infty E_n(t)(n/t)^n e^{-ns/t}dt\right]ds\ ,$$

where $n > T|\omega|$. Now,

$$J \equiv \frac{1}{(n-1)!}\int_0^\infty E_n(t)(n/t)^n e^{-ns/t}dt$$
$$= \frac{1}{(n-1)!}\int_0^{n/|\omega|}(n/t-\omega)^n e^{-ns/t}dt = \frac{ne^{-\omega s}}{(n-1)!}\int_{|\omega|-\omega}^\infty \frac{t^n}{(t+\omega)^2}e^{-st}dt\ ;$$

and a simple calculus shows that $J \leqq (n/(n-1))e^{-\omega s}s^{1-n}$ if $\omega \geqq 0$, and $J \leqq 4\,(n/(n-1))e^{-\omega s}s^{1-n}$ if $\omega < 0$. Therefore

$$\int_0^T E_n(t)\|T(t;n)x\|dt \leqq 4\frac{n}{n-1}\int_0^\infty e^{-\omega s}f(s,x)ds \quad \text{for } n > T|\omega|\ .$$

Passing to the limit as $n \to \infty$, we see from the Fatou lemma that

$$\int_0^T e^{-\omega t}\liminf_{n\to\infty}\|T(t;n)x\|dt \leqq 4\int_0^\infty e^{-\omega s}f(s,x)ds\ .$$

Since T is arbitrary, we obtain the desired conclusion. Q.E.D.

LEMMA 3.2. *Let A be a closed linear operator with domain and range in X. If we assume* (i), (ii) *in Lemma* 3.1 *and* (ii_3) *for each $x \in X$, $f(t,x)$ is bounded on every compact subset of* $(0,\infty)$, *then for each $\varepsilon > 0$ there exist $M_\varepsilon > 0$ and $\lambda_0 = \lambda_0(\varepsilon)$ such that*

(3.1) $\|\lambda^n R(\lambda;A)^n\| \leqq M_\varepsilon$ for $\lambda > \lambda_0$ *and n with* $n/\lambda \in [\varepsilon, 1/\varepsilon]$.

PROOF. Let $x \in X$ and $\lambda > 2|\omega|$. Clearly

$$\|\lambda^n R(\lambda;A)^n x\| \leqq \frac{\lambda^n}{(n-1)!}\int_0^\infty e^{-(\lambda-\omega)t}t^{n-1}e^{-\omega t}f(t,x)dt \equiv I\ .$$

Note that the function $e^{-(\lambda-\omega)t}t^{n-1}(n \geqq 1)$ is increasing on $[0,\alpha]$ and decreasing on $[\alpha,\infty)$, where $\alpha = (n-1)/(\lambda-\omega)$. Let δ and η be arbitrary numbers with $0 < \delta < 1 < \eta$, and divide the integral domain as follows:

$$I = \frac{\lambda^n}{(n-1)!}\left[\int_0^{\delta\alpha} + \int_{\delta\alpha}^{\eta\alpha} + \int_{\eta\alpha}^\infty\right] \equiv I_1 + I_2 + I_3\ .$$

Then

$$I_1 \leqq \frac{\lambda^n}{(n-1)!}e^{-(\lambda-\omega)\delta\alpha}(\delta\alpha)^{n-1}K(x) = \frac{e^{-(n-1)\delta}}{(n-1)!}\lambda(\lambda\alpha)^{n-1}\delta^{n-1}K(x)\ ,$$

$$I_3 \leqq \frac{e^{-(n-1)\eta}}{(n-1)!}\lambda(\lambda\alpha)^{n-1}\eta^{n-1}K(x), \text{ where } K(x) = \int_0^\infty e^{-\omega t}f(t, x)dt .$$

Since $\alpha\lambda = (n-1)(1+\omega/(\lambda-\omega)) \leqq 2(n-1)$, we have

$$I_1 \leqq \frac{(n-1)^{n-1}}{(n-1)!}\lambda e^{-(n-1)\delta}(2\delta)^{n-1}K(x) .$$

By virtue of the Stirling formula, we obtain

$$I_1 \leqq \frac{e}{\sqrt{2\pi}}\frac{\lambda}{\sqrt{n}}(2\delta e^{1-\delta})^{n-1}K(x) . \tag{3.2}$$

Similarly as in the above, we have

$$I_3 \leqq \frac{e}{\sqrt{2\pi}}\frac{\lambda}{\sqrt{n}}(2\eta e^{1-\eta})^{n-1}K(x) . \tag{3.3}$$

Let $0 < \varepsilon < 1$ and let $n/\lambda \in [\varepsilon, 1/\varepsilon]$. Since $\lambda \leqq n/\varepsilon$, it follows from (3.2) and (3.3) that

$$I_1 \leqq \frac{e}{\sqrt{2\pi}\ \varepsilon}\sqrt{n}(2\delta e^{1-\delta})^{n-1}K(x),\ I_3 \leqq \frac{e}{\sqrt{2\pi}\ \varepsilon}\sqrt{n}(2\eta e^{1-\eta})^{n-1}K(x) .$$

Choose $\delta \in (0, 1)$ and $\eta \in (1, \infty)$ such that $2\delta e^{1-\delta} < 1$ and $2\eta e^{1-\eta} < 1$. Since $\sqrt{n}(2\delta e^{1-\delta})^{n-1}$ and $\sqrt{n}(2\eta e^{1-\eta})^{n-1}$ are bounded with respect to n, there is a $K_\varepsilon > 0$ such that

$$I_1 + I_3 = \frac{\lambda^n}{(n-1)!}\left[\int_0^{\delta\alpha} + \int_{\eta\alpha}^\infty\right] \leqq K_\varepsilon K(x) . \tag{3.4}$$

Finally we estimate

$$I_2 = \frac{\lambda^n}{(n-1)!}\int_{\delta\alpha}^{\eta\alpha} e^{-\lambda t}t^{n-1}f(t, x)dt .$$

It is easy to see that $\delta\varepsilon/4 \leqq \delta\alpha \leqq \eta\alpha \leqq 2\eta/\varepsilon$ for $n \geqq 2$. Set $\lambda_0 = \lambda_0(\varepsilon) = \max(2/\varepsilon, 2|\omega|)$. Then for $\lambda > \lambda_0$ and n with $n/\lambda \in [\varepsilon, 1/\varepsilon]$,

$$\begin{aligned} I_2 &\leqq \frac{\lambda^n}{(n-1)!}\int_{\delta\varepsilon/4}^{2\eta/\varepsilon} e^{-\lambda t}t^{n-1}f(t, x)dt \leqq K(\varepsilon, x)\frac{\lambda^n}{(n-1)!}\int_0^\infty e^{-\lambda t}t^{n-1}dt \\ &= K(\varepsilon, x), \text{ where } K(\varepsilon, x) = \sup\{f(t, x); \delta\varepsilon/4 \leqq t \leqq 2\eta/\varepsilon\} . \end{aligned}$$

Combining this with (3.4), for every $x \in X$ we have

$$||\lambda^n R(\lambda; A)^n x|| \leqq K_\varepsilon\int_0^\infty e^{-\omega t}f(t, x)dt + K(\varepsilon, x)$$

for $\lambda > \lambda_0$ and n with $n/\lambda \in [\varepsilon, 1/\varepsilon]$. By the uniform boundedness principle, there exists an $M_\varepsilon > 0$ such that $||\lambda^n R(\lambda; A)^n|| \leqq M_\varepsilon$ for $\lambda > \lambda_0$ and n with

$n/\lambda \in [\varepsilon, 1/\varepsilon]$. Q.E.D.

We now prove Theorem 1.

PROOF OF THEOREM 1. Suppose first that $\{T(t); t \geqq 0\}$ is of class $(0, A)$. Then the complete infinitesimal generator A satisfies the condition (i). Since $R(\lambda; A)x = \int_0^\infty e^{-\lambda t} T(t)x dt$ for $x \in X$ and $\lambda > \omega_0$ (= the type of $\{T(t); t \geqq 0\}$), (iii) follows from the condition $(0, A)$ together with the uniform boundedness theorem. Choose an $\omega > \omega_0$ and set $f(t, x) = \| T(t)x \|$ for $x \in X$ and $t > 0$. Then (ii) and (iv$_1$) are valid. (Note that $R(\lambda; A)^n x = 1/(n-1)! \int_0^\infty e^{-\lambda t} t^{n-1} T(t)x dt$ for $x \in X$ and $\lambda > \omega$.)

Suppose next that (i)–(iii) and (iv$_1$) are satisfied. It follows from Lemmas 3.1 and 3.2 that the condition (iv$_2$) holds true. Hence, by virtue of Theorem A, A is the complete infinitesimal generator of a semi-group $\{T(t); t \geqq 0\}$ of class $(C_{(1)})$ and

$$T(t)x = \lim_{n \to \infty} T(t; n)x \text{ for } x \in D(A) \text{ and } t \geqq 0 . \tag{3.5}$$

Let $0 < \varepsilon < 1$. It follows from (iv$_2$)–(a′) that if $n > \lambda_0/\varepsilon$, then

$$\| T(t; n) \| = \left\| \left[\frac{n}{t} R\left(\frac{n}{t}; A \right) \right]^n \right\| \leqq M_\varepsilon \text{ for } t \in [\varepsilon, 1/\varepsilon] . \tag{3.6}$$

Since $D(A)$ is dense in X, (3.5) and (3.6) imply that

$$T(t)x = \lim_{n \to \infty} T(t; n)x \text{ for } x \in X \text{ and } t > 0 .$$

Hence we see from (iv$_2$)–(c′) that

$$\int_0^\infty e^{-\omega t} \| T(t)x \| dt < \infty \text{ for } x \in X . \tag{3.7}$$

We next want to show

$$\lim_{\lambda \to \infty} \lambda \int_0^\infty e^{-\lambda t} T(t)x dt = x \text{ for } x \in X . \tag{3.8}$$

Since $\{T(t); t \geqq 0\}$ is of class $(C_{(1)})$, $R(\lambda; A)[X] (= D(A)) \subset \Sigma$ and $R(\lambda; A)x = \int_0^\infty e^{-\lambda t} T(t)x dt$ for $x \in X_0 = \bigcup_{t>0} T(t)[X]$ and sufficiently large λ (see Lemma 2.1). Therefore

$$T(h)R(\lambda; A)x = \int_0^\infty e^{-\lambda t} T(t+h)x dt = e^{\lambda h} \int_h^\infty e^{-\lambda t} T(t)x dt$$

for $h > 0$ and $x \in X$. Letting $h \to 0+$, it follows from $R(\lambda; A)[X] \subset \Sigma$ that

$$R(\lambda; A)x = \int_0^\infty e^{-\lambda t} T(t)x dt$$

for $x \in X$ and sufficiently large λ. Further, by (iii), $\|\lambda R(\lambda; A)x - x\| = \|R(\lambda; A)Ax\| \leqq O(1/\lambda)\|Ax\| \to 0$ for $x \in D(A)$ and hence $\|\lambda R(\lambda; A)x - x\| \to 0$ for $x \in X$ as $\lambda \to \infty$. Thus we obtain (3.8), and hence $\{T(t); t \geqq 0\}$ is of class $(0, A)$. Q.E.D.

To prove Theorem 2 we prepare the following

LEMMA 3.3. *Let A be a closed linear operator with domain and range in X.*

Suppose that

(i) *there is a real ω such that $\{\lambda; \lambda > \omega\} \subset \rho(A)$,*

(ii) $\|R(\lambda; A)\| = O(1/\lambda)$ *as* $\lambda \to \infty$,

(iii) *there exists a non-negative measurable function $f(t)$ on $(0, \infty)$ satisfying the following properties*

(iii$_1$) $\int_0^\infty e^{-\omega t} f(t) dt < \infty$,

(iii$_2$) $\|R(\lambda; A)^n\| \leqq 1/(n-1)! \int_0^\infty e^{-\lambda t} t^{n-1} f(t) dt$ *for* $\lambda > \omega$ *and* $n \geqq 1$.

If we define $T_\lambda(t)$ by $T_\lambda(t) = \{\lambda R(\lambda; A)\}^{[\lambda t]}$ for $\lambda > \max(0, \omega)$ and $t \geqq 0$, where $[\lambda t]$ denotes the integral part of λt, then

(i′) *there is a $\lambda_1 > 0$ such that*

$$\int_0^\infty e^{-\mu t} \| T_\lambda(t) \| dt \leqq 1 + \int_0^\infty e^{-\omega t} f(t) dt \quad \text{for } \lambda > \lambda_1 ,$$

(ii′) *there exist $M > 0$ and $\lambda_0 > 0$ such that*

$$\| T_\lambda(t) \| \leqq M\left(1 + \int_0^\infty e^{-\omega s} f(s) ds\right)^2 e^{\mu t}/t^2 \quad \text{for } t > 0 \text{ and } \lambda > \lambda_0 ,$$

where $\mu = |\omega| + 1$.

PROOF. Let $\lambda > \max(0, \omega)$. Since

$$\| T_\lambda(t) \| \leqq \frac{\lambda^{[\lambda t]}}{([\lambda t] - 1)!} \int_0^\infty e^{-\lambda s} s^{[\lambda t]-1} f(s) ds = \frac{\lambda^k}{(k-1)!} \int_0^\infty e^{-\lambda s} s^{k-1} f(s) ds$$

for $k/\lambda \leqq t < (k+1)/\lambda$, $k = 1, 2, \cdots$, we obtain

$$\int_0^\infty e^{-\mu t} \| T_\lambda(t) \| dt = \int_0^{1/\lambda} e^{-\mu t} dt + \sum_{k=1}^\infty \int_{k/\lambda}^{(k+1)/\lambda} e^{-\mu t} \| T_\lambda(t) \| dt$$

$$\leqq 1/\lambda + \sum_{k=1}^\infty \int_{k/\lambda}^{(k+1)/\lambda} e^{-\mu t} \left[\frac{\lambda^k}{(k-1)!} \int_0^\infty e^{-\lambda s} s^{k-1} f(s) ds \right] dt$$

$$\leqq 1/\lambda + \sum_{k=1}^\infty e^{-\mu k/\lambda} \frac{\lambda^{k-1}}{(k-1)!} \int_0^\infty e^{-\lambda s} s^{k-1} f(s) ds$$

$$\leqq 1/\lambda + \int_0^\infty \exp\left[-\mu s\frac{1-e^{-\mu/\lambda}}{\mu/\lambda}\right] f(s)ds\,.$$

Choose a $\lambda_1 \geqq \max(1, \omega)$ such that $(1-e^{-\mu/\lambda})/(\mu/\lambda) > |\omega|/\mu$ for $\lambda > \lambda_1$. Then we have

$$\int_0^\infty e^{-\mu t}\|T_\lambda(t)\|dt \leqq 1 + \int_0^\infty e^{-|\omega|s} f(s)ds \leqq 1 + \int_0^\infty e^{-\omega t} f(t)dt$$

for $\lambda > \lambda_1$.

We next prove (ii′). By the assumption (ii) there exist $M \geqq 1$ and $\lambda_2 > \max(0, \omega)$ such that $\|\lambda R(\lambda; A)\| \leqq M$ for $\lambda \geqq \lambda_2$. Since $[\lambda(t+s)] - ([\lambda t] + [\lambda s]) = 0$ or 1 for every $t, s \geqq 0$ and $\lambda > 0$, we obtain

$$\begin{aligned} &\|T_\lambda(t+s)\| = \|\{\lambda R(\lambda; A)\}^{[\lambda(t+s)]}\| \\ \text{(3.9)}\qquad &\leqq M\|\{\lambda R(\lambda; A)\}^{[\lambda t]}\{\lambda R(\lambda; A)\}^{[\lambda s]}\| \leqq M\|T_\lambda(t)\|\ \|T_\lambda(s)\| \end{aligned}$$

for $\lambda \geqq \lambda_2$ and $t, s \geqq 0$.

Let $\lambda > \lambda_0 \equiv \max(\lambda_1, \lambda_2)$ and set $g(t) = e^{-\mu t}\|T_\lambda(t)\|$. Then (3.9) implies that $2g(t) \leqq 2Mg(t-s)g(s) \leqq M([g(t-s)]^2 + [g(s)]^2)$ and hence

$$2g(t)^{1/2} \leqq M^{1/2}\{g(t-s) + g(s)\} \quad \text{for} \quad 0 \leqq s \leqq t\,.$$

Now

$$\begin{aligned} t[g(t)]^{1/2} &= 2\int_0^{t/2} g(t)^{1/2}ds \leqq M^{1/2}\int_0^{t/2}\{g(t-s) + g(s)\}ds \\ &= M^{1/2}\int_0^t g(s)ds \leqq M^{1/2}\left(1 + \int_0^\infty e^{-\omega s}f(s)ds\right) \end{aligned}$$

by (i′). Therefore we have the conclusion. Q.E.D.

PROOF OF THEOREM 2. If A is the complete infinitesimal generator of a semi-group $\{T(t); t \geqq 0\}$ of class $(1, A)$, then (i)–(iii) and (v_1) are valid with $f(t) = \|T(t)\|$ and $\omega > \omega_0$.

Suppose next that (i)–(iii) and (v_1) are satisfied. Then (v_2) holds true. In fact, similarly as in the proof of Lemma 3.1, (v_2)–(b′), (c′) follow from (i), (ii) and (v_1). By virtue of Lemma 3.3,

$$\|\{\lambda R(\lambda; A)\}^{[\lambda t]}\| \leqq Ke^{\mu t}/t^2 \text{ for } t > 0 \text{ and } \lambda > \lambda_0\,,$$

where $K = M\left(1 + \int_0^\infty e^{-\omega s}f(s)ds\right)^2$ and M, λ_0, μ are constants in Lemma 3.3 (ii′). If we set $M_\varepsilon = Ke^{\mu/\varepsilon}/\varepsilon^2$ for $\varepsilon > 0$, then

$$\|\{\lambda R(\lambda; A)\}^{[\lambda t]}\| \leqq M_\varepsilon \text{ for } \varepsilon \leqq t \leqq 1/\varepsilon \text{ and } \lambda > \lambda_0$$

and hence (v_2)–(a′) is obtained. Consequently it follows from Theorem 1 that A is the complete infinitesimal generator of a semi-group $\{T(t); t \geqq 0\}$

of class $(0, A)$. Since $T(t)x = \lim_{n\to\infty} T(t; n)x$ for $x \in X$ and $t > 0$ (see the proof of Theorem 1), (v_2)-(c') implies that $\int_0^\infty e^{-\omega t}\| T(t)\| dt < \infty$. Thus $\{T(t); t \geqq 0\}$ is of class $(1, A)$. Q.E.D.

REMARK. The class (A) of semi-groups was introduced by Phillips, and he showed that if $\{T(t); t \geqq 0\}$ is of class (A) then $\lim_{t\to 0+} T(t)x = x$ for $x \in D(A^2)$, where A is the complete infinitesimal generator of $\{T(t); t \geqq 0\}$ (see [2, 6]). This implies that $(A) \subset (C_{(2)})$. And a generation theorem for semi-groups of class (A) is also obtained from Theorem A (see [4]).

REFERENCES

[1] W. Feller, On the generation of unbounded semi-group of bounded linear operators, Ann. of Math., 58 (1953), 166-174.

[2] E. Hille and R. S. Phillips, Functional analysis and semi-groups, Amer. Math. Soc. Colloq. Publ., (1957).

[3] I. Miyadera, On the generation of strongly ergodic semi-groups of operators, II, Tôhoku Math. J., 6 (1954), 231-242.

[4] S. Oharu, Semi-groups of linear operators in a Banach space, Publ. R.I.M.S. Kyoto Univ., 7 (1971), 205-260.

[5] R. S. Phillips, An inversion formula for Laplace Transforms and semi-groups of linear operators, Ann. of Math., 59 (1954), 325-356.

[6] R. S. Phillips, Semi-groups of operators, Bull. Amer. Math. Soc., 61 (1955), 16-33.

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